

Lattices in the metabelian nilpotent Lie group $\mathbb{R}^n \rtimes \mathbb{R}^m$

Réseaux dans les groupes nilpotents métabéliens de type $\mathbb{R}^n \rtimes \mathbb{R}^m$

Béchir Dali and Yasmin Ben Soula

Department of Mathematics, Faculty of Sciences of Bizerte, University of Carthage, 7021 Jarzouna, Bizerte, Tunisia
bechir.dali@fss.rnu.tn, bechir.dali@fsb.ucar.tn, yassousoula@gmail.com

ABSTRACT. The main objective of this paper is to characterize a class of lattices, called splittable lattices, in the nilpotent semidirect product Lie group $G = \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$. Here, η denotes the representation of the abelian Lie group $\mathbb{R}^m$ in $GL_n(\mathbb{R})$ defined by $\eta(t_1, \dots, t_m) = \exp(\sum_{i=1}^m t_i N_i)$ with $(N_i)_{1 \leq i \leq m}$ is a set of pairwise commuting nilpotent matrices in $\mathbb{Q}^{n \times n}$. As an application we study the case of the nilpotent Lie group $F_{n,m} = \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$ where $N_i = J^i$ with $J = \sum_{k=1}^{n-1} E_{k+1,k}$ where $E_{k+1,k} \in \mathbb{R}^{n \times n}$ denotes the matrix given by the Kronecker delta, namely, $(E_{k+1,k})_{i,j} = \delta_{i,k+1} \delta_{j,k}$.

KEYWORDS. lattice, nilpotent Lie group, metabelian group, exponential mapping

1. Introduction

Discrete groups play an important role in several topics linking the geometric structures on manifolds to the representation theory. For instance, it is well known that if Γ is a countable discrete group with its corresponding left regular representation $\pi : \Gamma \rightarrow \mathcal{B}(\ell^2(\Gamma))$, then the Von Neumann algebra $(\pi(\Gamma))'' \subseteq \mathcal{B}(\ell^2(\Gamma))$ is a factor (i.e., its center is trivial) if and only if the group Γ has infinite conjugacy classes, in the sense that the conjugacy class $\{\gamma\beta\gamma^{-1} : \gamma \in \Gamma\}$ is infinite for every $\beta \in \Gamma \setminus \{1_{\Gamma}\}$, where 1_{Γ} is the unit element of Γ (see [6]). The above fact is specific to discrete groups.

By a lattice in a locally compact group G , we mean a discrete subgroup Γ such that the homogeneous space G/Γ carries a finite G -invariant Borel measure. If in addition G/Γ is compact, Γ is said to be a uniform lattice. A necessary condition for a group to contain a lattice is that the group must be unimodular. Recall that a Lie group G is called unimodular if $\text{tr}(\text{ad}_X) = 0$ holds for all $X \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G . This allows for an easy construction of groups without lattices, for example the group of invertible upper triangular matrices or the affine groups, it is also not very hard to find unimodular groups without lattices (for instance see [14, 15]). For nilpotent Lie groups (which are unimodular) the theory becomes simpler than in the general case, and stays similar to the case of abelian groups.

The criterion of the existence of a lattice in a nilpotent Lie group is due to Malcev.

Theorem 1.1. ([13]). *A simply-connected nilpotent Lie group G admits a lattice if and only if there exists a basis $(X_1, \dots, X_n)$ of the Lie algebra $\mathfrak{g}$ of G such that the structure constants $C_{i,j}^k$ arising in the brackets*

$$[X_i, X_j] = \sum_{i,j} C_{i,j}^k X_k$$

are rational numbers. More precisely we have:

- (i) Let $\mathfrak{g}$ have a basis with respect to which the structure constants are rational and let $\mathfrak{g}_{\mathbb{Q}}$ be the vector space over $\mathbb{Q}$ spanned by this basis. Then, if L is any lattice of maximal rank in $\mathfrak{g}$ contained in $\mathfrak{g}_{\mathbb{Q}}$, the group generated by $\exp(L)$ is a lattice in G .
- (ii) If Γ is a lattice in G , then the $\mathbb{Z}$ -span of $\exp^{-1}(\Gamma)$ is a lattice L of maximal rank in the vector space $\mathfrak{g}$ such that the structure constants of $\mathfrak{g}$ with respect to any basis contained in L belong to $\mathbb{Q}$.

Malcev's Rigidity Theorem ([12]) implies that the algebraic structure of uniform (co-compact) discrete subgroups completely determines the global geometry of connected and simply connected nilpotent Lie groups. More precisely let G_1, G_2 be connected and simply connected nilpotent Lie groups. Let $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ be discrete uniform subgroups (lattices). If $\varphi : \Gamma_1 \rightarrow \Gamma_2$ is an abstract group isomorphism, then there exists a unique continuous isomorphism $\bar{\varphi} : G_1 \rightarrow G_2$ such that the restriction of $\bar{\varphi}$ to Γ_1 is exactly φ ($\bar{\varphi}|_{\Gamma_1} = \varphi$).

By a nilmanifold, we mean a compact manifold obtained as a quotient of simply connected nilpotent Lie group G by a lattice Γ . In this setting, wavelet applications shift from the standard flat Euclidean environment to a non-commutative geometric framework. For instance in harmonic analysis, wavelets and lattices can be studied within the context of Lie groups by leveraging the group's inherent structure for translation and dilation, allowing for the construction of wavelet systems on stratified Lie groups and the investigation of discrete decompositions via frames on function spaces like $L^2(G)$. Lattices, which are discrete subgroups, play a crucial role in defining the sampling points for these wavelets and ensuring the existence and properties of frames. Wavelets can be constructed on the homogeneous spaces G/Γ that are formed by taking a quotient of a Lie group G by a lattice Γ . The commensurability of lattices can be used to construct wavelets by leveraging the fact that the lattices are "closely related". This is especially true when using the decomposition of a split solvable Lie group and the commensurability of lattices provides the framework for building a complete wavelet system. Many other applications of nilmanifolds appears in Non-Commutative Multiresolution Analysis (MRA), Time-Frequency Analysis (Heisenberg group), Characterization of Function Spaces, Coherent States and Phase Space, Numerical Approximation (Cascade Algorithms), and so on (for instance see [1, 2, 7, 8]).

In this paper we investigate the nilpotent Lie group $G = \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$ where η is a continuous representation of the topological additive group $\mathbb{R}$ in $SL_n(\mathbb{R})$ given by

$$\eta(t) = \exp\left(\sum_{i=1}^m t_i N_i\right), \quad t = (t_1, \dots, t_m) \in \mathbb{R}^m.$$

Here $(N_i)_{1 \leq i \leq m}$ is a family of linearly independent pairwise commuting nilpotent matrices in $\mathbb{Q}^{n \times n}$. The map $\exp(A) = \sum_{n \geq 0} \frac{A^n}{n!}$ represents the standard matrix exponential. In this framework, we focus our investigation on the class of *splittable lattices* in G . We say that a lattice Γ in G is splittable if it inherits the semidirect structure of G , that is, Γ is isomorphic to the lattice $\Gamma_1 \rtimes_{\eta} \Gamma_2$ where Γ_1 (resp. Γ_2) is a lattice in $\mathbb{R}^n$ (resp. in $\mathbb{R}^m$). To achieve the classification of this class of lattices, we determine $\text{End}(G)$ (the set of homomorphisms of G) and the automorphism group $\text{Aut}(G)$, providing the necessary framework to fully characterize the splittable lattices in G . As an application, we consider the so called m -filiform nilpotent Lie group $F_{n,m} := \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$ with

$$\eta(t) = \exp\left(\sum_{i=1}^m t_i J^i\right), \quad t = (t_1, \dots, t_m) \in \mathbb{R}^m,$$

where $J = E_{2,1} + \dots + E_{n,n-1}$ with $E_{i,j}$ is the n -square matrix with only one non zero coefficient, which is 1 in the i -line and the j -column, that is

$$J = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Note that if $m = 1$ then $F_{n,1}$ is the generic filiform nilpotent Lie group $\mathbb{R}^n \rtimes \mathbb{R}$. The paper is organized as follows. In Section 2, we review the foundational tools and general properties required throughout this work. In Section 3, we investigate splittable lattices in the nilpotent Lie group $G = \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$, where $\eta : \mathbb{R}^m \rightarrow \text{SL}_n(\mathbb{R})$ is the continuous representation of the topological additive group given by $\eta(t) = \exp(\sum_{i=1}^m t_i N_i)$. We then describe the monoid $\text{End}(G)$ and analyze the commensurability of two splittable lattices in G . Finally, Section 4 restricts our attention to the setting where $N_i = J^i$ for $i = 1, \dots, m$, applying the general results established in Section 3 to the specific family of groups $F_{n,m}$.

2. Basic notions and Background material

2.1. Semidirect product of groups

The paper deals with semidirect products $H \rtimes_{\mu} K$ of (Lie) groups H and K . This construction is relative to an action $\mu : K \times H \rightarrow H$ with the group operation defined by the formula

$$(h_1, k_1)(h_2, k_2) = (\mu(k_1, h_2), k_1 k_2) \quad \forall (h_i, k_i) \in H \times K, \quad i = 1, 2.$$

Equivalently, this corresponds to a (smooth) homomorphism

$$\alpha : K \rightarrow \text{Aut}(H), \quad k \mapsto \alpha(k) := \mu(k, \cdot),$$

the multiplication rule on $H \rtimes_{\mu} K$ is defined as

$$(h_1, k_1)(h_2, k_2) = (h_1 \alpha(k_1)(h_2), k_1 k_2) \quad \forall (h_i, k_i) \in H \times K, \quad i = 1, 2..$$

Denote the Lie algebras of H and K by $\mathfrak{h}$ and $\mathfrak{k}$ respectively and let $\phi := (d\mu_1)_{e_H} : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{k})$, where $\mu_1 : K \rightarrow \text{Aut}(\mathfrak{h})$ is given by

$$\mu_1(k) = (d\mu(k, \cdot))_e = \text{Ad}_k^{H \rtimes_{\mu} K}.$$

The Lie algebra of $H \rtimes_{\mu} K$ is $\mathfrak{h} \rtimes_{\phi} \mathfrak{k}$ with the Lie bracket is given by

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2]_{\mathfrak{h}} + \phi(Y_1)X_2 - \phi(Y_2)X_1, [Y_1, Y_2]_{\mathfrak{k}}),$$

for any $(X_1, Y_1), (X_2, Y_2) \in \mathfrak{h} \times \mathfrak{k}$.

Specifically, let $(N_j)_{j=1}^m \subset \mathbb{Q}^{n \times n}$ be a family of linearly independent, pairwise commuting nilpotent matrices with $m < n$. We consider the semidirect product group $G = \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$ equipped with the product rule

$$(x, t)(y, s) = (x + \eta(t)y, t + s) \quad x, y \in \mathbb{R}^n, t, s \in \mathbb{R}^m$$

where η is the continuous representation of the topological additive (abelian) group $\mathbb{R}^m$ given by

$$\eta : \mathbb{R}^m \rightarrow GL_n(\mathbb{R}), \quad t = (t_1, \dots, t_m) \mapsto \eta(t) = \exp(t \cdot N),$$

where $t \cdot N = \sum_{i=1}^m t_i N_i$. Accordingly, the inverse of an element $(x, t) \in G$ takes the form

$$(x, t)^{-1} = (-\eta(-t)x, -t).$$

The Lie algebra $\mathfrak{g} = \mathbb{R}^n \rtimes_{d\eta} \mathbb{R}^m$ of the Lie group G is equipped with the Lie bracket

$$[(X, T), (Y, S)] = \left(\sum_{j=1}^m t_j N_j Y - \sum_{j=1}^m s_j N_j X, 0 \right)$$

for all $X, Y \in \mathbb{R}^n$ and $T = (t_1, \dots, t_m), S = (s_1, \dots, s_m) \in \mathbb{R}^m$. Considering $\mathfrak{g}$ as the vector space $\mathbb{R}^n \oplus \mathbb{R}^m$, the adjoint endomorphisms $\text{ad}_X \equiv \text{ad}_{(X,0)}$ and $\text{ad}_T \equiv \text{ad}_{(0,T)}$ can be expressed in block-matrix form as

$$\text{ad}_X = \begin{pmatrix} 0 & \varphi_X \\ 0 & 0 \end{pmatrix}, \quad \text{ad}_T = \begin{pmatrix} \sum_{i=1}^m t_i N_i & 0 \\ 0 & 0 \end{pmatrix},$$

where $\varphi_X : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the linear mapping (which we identify with its matrix) given by

$$\varphi_X(s) = - \sum_{i=1}^m s_i N_i X.$$

It follows directly from these formulas that $\text{spec}(\text{ad}_{(X,T)}) = \{0\}$, confirming that G is a nilpotent Lie group (see [5]). Furthermore, since $\text{ad}_X^2 = 0$, a straightforward computation yields the global adjoint representation:

$$\text{Ad}_{(x,t)} = \begin{pmatrix} \exp(t \cdot N) & \Phi_{(x,t)} \\ 0 & I_m \end{pmatrix}.$$

Consequently, the Lie group G is unimodular because $|\det \text{Ad}_{(x,t)}| = 1$ for all $(x, t) \in G$.

We choose a basis $(X_1, \dots, X_n, T_1, \dots, T_m)$ for $\mathfrak{g}$ whose non-trivial Lie brackets are determined by

$$[T_k, X_j] = N_k X_j, \quad 1 \leq j \leq n, \quad 1 \leq k \leq m.$$

For our purposes, we rely on the following simultaneous reduction theorem.

Theorem 2.1. ([4]) *Let $A_1, \dots, A_m \in \mathfrak{gl}(n, \mathbb{R})$ be commuting matrices. Then there exist $B \in GL(n, \mathbb{R})$, $d_s \in \mathbb{N}$, and $\mathbb{K}_s \in \{\mathbb{R}, \mathbb{C}\}$ (for $s = 1, \dots, l$) such that*

$$\sum_{s=1}^l d_s \dim_{\mathbb{R}} \mathbb{K}_s = n$$

and for $j = 1, \dots, k$,

$$T_j = B A_j B^{-1} = \begin{pmatrix} T_{j,1} & 0 & \dots & 0 \\ 0 & A_{j,1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & T_{j,l} \end{pmatrix}$$

with blocks $T_{j,s} \in \mathbb{K}_s \mathbf{1}_{\mathbf{d}_s} + \mathcal{N}(\mathbf{d}_s, \mathbb{K}_s)$. If the spectra of $A_1, \dots, A_m$ are known, B is explicitly computable by repeated applications of Gaussian elimination. One has $\mathbb{K}_1 = \dots = \mathbb{K}_l = \mathbb{R}$ if and only if $\text{spec}(A_s) \subset \mathbb{R}$ for all $1 \leq s \leq l$.

Since the nilpotent matrices $(N_j)_{1 \leq j \leq m}$ have completely real spectra, Theorem 2.1 implies that they can be simultaneously triangularized over $\mathbb{Q}$. Consequently, we may assume without loss of generality that each N_j is a strictly lower triangular matrix.

2.2. Nilmanifolds and lattices in nilpotent Lie groups

Let G be a connected, simply connected real nilpotent Lie group, and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or G) has a rational structure if there is a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$ -basis $(X_1, \dots, X_n)$ with rational structure constants. Now a discrete subgroup Γ is called uniform in G if the quotient space G/Γ is compact. A nilmanifold is a compact homogeneous space G/Γ , where G is a connected and simply-connected nilpotent Lie group and Γ a lattice in G , i.e. a discrete co-compact subgroup. Every connected and simply connected nilpotent Lie group is diffeomorphic to $\mathbb{R}^m$ (see e.g. [16]), hence nilmanifolds are aspherical and their fundamental group is isomorphic to the considered lattice. In the special case when $G = \mathbb{R}^n$, the definition of nilmanifolds reduces to the definition of tori. Since simply connected nilpotent Lie groups are diffeomorphic to $\mathbb{R}^n$ (see [11, 16]), nilmanifolds are quotients of $\mathbb{R}^n$ by a left action of a discrete nilpotent group Γ .

For a given lattice Γ in a connected and simply-connected nilpotent Lie group G , the subset $\exp^{-1}(\Gamma)$ need not to be an additive subgroup of the Lie algebra $\mathfrak{g}$ (see [3]). If Γ is a lattice in G such that $\exp^{-1}(\Gamma)$ is a subgroup in the Lie algebra $\mathfrak{g}$ of G , we call Γ a lattice subgroup.

According to the criterion stated above, the rational entries of the matrices $(N_j)_{j=1}^m$ equip the Lie algebra $\mathfrak{g}$ with a rational structure. It follows immediately that G admits a lattice. We focus our investigation on the class of splittable lattices in G . Recall that a lattice $\Gamma \subset G$ is splittable if it can be decomposed as $\Gamma_1 \rtimes_{\eta} \Gamma_2$, where $\Gamma_1 \subset \mathbb{R}^n$ and $\Gamma_2 \subset \mathbb{R}^m$ are themselves lattices.

3. Splittable lattices in $G = \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$

In this section we focus our study on the class of *splittable lattices* in G .

Definition 3.1. Let $(\sigma, \tau) \in GL_n(\mathbb{R}) \times GL_m(\mathbb{R})$, we shall say that the pair (σ, τ) is G -compatible if

$$\sigma \exp \left(\tau^{(j)} \cdot N \right) \sigma^{-1} \in SL_n(\mathbb{Z})$$

for all $j = 1, \dots, m$, where $\tau^{(j)}$ denotes the j^{th} column of τ .

Remark 3.2. • If (σ, τ) is G -compatible then for any $(\lambda, k) \in \mathbb{R}^* \times \mathbb{Z}^*$, the pair $(\lambda\sigma, k\tau)$ is also G -compatible.

• For any $\sigma \in GL_n(\mathbb{Q})$ it exists $\tau \in GL_m(\mathbb{Z})$ such that (σ, τ) is G -compatible.

Theorem 3.3. Let (σ, τ) be a G -compatible pair then

$$L_{(\sigma, \tau)} = \sigma^{-1} \mathbb{Z}^n \rtimes_{\eta} \tau \mathbb{Z}^m$$

is a lattice in G

Proof. Let $(\sigma^{-1}v, \tau(p)), (\sigma^{-1}w, \tau(q)) \in L_{(\sigma, \tau)}$ with $\tau := (\tau_{i,j})_{1 \leq i, j \leq m}$. Using the inverse property and applying the group multiplication rule, we compute

$$\begin{aligned} (\sigma^{-1}v, \tau(p))(\sigma^{-1}w, \tau(q))^{-1} &= (\sigma^{-1}v, \tau(p))(-\exp(-\tau(q) \cdot N)\sigma^{-1}w, -\tau(q)) \\ &= (\sigma^{-1}v - \exp(\tau(p) \cdot N)\exp(-\tau(q) \cdot N)\sigma^{-1}w, \tau(p) - \tau(q)) \\ &= (\sigma^{-1}(v - \sigma \exp(\tau(p - q) \cdot N)\sigma^{-1}w), \tau(p - q)). \end{aligned}$$

Thus

$$(\sigma^{-1}v, \rho(p))(\sigma^{-1}w, \rho(q))^{-1} \in L_{(\sigma, \rho)} \quad \forall v, w \in \mathbb{Z}^n, p, q \in \mathbb{Z}^m.$$

Hence $L_{(\sigma, \rho)}$ is a discrete subgroup of G .

We show now that $L_{(\sigma, \tau)}$ is co-compact. Precisely, we will prove that

$$G/L_{(\sigma, \tau)} = \{[(x, t)] : (x, t) \in (\sigma^{-1}[0, 1]^n) \times (\tau[0, 1]^m)\},$$

where $[(x, t)]$ denotes the class of (x, t) modulo $L_{(\sigma, \tau)}$.

Let (x, t) be an arbitrary element of G . Take $y \in [0, 1]^m$ and $k \in \mathbb{Z}^m$ such that

$$\tau^{-1}t = y + k.$$

Further, let $z \in [0, 1]^n, l \in \mathbb{Z}^n$ be such that

$$\sigma \exp(-\tau(k) \cdot N)x = z + l.$$

Let's rewrite the expression of $(x, t)(\sigma^{-1}z, \tau y)^{-1}$.

We have

$$\begin{aligned} (x, t)(\sigma^{-1}z, \tau y)^{-1} &= (x, t)(-\exp(-(\tau y) \cdot N)\sigma^{-1}z, -\tau y) \\ &= (x - \exp((t - \tau y) \cdot N)\sigma^{-1}z, \tau(\tau^{-1}t - y)). \end{aligned}$$

Since $t - \tau y = \tau k$, this reduces to

$$(x, t)(\sigma^{-1}z, \tau y)^{-1} = (x - \exp(\tau(k) \cdot N)\sigma^{-1}z, \tau k).$$

Using the expression $x = \exp(\tau(k) \cdot N)\sigma^{-1}(z + l)$, we have

$$(x, t)(\sigma^{-1}z, \tau y)^{-1} = (\exp(\tau(k) \cdot N)\sigma^{-1}l, \tau k).$$

The right-hand side can be rewritten as:

$$(\sigma^{-1}\sigma \exp(\tau(k) \cdot N)\sigma^{-1}l, \tau k) \in L_{(\sigma, \tau)},$$

since $l \in \mathbb{Z}^n$ and $k \in \mathbb{Z}^m$. It follows that

$$[(x, t)] = [(\sigma^{-1}z, \tau y)]$$

This shows that

$$G/L_{(\sigma, \tau)} = \{[(x, t)] : (x, t) \in \sigma^{-1}([0, 1]^n) \times \tau([0, 1]^m)\}$$

and hence $L_{(\sigma, \tau)}$ is co-compact. Therefore, $L_{(\sigma, \tau)}$ is a lattice in G .

□

3.1. On the Endomorphism Monoid of G

The goal of this subsection is to completely determine the endomorphisms of the group G .

Proposition 3.4. *Let ϕ be an endomorphism of the Lie group G and let $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\beta : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\gamma : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\delta : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be defined as follows*

$$\begin{aligned}\alpha &= \pi_{\mathbb{R}^n} \circ \phi \circ j_{\mathbb{R}^n} & \beta &= \pi_{\mathbb{R}^n} \circ \phi \circ j_{\mathbb{R}^m} \\ \gamma &= \pi_{\mathbb{R}^m} \circ \phi \circ j_{\mathbb{R}^n} & \delta &= \pi_{\mathbb{R}^m} \circ \phi \circ j_{\mathbb{R}^m},\end{aligned}$$

where π (resp. j) denotes the projection (resp. the injection) onto (resp. from) the indicated parts of the semidirect product. That is

$$\phi(x, 0) = (\alpha(x), \gamma(x)) \text{ and } \phi(0, t) = (\beta(t), \delta(t)), \quad \forall (x, t) \in G.$$

Then the following conditions hold:

- 1) $\delta \in \text{End}(\mathbb{R}^m)$ and $\gamma \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$,
- 2) $\alpha(x) = \sum_{k \geq 0} \frac{(\gamma(x) \cdot N)^k}{(k+1)!} Ax$ where $A \in \mathbb{R}^{n \times n}$,
- 3) $\beta(t) = \sum_{k \geq 0} \frac{(\delta(t) \cdot N)^k}{(k+1)!} Bt$ where $B \in \mathbb{R}^{n \times m}$.

Proof. Let ϕ be an endomorphism of G , thus we can write

$$\phi(x, 0) = (\alpha(x), \gamma(x)) \text{ and } \phi(0, t) = (\beta(t), \delta(t)),$$

with

$$\begin{aligned}\alpha &= \pi_{\mathbb{R}^n} \circ \phi \circ j_{\mathbb{R}^n} & \beta &= \pi_{\mathbb{R}^n} \circ \phi \circ j_{\mathbb{R}^m} \\ \gamma &= \pi_{\mathbb{R}^m} \circ \phi \circ j_{\mathbb{R}^n} & \delta &= \pi_{\mathbb{R}^m} \circ \phi \circ j_{\mathbb{R}^m},\end{aligned}$$

Now since $(x, t) = (x, 0)(0, t)$ and ϕ is an endomorphism of G , one derives the formula

$$\phi(x, t) = (\alpha(x) + \exp(\gamma(x) \cdot N)\beta(t), \gamma(x) + \delta(t)). \quad (1)$$

On the other hand, since $(0, t)(0, s) = (0, t+s)$ and $\phi \in \text{End}(G)$ therefore

$$\begin{cases} \delta(t+s) &= \delta(t) + \delta(s) \\ \beta(t+s) &= \beta(t) + \exp(\delta(t) \cdot N)\beta(s) \end{cases} \quad (2)$$

From this, it derives that $\delta \in \text{End}(\mathbb{R}^m)$. Now concerning the function β , one has

$$\beta(t+s) = \beta(t) + \exp(\delta(t) \cdot N)\beta(s), \quad (3)$$

which implies that $\beta(0) = 0$, and thus we have

$$\begin{aligned}\beta(t+s) - \beta(t) &= \eta(\delta(t))(\beta(s)) \\ &= \eta(\delta(t))(\beta(s) - \beta(0)).\end{aligned}$$

Therefore we get

$$\begin{aligned} D\beta(t)(s) &= \lim_{\lambda \rightarrow 0} \frac{\beta(t + \lambda s) - \beta(t)}{\lambda} \\ &= \exp(\delta(t) \cdot N) \lim_{\lambda \rightarrow 0} \frac{\beta(\lambda s) - \beta(0)}{\lambda} \\ &= \exp(\delta(t) \cdot N) D\beta(0)(s) \end{aligned}$$

That is,

$$D\beta(t) = \exp(\delta(t) \cdot N) D\beta(0), \quad \forall t \in \mathbb{R}^m.$$

On the other hand, the Taylor formula gives

$$\beta(t) = \beta(0) + \int_0^1 D\beta(\lambda t) t d\lambda = \left(\int_0^1 \exp(\lambda \delta(t) \cdot N) d\lambda \right) D\beta(0) t.$$

Now we shall compute $\int_0^1 \exp(\lambda \delta(t) \cdot N) d\lambda$. Recall that

$$\exp(\lambda \delta(t) \cdot N) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (\delta(t) \cdot N)^k$$

and hence

$$\int_0^1 \exp(\lambda \delta(t) \cdot N) d\lambda = \sum_{k=0}^{\infty} \left(\int_0^1 \frac{\lambda^k}{k!} d\lambda \right) (\delta(t) \cdot N)^k = \sum_{k=0}^{\infty} \frac{(\delta(t) \cdot N)^k}{(k+1)!}.$$

Therefore

$$\beta(t) = \sum_{k=0}^{\infty} \frac{(\delta(t) \cdot N)^k}{(k+1)!} B t, \quad \text{with } B = D\beta(0) \in \mathbb{R}^{n \times m}.$$

The converse follows from the preceding calculations.

On the other hand, since $\phi \in \text{End}(G)$ and $(x, 0)(y, 0) = (x + y, 0)$, thus

$$\begin{cases} \gamma(x + y) &= \gamma(x) + \gamma(y) \\ \alpha(x + y) &= \alpha(x) + \exp(\gamma(x) \cdot N) \alpha(y) \end{cases} \quad (4)$$

From this one derives $\gamma \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

On the other hand, since ϕ is an endomorphism of G , then:

$$\phi((x, t)(y, s)) = \phi(x, t) \phi(y, s). \quad (5)$$

The left-hand side of (5) is

$$\begin{aligned} \phi((x, t)(y, s)) &= \phi(x + \exp(t \cdot N)y, t + s) \\ &= \left(\alpha(x + \exp(t \cdot N)y) \right. \\ &\quad \left. + \exp(\gamma(x + \exp(t \cdot N)y) \cdot N) \beta(t + s), \right. \\ &\quad \left. \gamma(x + \exp(t \cdot N)y) + \delta(t + s) \right), \end{aligned}$$

while the right-hand side gives

$$\begin{aligned}\phi(x, t)\phi(y, s) &= \left(\alpha(x) + \exp(\gamma(x) \cdot N)\beta(t), \gamma(x) + \delta(t) \right) \\ &\quad \times \left(\alpha(y) + \exp(\gamma(y) \cdot N)\beta(s), \gamma(y) + \delta(s) \right) \\ &= \left(\alpha(x) + \exp(\gamma(x) \cdot N)\beta(t) + \exp((\gamma(x) + \delta(t)) \cdot N)\alpha(y) \right. \\ &\quad \left. + \exp((\gamma(x) + \gamma(y) + \delta(t)) \cdot N)\beta(s), \gamma(x) + \gamma(y) + \delta(t)\delta(s) \right).\end{aligned}$$

We now determine the explicit expression for α . By (4), one has

$$\alpha(x + y) = \alpha(x) + \exp(\gamma(x) \cdot N)\alpha(y). \quad (6)$$

Setting $y = 0$ in this equation, we obtain $\alpha(0) = 0$ and analogously to the case of β , for α we find

$$D\alpha(x) = \exp(\gamma(x) \cdot N)D\alpha(0). \quad (7)$$

On other hand, by Taylor's Theorem, we can write

$$\begin{aligned}\alpha(x) - \alpha(0) &= \int_0^1 D\alpha(\lambda x)(x)d\lambda \\ &= \int_0^1 \exp(\lambda(\gamma(x) \cdot N))D\alpha(0)(x)d\lambda \\ &= \int_0^1 \exp(\lambda(\gamma(x) \cdot N))d\lambda D\alpha(0)(x).\end{aligned}$$

Therefore

$$\alpha(x) = \sum_{k \geq 0} \frac{(\gamma(x) \cdot N)^k}{(k+1)!} Ax, \quad \text{for some } A \in \mathbb{R}^{n \times n}. \quad (8)$$

□

Throughout the remainder of this paper, we shall adopt the following notation

$$\begin{aligned}Q(t) &:= \sum_{k \geq 0} \frac{(t \cdot N)^k}{(k+1)!} \quad \forall t = (t_1, \dots, t_m) \in \mathbb{R}^m, \\ x_w &:= \langle x, w \rangle \quad \forall x, w \in \mathbb{R}^n.\end{aligned}$$

One can easily verify that the endomorphism $Q(t)$ obeys the following relations

- $Q(0) = I_n$ (the identity matrix in $\mathbb{R}^{n \times n}$).
- $Q(t)$ is a unipotent endomorphism of $\mathbb{R}^n$.
- $(t \cdot N)Q(t) = Q(t)(t \cdot N) = \exp(t \cdot N) - I_m, \quad \forall t \in \mathbb{R}^m$.

With this in mind, we can express the mappings as follows

$$\begin{aligned}\alpha(x) &= Q(\gamma(x))Ax = Q(x_{e_1}v)Ax \quad \text{for some } A \in \mathbb{R}^{n \times n} \forall x \in \mathbb{R}^n, \\ \gamma(x) &= \langle x, e_1 \rangle v = x_{e_1}v \quad \forall x \in \mathbb{R}^n, \text{ with } v \in \mathbb{R}^m, \\ \delta(t) &= \Delta t \quad \forall t \in \mathbb{R}^m, \text{ with } \Delta \in \mathbb{R}^{m \times m}, \\ \beta(t) &= Q(\Delta t)Bt \quad \forall t \in \mathbb{R}^m, \text{ with } B \in \mathbb{R}^{n \times m}.\end{aligned}$$

Our next goal is to establish the converse of Proposition 3.4. To this end, recall that any endomorphism ϕ of G satisfies $\phi(x, t) = \phi(x, 0)\phi(0, t)$ and hence one derives that

$$\phi(x, t) = (Q(x_{e_1}v)Ax + \exp(x_{e_1}v \cdot N)Q(\Delta t)Bt, \Delta t + x_{e_1}v), \quad \forall (x, t) \in G, \quad (9)$$

where

$$A \in \mathbb{R}^{n \times n}, \Delta \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{n \times m}, v \in \mathbb{R}^m.$$

We now state the following

Proposition 3.5. *Let ϕ be an endomorphism of the Lie group G and let α, β, γ and δ be defined as in Proposition 3.4, that is*

$$\phi(x, 0) = (\alpha(x), \gamma(x)) \text{ and } \phi(0, t) = (\beta(t), \delta(t)), \quad \forall (x, t) \in G.$$

Then ϕ is an endomorphism of G if and only if

$$1) \delta \in \text{End}(\mathbb{R}^m) \text{ and } \gamma \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \text{ with } \gamma \circ N_i = 0 \text{ for all } i = 1, \dots, m,$$

$$2) \alpha(x) = \sum_{k \geq 0} \frac{(\gamma(x) \cdot N)^k}{(k+1)!} Ax \text{ where } A \in \mathbb{R}^{n \times n} \text{ satisfying}$$

$$(\Delta^{(i)} \cdot N)A - AN_i = 0 \text{ on } \text{span}\{e_2, \dots, e_n\}$$

and

$$((\Delta^{(i)} \cdot N)A - AN_i)e_1 = (v \cdot N)B^{(i)}$$

where $\Delta^{(i)}$ (resp. $B^{(i)}$) denotes the i^{th} -column of Δ (resp. B)

$$3) \beta(t) = \sum_{k \geq 0} \frac{(\delta(t) \cdot N)^k}{(k+1)!} Bt.$$

Proof. If $\phi \in \text{End}(G)$ then ϕ has the form given in Proposition 3.4, on other hand, equation (3.5) implies

$$\gamma(\exp(t \cdot N)y) = \gamma(y), \quad \forall (t, y) \in \mathbb{R}^m \times \mathbb{R}^n, \quad (10)$$

Therefore, differentiating with respect to t_i and setting $t_1 = \dots = t_m = 0$ in (10), we obtain

$$\gamma \circ N_i = 0 \quad \forall i = 1, \dots, m. \quad (11)$$

Conversely, if γ satisfies (11) then (10) is satisfied. Finally, equation (3.5) gives

$$Q(x_{e_1}v)(\exp(\delta(t) \cdot N)A - A \exp(t \cdot N))x = Q(\Delta t)(\exp(x_{e_1}v \cdot N) - I_n)Bt \quad \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}^m. \quad (12)$$

This implies that for any $x \in \text{span}\{e_2, \dots, e_n\}$, we have

$$\exp(\delta(t) \cdot N)A - A \exp(t \cdot N) = 0$$

and hence, by taking the derivative with respect t_i and letting $t_1 = \dots = t_m = 0$ we obtain

$$(\Delta^{(i)} \cdot N)A - AN_i = 0, \quad i = 1, \dots, m \quad (13)$$

On the other hand, for $x = se_1$, taking the derivative with respect to s and setting $s = 0$ gives

$$(\exp(\delta(t) \cdot N)A - A \exp(t \cdot N))e_1 = Q(\delta(t))(v \cdot N)Bt. \quad (14)$$

By differentiating the previous equation with respect to t_i and imposing the condition $t_1 = \dots = t_m = 0$, we get

$$((\Delta^{(i)} \cdot N)A - AN_i)e_1 = (v \cdot N)B^{(i)}, \quad i = 1, \dots, m. \quad (15)$$

Here, $\Delta^{(i)}$ and $B^{(i)}$ denote the i -th columns of the matrices associated with δ and B . Conversely, if (13), (14) and (15) are satisfied then (12) holds. $\square$

Lemma 3.6. *Let $\phi \in \text{End}(G)$ given as in (9).*

If $\phi = (\alpha + \beta, \delta) \in \text{End}(G)$ then ϕ is an automorphism of G if and only if $A \in GL_n(\mathbb{R})$ and $\delta \in GL_m(\mathbb{R})$.

Proof. If $\phi = (\alpha + \beta, \delta) \in \text{Aut}(G)$ that is

$$\phi(x, t) = (Q(x_{e_1}v)Ax + Q(\Delta t)Bt, \Delta t),$$

then $\Delta \in GL_m(\mathbb{R})$ and if $A \notin GL_n(\mathbb{R})$ we see that

$$\phi(x, 0) = (0, 0) \quad \forall x \in \ker A$$

which shows that ϕ is not injective. Conversely, it is easy verified that any endomorphism $\phi = (\alpha + \beta, \delta) \in \text{End}(G)$ satisfying $A \in GL_n(\mathbb{R})$ and $\Delta \in GL_m(\mathbb{R})$ is automatically an automorphism of G . $\square$

Theorem 3.7. *Let (σ, τ) and (ν, ρ) be two G -compatible pairs. The lattices $L_{(\sigma, \tau)} = \sigma^{-1}\mathbb{Z}^n \rtimes_{\eta} \tau\mathbb{Z}^m$ and $L_{(\nu, \rho)} = \nu^{-1}\mathbb{Z}^n \rtimes_{\eta} \rho\mathbb{Z}^m$ differ by an automorphism $\phi := (\alpha + \beta, \delta)$ of G if and only if*

$$\nu\alpha\sigma^{-1} \in GL_n(\mathbb{Z}) \quad \text{and} \quad \rho^{-1}\delta\tau \in GL_m(\mathbb{Z}).$$

Proof. Assume that there is an automorphism ϕ of G with $\phi = (\alpha + \beta, \delta)$ and such that $\phi(L_{(\sigma, \tau)}) = L_{(\nu, \rho)}$. Then

$$\alpha(\sigma^{-1}\mathbb{Z}^n) = \nu^{-1}\mathbb{Z}^n \quad \text{and} \quad \delta(\tau\mathbb{Z}^m) = \rho\mathbb{Z}^m,$$

it follows that

$$\nu\alpha\sigma^{-1} \in SL_n(\mathbb{Z}) \quad \text{and} \quad \rho^{-1}\delta\tau \in SL_m(\mathbb{Z})$$

Conversely, suppose that there exist $\phi := (\alpha, \delta) \in \text{Aut}(G)$ such that

$$\nu\alpha\sigma^{-1} \in SL_n(\mathbb{Z}) \quad \text{and} \quad \rho^{-1}\delta\tau \in SL_m(\mathbb{Z})$$

Then we have

$$\begin{aligned}\alpha(\sigma^{-1}\mathbb{Z}^n) &= \nu^{-1}(\nu\alpha\sigma^{-1}\mathbb{Z}^n) \\ &= \nu^{-1}\mathbb{Z}^n,\end{aligned}$$

and

$$\begin{aligned}\delta(\tau\mathbb{Z}^m) &= \rho(\rho^{(-1)}\delta\tau\mathbb{Z}^m) \\ &= \rho\mathbb{Z}^m.\end{aligned}$$

Thus

$$\phi(L_{(\sigma,\tau)}) = L_{(\nu,\rho)}$$

□

3.2. Commensurability of splittable lattices in G

In this subsection, we investigate the commensurability of lattices in G . Two lattices Γ and Γ' are said to be commensurable if their intersection has finite index in both subgroups, meaning that

$$[\Gamma : \Gamma \cap \Gamma'] < \infty \quad \text{and} \quad [\Gamma' : \Gamma \cap \Gamma'] < \infty.$$

For a comprehensive treatment of this property, we refer the reader to [17].

Proposition 3.8. *Let (σ, τ) and (ν, ρ) be two G -compatible pairs. The lattices $L_{(\sigma,\tau)} = \sigma^{-1}\mathbb{Z}^n \rtimes_{\eta} \tau\mathbb{Z}^m$ and $L_{(\nu,\rho)} = \nu^{-1}\mathbb{Z}^n \rtimes_{\eta} \rho\mathbb{Z}^m$ are commensurable if and only if*

$$(\text{rank}_{\mathbb{Z}}(\sigma^{-1}\mathbb{Z}^n \cap \nu^{-1}\mathbb{Z}^n), \text{rank}_{\mathbb{Z}}(\tau\mathbb{Z}^m \cap \rho\mathbb{Z}^m)) = (n, m).$$

Proof. Recall that the intersection $L_{(\sigma,\tau)} \cap L_{(\nu,\rho)}$ is given by

$$L_{(\sigma,\tau)} \cap L_{(\nu,\rho)} = (\sigma^{-1}\mathbb{Z}^n \cap \nu^{-1}\mathbb{Z}^n) \times (\tau\mathbb{Z}^m \cap \rho\mathbb{Z}^m).$$

Let

$$n_0 = \text{rank}_{\mathbb{Z}}(\sigma^{-1}\mathbb{Z}^n \cap \nu^{-1}\mathbb{Z}^n), \quad m_0 = \text{rank}_{\mathbb{Z}}(\tau\mathbb{Z}^m \cap \rho\mathbb{Z}^m).$$

Let $v_1, \dots, v_n \in \sigma^{-1}\mathbb{Z}^n$ and $p_1, \dots, p_{n_0} \in \mathbb{N}$ such that $(v_1, \dots, v_n)$ is a basis of $\sigma^{-1}\mathbb{Z}^n$ and $(p_1 v_1, \dots, p_{n_0} v_{n_0})$ is a basis of $(\sigma^{-1}\mathbb{Z}^n \cap \nu^{-1}\mathbb{Z}^n)$. Let $w_1, \dots, w_m \in \tau\mathbb{Z}^m$ and $q_1, \dots, q_{m_0} \in \mathbb{N}$ such that $(w_1, \dots, w_m)$ is a basis of $\tau\mathbb{Z}^m$ and $(q_1 w_1, \dots, q_{m_0} w_{m_0})$ is a basis of $(\tau\mathbb{Z}^m \cap \rho\mathbb{Z}^m)$.

Suppose that $n_0 < n$, in this case we get

$$(0_{\mathbb{Z}^{n-1}} \times \mathbb{Z}v_n) \cap (\sigma^{-1}\mathbb{Z}^n \cap \nu^{-1}\mathbb{Z}^n) = \{0\},$$

and it follows that $\{[(kv_n, 0)], k \in \mathbb{Z}\}$ is an infinite subset of $L_{(\sigma,\tau)}/L_{(\sigma,\tau)} \cap L_{(\nu,\rho)}$, thus $L_{(\sigma,\tau)}$ and $L_{(\nu,\rho)}$ are non commensurable, here $[(kv_n, 0)]$ is the class of $(kv_n, 0)$ modulo $L_{(\sigma,\tau)} \cap L_{(\nu,\rho)}$.

Similarly, if $m_0 < m$ then $\{[(0, lw_m)], l \in \mathbb{Z}\}$ is an infinite subset of $L_{(\sigma,\tau)}/L_{(\sigma,\tau)} \cap L_{(\nu,\rho)}$. This shows the non-commensurability.

Suppose now that $n_0 = n$ and $m_0 = m$. We prove that $L_{(\sigma,\tau)}/L_{(\sigma,\tau)} \cap L_{(\nu,\rho)}$ is finite. Let $(x, t) \in L_{(\sigma,\tau)}$ with $t = t_1 w_1 + \dots + t_m w_m \in \tau \mathbb{Z}^m$, where $(t_1, \dots, t_m) \in \mathbb{Z}^m$. For each $i = 1, \dots, m$, write

$$t_i = s_i + k_i q_i,$$

with $(s_i, k_i) \in \mathbb{N} \times \mathbb{Z}$ such that $0 \leq s_i < q_i$. Then,

$$t = s + \kappa, \quad \text{where} \quad s = \sum_{j=1}^m s_j w_j \text{ and } \kappa = \sum_{j=1}^m k_j q_j w_j.$$

Here, $s \in \tau \mathbb{Z}^m$ and $\kappa \in \tau \mathbb{Z}^m \cap \rho \mathbb{Z}^m$.

Since $(0, -\kappa), (x, 0) \in L_{(\sigma,\tau)}$, we have

$$(\eta(-\kappa)x, -\kappa) = (0, -\kappa)(x, 0) \in L_{(\sigma,\tau)}.$$

Thus $\eta(-\kappa)x \in \sigma^{-1} \mathbb{Z}^n$, and we can write

$$\eta(-\kappa)x = \xi + \zeta,$$

where

$$\xi = \sum_{i=1}^n r_i v_i \in \sigma^{-1} \mathbb{Z}^n \text{ and } \zeta = \sum_{i=1}^n l_i p_i v_i \in \sigma^{-1} \mathbb{Z}^n \cap \nu^{-1} \mathbb{Z}^n,$$

with $r_i \in \mathbb{N}$, $0 \leq r_i < p_i$ and $l_i \in \mathbb{Z}$.

Now, observe that $(\xi, s) \in L_{(\sigma,\tau)}$ and

$$\begin{aligned} (x, t)(\xi, s)^{-1} &= (x, t)(-\eta(-s)\xi, -s) \\ &= (x - \eta(t - s)\xi, t - s) \\ &= (x - \eta(\kappa)\xi, \kappa). \end{aligned}$$

Hence,

$$(x, t)(\xi, s)^{-1} = (\eta(\kappa)\xi, \kappa).$$

Since the right-hand side belongs to $L_{(\sigma,\tau)} \cap L_{(\nu,\rho)}$, it follows that

$$[(x, t)] = [(\xi, s)].$$

This proves that

$$L_{(\sigma,\tau)}/L_{(\sigma,\tau)} \cap L_{(\nu,\rho)} = \left\{ \left[\left(\sum_{i=1}^n r_i v_i, \sum_{j=1}^m s_j w_j \right) \right], r_i, s_j \in \mathbb{N}, r_i < p_i, s_j < q_j \right\}$$

is finite. A similar argument shows that $L_{(\nu,\rho)}/L_{(\sigma,\tau)} \cap L_{(\nu,\rho)}$ is also finite. Therefore, the lattices $L_{(\sigma,\tau)}$ and $L_{(\nu,\rho)}$ are commensurable. $\square$

Lemma 3.9. *Let $A \in GL_n(\mathbb{R})$. The following assertions are equivalent:*

$$i) \text{ rank}_{\mathbb{Z}}(A\mathbb{Z}^n \cap \mathbb{Z}^n) = n$$

$$ii) A \in GL_n(\mathbb{Q})$$

Proof. Assume that $\text{rank}_{\mathbb{Z}}(A\mathbb{Z}^n \cap \mathbb{Z}^n) = n$, therefore it exists $B \in \mathbb{Z}^{n \times n} \cap GL_n(\mathbb{Q})$ such that

$$A\mathbb{Z}^n \cap \mathbb{Z}^n = B\mathbb{Z}^n.$$

This containment implies $B\mathbb{Z}^n \subset A\mathbb{Z}^n$, which yields $A^{-1}B\mathbb{Z}^n \subset \mathbb{Z}^n$. It follows that $A^{-1}B \in \mathbb{Z}^{n \times n} \subset GL_n(\mathbb{Q})$. Since both B and $A^{-1}B$ are in $GL_n(\mathbb{Q})$, we conclude that $A \in GL_n(\mathbb{Q})$.

Conversely, suppose $A \in GL_n(\mathbb{Q})$. Then there exists a non-zero integer c such that $cA \in \mathbb{Z}^{n \times n}$. Furthermore, because A is invertible, $\text{rank}_{\mathbb{Z}}((cA)\mathbb{Z}^n) = n$. Since $cA\mathbb{Z}^n \subset A\mathbb{Z}^n \cap \mathbb{Z}^n \subset \mathbb{Z}^n$, it follows immediately that $\text{rank}_{\mathbb{Z}}(A\mathbb{Z}^n \cap \mathbb{Z}^n) = n$. □

Theorem 3.10. *Let (σ, τ) and (ν, ρ) be two G -compatible pairs. The lattices $\sigma^{-1}\mathbb{Z}^n \rtimes_{\eta} \tau\mathbb{Z}^m$ and $\nu^{-1}\mathbb{Z}^n \rtimes_{\eta} \rho\mathbb{Z}^m$ are commensurable if and only if $\nu\sigma^{-1} \in GL_n(\mathbb{Q})$ and $\rho^{-1}\tau \in GL_m(\mathbb{Q})$.*

Proof. According to Proposition 3.8, the lattices $L_{(\sigma, \tau)}$ and $L_{(\nu, \rho)}$ are commensurable if and only if

$$\text{rank}_{\mathbb{Z}}(\sigma^{-1}\mathbb{Z}^n \cap \nu^{-1}\mathbb{Z}^n) = n \quad \text{and} \quad \text{rank}_{\mathbb{Z}}(\tau\mathbb{Z}^m \cap \rho\mathbb{Z}^m) = m.$$

This means that $\text{rank}_{\mathbb{Z}}(\nu\sigma^{-1}\mathbb{Z}^n \cap \mathbb{Z}^n) = n$ and $\text{rank}_{\mathbb{Z}}(\rho^{-1}\tau\mathbb{Z}^m \cap \mathbb{Z}^m) = m$. According to Lemma 3.9, this is equivalent to the fact that $\nu\sigma^{-1} \in GL_n(\mathbb{Q})$ and $\rho^{-1}\tau \in GL_m(\mathbb{Q})$. □

4. Application: splittable lattices in the m -filiform nilpotent Lie group $F_{n,m} = \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$

This section provides an application of the results established in the previous section. More precisely, we assume that $N = (N_1, \dots, N_m)$ with $N_i = J^i$ for $i = 1, \dots, m$ ($1 \leq m < n$), where $J = E_{2,1} + \dots + E_{n,n-1}$. Here, $E_{i,j}$ denotes the standard elementary matrix in $\mathbb{R}^{n \times n}$ whose only non-zero entry is a 1 at the i -th row and the j -th column.

We specialize our notation by setting $G = F_{n,m} = \mathbb{R}^n \rtimes_{\eta} \mathbb{R}^m$, which we call the m -filiform nilpotent Lie group. When $m = 1$, this construction yields the classical filiform nilpotent Lie group $F_{n,1} = \mathbb{R}^n \rtimes \mathbb{R}$.

The Lie group $F_{n,m}$ can be realized as a subgroup of $GL_{n+m}(\mathbb{R})$ consisting of unipotent lower-triangular matrices:

$$F_{n,m} = \left\{ (x, t) := \left(\begin{array}{c|c} I_m & 0_{m,n} \\ \hline x_1 & \exp(\sum_{i=1}^m t_i J^i) \\ \vdots & \\ x_n & \end{array} \right) \mid t_1, \dots, t_m, x_1, \dots, x_n \in \mathbb{R} \right\}$$

The Lie algebra $\mathfrak{f}_{n,m}$ associated with $F_{n,m}$ can also be viewed as the subalgebra of $\mathfrak{gl}_{n+m}(\mathbb{R})$ consisting of strictly lower-triangular matrices:

$$\mathfrak{f}_{n,m} = \left\{ (X, T) := \left(\begin{array}{c|c} 0_m & 0_{m,n} \\ \hline x_1 & \sum_{i=1}^m t_i J^i \\ \vdots & \\ x_n & \end{array} \right) \mid t_1, \dots, t_m, x_1, \dots, x_n \in \mathbb{R} \right\}$$

$\mathfrak{f}_{n,m}$ is a n -step nilpotent Lie algebra with basis $(X_1, \dots, X_n, T_1, \dots, T_m)$ and non trivial brackets

$$[T_j, X_i] = X_{i+j}, \quad i = 1, \dots, n, j = 1, \dots, m, i + j \leq n.$$

The exponential mapping $\exp : \mathfrak{f}_{n,m} \rightarrow f_{n,m}$ is given by

$$\exp(X + T) = (x, t)$$

where

$$(x, t) = \left(\sum_{k=0}^m \frac{T^k}{(k+1)!} X, t \right), \quad \text{with} \quad t = (t_1, \dots, t_m) \equiv T = \sum_{i=1}^m t_i J^i.$$

4.1. The Endomorphism Monoid $\text{End}(F_{n,m})$

As $N = (N_1, \dots, N_m)$ where $N_i = J^i$ (for $i = 2, \dots, m$), then Proposition 3.4 and Proposition 3.5 give

Proposition 4.1. *Let ϕ be an endomorphism of the Lie group $F_{n,m}$ and assume α, β, γ , and δ are given as in Propositions 3.4 and 3.5. Then, $\phi \in \text{End}(F_{n,m})$ if and only if the following relations are satisfied:*

1) $\delta \in \text{End}(\mathbb{R}^m)$, that is, $\delta(t) = \Delta t$ for some $\Delta \in \mathbb{R}^{m \times m}$ and $\gamma \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is given by $\gamma(x) = \langle x, e_1 \rangle v$ for some $v \in \mathbb{R}^m$ where $(e_1, \dots, e_n)$ is the standard basis in $\mathbb{R}^n$ and $\langle, \rangle$ represents the standard inner product on $\mathbb{R}^n$.

2) $\alpha(x) = \sum_{k \geq 0} \frac{(\gamma(x) \cdot N)^k}{(k+1)!} Ax$ where $A \in \mathbb{R}^{n \times n}$,

3) $\beta(t) = \sum_{k \geq 0} \frac{(\delta(t) \cdot N)^k}{(k+1)!} Bt$ where $B \in \mathbb{R}^{m \times n}$ such that

$$(\Delta^{(i)} \cdot N)A - AN_i = 0 \text{ on } \text{span}\{e_2, \dots, e_n\}$$

and

$$((\Delta^{(i)} \cdot N)A - AN_i)e_1 = (v \cdot N)B^{(i)}.$$

Here, $\Delta^{(i)}$ and $B^{(i)}$ denote the i -th columns of the matrices Δ and B , respectively.

Proof. In Proposition 3.5, the relation (11) can be equivalently rewritten as

$$\gamma \circ N_i = \gamma \circ J^i = 0, \quad \forall i = 1, \dots, m$$

and hence

$$\gamma \circ J^i = 0 \quad \forall i = 1, \dots, m. \tag{16}$$

Therefore, letting $(e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^n$, then (16) gives

$$\gamma(e_2) = \dots = \gamma(e_n) = 0,$$

and thus it exists $v \in \mathbb{R}^m$ such that

$$\gamma(x) = \langle x, e_1 \rangle v \quad \forall x \in \mathbb{R}^n,$$

where $\langle , \rangle$ is the standard inner product of $\mathbb{R}^n$.

□

Lemma 4.2. *Let $\sigma \in GL_n(\mathbb{R})$, then $\sigma^{-1}\mathbb{Z}^n \rtimes \mathbb{Z}^m$ is a lattice in $F_{n,m}$ if and only if $\sigma \exp(J)\sigma^{-1} \in SL_n(\mathbb{Z})$.*

Proof. The $F_{n,m}$ –compatibility applied for the pair (σ, I_m) gives

$$\sigma \exp(J^i)\sigma^{-1} \in SL_n(\mathbb{Z}) \quad \forall i = 1, \dots, m.$$

And this occurs if and only if $\sigma \exp(J)\sigma^{-1} \in SL_n(\mathbb{Z})$.

□

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