

Hardy-Sobolev Critical Equations with Totally Geodesic Singularities: Existence via the Mountain Pass Theorem

Équations critiques de Hardy–Sobolev avec singularités totalement géodésiques : existence par le théorème du col de montagne

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ABSTRACT. We consider a compact Riemannian manifold (M, g) of dimension $N \geq 3$ and Σ a closed totally geodesic submanifold of dimension $1 \leq k \leq N - 2$, and $h : M \rightarrow \mathbb{R}$ is a continuous function such that the linear operator $-\Delta_g + h$ is coercive. We study existence of positive solutions $u \in H^1(M)$ to the following nonlinear PDE with two Hardy-Sobolev critical exponents:

$$(0,1) \quad -\Delta_g u + hu = \lambda \rho_{\Sigma}^{-s_1} u^{2_{s_1}^* - 1} + \rho_{\Sigma}^{-s_2} u^{2_{s_2}^* - 1} \quad \text{in } (M, g)$$

where λ is a positive parameter, $0 < s_2 < s_1 < 2$, the $2_{s_i}^* := \frac{2(N-s_i)}{N-2}$ ($i = 1, 2$) are two critical Hardy-Sobolev exponents and $\rho_{\Sigma} : M \rightarrow \mathbb{R}$ is the distance function to Σ . In this paper, we give sufficient condition depending on the local geometries of the submanifold Σ and the manifold M , for the existence of mountain pass solution to $(0,1)$.

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KEYWORDS. Mountain Pass Solution, Two Hardy-Sobolev critical exponents, Scalar Curvature, Riemannian curvature tensor, Submanifold.

1. Introduction

Weighted interpolation inequalities of Caffarelli–Kohn–Nirenberg type and their critical limits, the Hardy and Sobolev inequalities, are central tools in the variational study of semilinear elliptic problems with singular weights. In Euclidean space the family of Hardy–Sobolev inequalities gives, for $s \in [0, 2]$, the critical integrability exponent

$$2^*(s) = \frac{2(N-s)}{N-2},$$

and a typical Hardy–Sobolev inequality reads (up to constants)

$$\left(\int_{\mathbb{R}^N} |x|^{-s} |u|^{2^*(s)} \right)^{2/2^*(s)} \leq C_{N,s} \int_{\mathbb{R}^N} |\nabla u|^2, \quad u \in D^{1,2}(\mathbb{R}^N),$$

with the extreme cases recovering the classical Hardy and Sobolev inequalities. The foundational analytic framework for these weighted inequalities was established in the work of Caffarelli, Kohn and Nirenberg [2].

Problems involving one or several Hardy–Sobolev critical terms exhibit two classical and intertwined difficulties: the *loss of compactness* of the Sobolev embedding at the critical exponent and the presence of a singular weight, which interacts with the local geometry near the singularity. The modern technical toolkit for handling the first difficulty is the concentration–compactness principle of P.-L. Lions [4, 5] and related profile decomposition techniques.

In this paper, we consider a compact Riemannian manifold (M, g) of dimension $N \geq 3$ and $\Sigma \subset M$ be a closed totally geodesic submanifold of dimension $1 \leq k \leq N - 2$. We study positive $H^1(M)$ -solutions of

$$-\Delta_g u + hu = \lambda \rho_\Sigma^{-s_1} u^{2_{s_1}^* - 1} + \rho_\Sigma^{-s_2} u^{2_{s_2}^* - 1} \quad \text{in } (M, g), \quad (1.1)$$

where $\rho_\Sigma : M \rightarrow \mathbb{R}_+$ denotes the distance to Σ , $0 < s_2 < s_1 < 2$, $\lambda > 0$, $h \in C^0(M)$ is such that $-\Delta_g + h$ is coercive and

$$\Delta_g := g^{ij} \left(\frac{\partial^2}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial}{\partial x_k} \right),$$

is the Laplace Beltrami operator with $(\Gamma_{ij}^k)_{1 \leq i, j, k \leq N}$ are the Christoffel symbols. The natural energy functional associated to (1.1) is

$$J(u) = \frac{1}{2} \int_M (|\nabla_g u|^2 + hu^2) dv_g - \frac{\lambda}{2_{s_1}^*} \int_M \rho_\Sigma^{-s_1} |u|^{2_{s_1}^*} dv_g - \frac{1}{2_{s_2}^*} \int_M \rho_\Sigma^{-s_2} |u|^{2_{s_2}^*} dv_g,$$

defined on $H^1(M)$. The functional typically has the standard *mountain-pass geometry*: $J(0) = 0$, there exists u_0 with $J(u_0) < 0$ and a small radius $r > 0$ such that $J|_{\partial B_r(0)} > 0$, see for instance Lemma 3.6 below. By the mountain-pass construction [6] one sets the minimax level

$$c^* = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J(\gamma(t)), \quad \Gamma = \{\gamma \in C([0, 1]; H^1(M)) : \gamma(0) = 0, \gamma(1) = u_0\}.$$

Under the classical Palais–Smale (PS) compactness condition the Ambrosetti–Rabinowitz theorem guarantees the existence of a critical point at level c^* . We therefore use the mountain-pass variational principle as the primary existence mechanism. Since the embeddings

$$H^1(M) \hookrightarrow L^{2_{s_i}^*}(M, \rho_\Sigma^{-s_i} dv_g)$$

are noncompact at the critical exponents, the functional J does not satisfy the Palais-Smale condition. Therefore, in general c^* might not be a critical value for J . To recover compactness, we consider

$$\begin{cases} -\Delta u(x) = \lambda \frac{u^{2_{s_1}^* - 1}(x)}{|z|^{s_1}} + \frac{u^{2_{s_2}^* - 1}(x)}{|z|^{s_2}} & \text{in } \mathbb{R}^N \\ u(x) > 0 & \text{in } \mathbb{R}^N \end{cases} \quad (1.2)$$

with $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. To obtain solutions of (1.2), we consider the functional $\Pi : \mathcal{D}^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\Pi(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2_{s_1}^*} \int_{\mathbb{R}^N} |z|^{-s_1} |u|^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^N} |z|^{-s_2} |u|^{2_{s_2}^*} dx.$$

Next, we set

$$\beta^* := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N), u \geq 0, u \neq 0} \max_{t \geq 0} \Pi(tu).$$

Then we get compactness provided $c^* < \beta^*$, see Lemma 3.5 below. Consequently, we have the following which is our main result.

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold of dimension $N \geq 3$, $\Sigma \subset M$ be a closed totally geodesic submanifold of dimension $1 \leq k \leq N - 2$ and $h : M \rightarrow \mathbb{R}$ is a continuous function such*

that the linear operator $-\Delta_g + h$ is coercive. We assume that there exists $x_0 \in \Sigma$, constants A_{N,s_1,s_2} , B_{N,s_1,s_2} , and C_{N,s_1,s_2} only depending on N , s_1 and s_2 such that

$$h(x_0) < A_{N,s_1,s_2}R_g(x_0) + B_{N,s_1,s_2}S_g(x_0) + C_{N,s_1,s_2}W_g(x_0), \quad (1.3)$$

where R_g and S_g are respectively the scalar curvatures of M and Σ ,

$$W_g(x_0) = \sum_{i=k+1}^N \sum_{a=1}^k R_{iaia}$$

measures the interaction between tangent and normal directions of the submanifold Σ inside the ambient manifold M and the R_{iaia} are components of the Riemannian curvature tensor. Then there exists $u \in H^1(M)$ positive solution of the following Euler-Lagrange equation

$$-\Delta_g u + hu = \lambda \rho_{\Sigma}^{-s_1} u^{2_{s_1}^* - 1} + \rho_{\Sigma}^{-s_2} u^{2_{s_2}^* - 1} \quad \text{in } (M, g).$$

Remark 1.2. Several comments on the geometric condition (1.3) and the role of the constants are in order.

- (i) **Geometric meaning of the curvature quantities.** The scalar curvature $R_g(x_0)$ measures the average sectional curvature of (M, g) at x_0 ; $S_g(x_0)$ is the intrinsic scalar curvature of Σ at x_0 ; and

$$W_g(x_0) = \sum_{i=k+1}^N \sum_{a=1}^k R_{iaia}$$

is a mixed curvature term that captures the interaction between the tangent bundle $T_{x_0}\Sigma$ and the normal bundle $N_{x_0}\Sigma$ inside $T_{x_0}M$. These three quantities appear naturally through the second-order metric expansion in Fermi coordinates (see Lemma 2.1).

- (ii) **Nature of the constants** A_{N,s_1,s_2} , B_{N,s_1,s_2} , C_{N,s_1,s_2} . These constants depend only on N , s_1 , and s_2 ; they are independent of the geometry of (M, g) and of Σ . They are determined by the Euclidean ground state w of the limit problem (1.2) through the explicit integrals

$$A_{N,s_1,s_2} = \frac{\Psi_1(w)}{\int_{\mathbb{R}^N} w^2 dx}, \quad B_{N,s_1,s_2} = \frac{\Psi_2(w)}{\int_{\mathbb{R}^N} w^2 dx}, \quad C_{N,s_1,s_2} = \frac{\Psi_3(w)}{\int_{\mathbb{R}^N} w^2 dx},$$

where Ψ_1, Ψ_2, Ψ_3 are the functionals defined in Section 5. Their sign reflects the geometric contributions of each curvature invariant to the energy expansion.

- (iii) **Interpretation of condition (1.3).** The condition requires that the weighted combination of curvature invariants at x_0 dominates $h(x_0)$. Geometrically, this ensures that concentrating a solution bubble near $x_0 \in \Sigma$ lowers the energy functional J below the critical threshold β^* . This is the intrinsic-manifold analogue of the role played by the mean curvature at a boundary singularity in the Euclidean results of Ghoussoub–Kang [7] and Ghoussoub–Robert [8]: in those works, negative mean curvature is the key geometric condition; here, the collective curvature data (R_g, S_g, W_g) plays an analogous role, reflecting the higher-codimensional nature of the singularity Σ .

(iv) **Explicit verification.** In concrete examples — e.g., when (M, g) is a space-form of positive curvature and Σ is a totally geodesic submanifold — condition (1.3) can be verified directly from the formulas for R_g , S_g , W_g , and the known decay of w , by using standard integration in spherical coordinates on $\mathbb{R}^N$.

This kind of problem is intensively studied in the Euclidean case. In particular, we let Ω be a domain of $\mathbb{R}^N$ such that $0 \in \partial\Omega$ and consider the equation

$$\begin{cases} -\Delta u(x) = \lambda \frac{u^{2_{s_1}^*-1}(x)}{|x|^{s_1}} + \frac{u^{2_{s_2}^*-1}(x)}{|x|^{s_2}} & \text{in } \Omega \\ u(x) > 0 & \text{in } \Omega, \end{cases} \quad (1.4)$$

where $0 \leq s_2 < s_1 < 2$, $\lambda \in \mathbb{R}$ and for $i = 1, 2$, the $2_{s_i}^* := \frac{2(N-s_i)}{N-2}$ are two critical Hardy-Sobolev exponents. When $s_2 = 0$ and $\lambda < 0$, then equation (1.4) has no nontrivial solution. For $\lambda > 0$, $0 < s_1 < 2$ and $s_2 = 0$, then using variational methods, Hsia Lin and Wadade [11] proved existence of solutions provided $N \geq 4$ and the mean curvature at the origin is negative. For the case $N = 3$, $\lambda \in \mathbb{R}$ and $0 < s_2 < s_1 < 2$, the equation (1.4) has a least-energy solution provided the mean curvature at the origin is negative, see [12].

Concerning the existence and non-existence of solution related to equation (1.4) in the half-space $\Omega = \mathbb{R}_+^N$, we refer to Bartsch-Peng and Zhang [1] for the case $0 < s_2 < s_1 = 2$ and $\lambda < \left(\frac{N-2}{2}\right)^2$; to Musina [13] when $N \geq 4$, $s_2 = 0$, $s_1 = 2$ and $0 < \lambda < \left(\frac{N-2}{2}\right)^2$ and to Hsia, Lin and Wadade [11] when $s_2 = 0$, $0 < s_1 < 2$ and $\lambda > 0$. For the case $N \geq 3$, $0 < s_2 < s_1 < 2$ and $\lambda \in \mathbb{R}$ then equation (1.2) has a least-energy solution $u \in H_0^1(\mathbb{R}_+^N)$, see for instance [12]. For generalizations and extensions, we refer to Diatta-Thiam [15], Thiam [16], Zhong-Zou [17], Ghoussoub-Kang [7] and references therein.

There are two main novelties in this work : the simultaneous treatment of *two distinct Hardy-Sobolev critical exponents* supported on the same singular set Σ and the analysis on a *compact Riemannian manifold* where the local geometry of M and Σ (scalar curvature of M , scalar curvature of Σ , and certain mixed curvature terms) enters explicitly in the comparison $c^* < \beta^*$. Earlier works in Euclidean or boundary-singularity settings have shown that curvature or boundary mean curvature can determine attainability of best constants and existence of minimizers [7–9].

The proof of Theorem 1.1 combines:

1. Limit problem and profile analysis : we study the Euclidean limit problem with cylindrical singular weight, establishing existence, symmetry, and sharp decay estimates for its ground state. These estimates are used to quantify the leading asymptotics of energy when one concentrates a bubble near Σ .
2. Local parametrization and metric expansion : around a point $x_0 \in \Sigma$ we introduce Fermi coordinates and expand the metric to second order; the expansion identifies how curvature terms enter into the energy expansion of localized test functions.
3. Test-function computation and energy comparison : using the scaled Euclidean ground state inserted into the Fermi chart (cut off at a suitable radius we compute the expansion of J to second

order in the concentration parameter ε , extract the geometric coefficients, and derive a curvature condition implying $c^* < \beta^*$).

Once $c^* < \beta^*$ is established, concentration–compactness yields compactness of PS sequences at level c^* , and the mountain-pass theorem provides a nontrivial positive solution. Our approach synthesizes ideas from the Hardy–Sobolev literature (sharp constants, curvature effects); the concentration–compactness methods, and the variational minimax methods. We refer to Caffarelli–Kohn–Nirenberg inequalities, the concentration–compactness framework [4, 5], and the works of Ghoussoub–Kang [7], Ghoussoub–Robert [8], and Chern–Lin [9].

The paper is organized as follows. In Section 2, we introduce Fermi coordinates and derive the metric expansion up to second order. In Section 3, we establish the compactness result and analyze the limit Euclidean ground state, recording its key decay and symmetry properties. In Section 4, we derive the integral estimates involving the ground state that are needed for the energy expansion. Section 5 builds the concentrating test functions, expands $J(u_\varepsilon)$ to second order, carries out the energy comparison $c^* < \beta^*$, and completes the proof of Theorem 1.1.

2. Parametrization

2.1. Fermi Coordinates

Let (M, g) be a compact Riemannian manifold of dimension $N \geq 3$, and let Σ be a totally geodesic submanifold of M with dimension $1 \leq k \leq N - 2$. Then given $x_0 \in M$, we have

$$T_{x_0}M = T_{x_0}\Sigma \oplus N_{x_0}\Sigma.$$

Next, we choose orthonormal basis $(E_1, \dots, E_k)$ for $T_{x_0}\Sigma$ and $(E_{k+1}, \dots, E_N)$ of $N_{x_0}\Sigma$. Then a neighborhood of x_0 in Σ can be parametrized via the exponential mapping

$$f : y \in B_{\mathbb{R}^k}(0, r) \mapsto f(y) = \exp_{x_0}^\Sigma \left(\sum_{a=1}^k y_a E_a \right) \in \Sigma,$$

where for $r > 0$, $B_{\mathbb{R}^k}(0, r)$ denotes the Euclidean ball of $\mathbb{R}^k$ centered at the origin and of radius r . This yields the coordinate vector field

$$X_a := f_* \left(\frac{\partial}{\partial y_a} \right).$$

For any tangent vector field E of the tangent space, the curve $s \mapsto \gamma_E(s) := \exp_{x_0}^\Sigma(sE)$ is a geodesic in Σ . Therefore $\nabla_{X_a} X_b \in N_p \Sigma$. Moreover, since Σ is totally geodesic, we have

$$\nabla_{X_a} X_b \big|_p = 0. \tag{2.1}$$

Next, we extend $(E_{k+1}, \dots, E_N)$ to an orthonormal frame $(X_{k+1}, \dots, X_N)$ in a neighborhood of x_0 in M via the mapping

$$F_{x_0*} : x \in Q_r := B_{\mathbb{R}^k}(0, r) \times B_{\mathbb{R}^{N-k}}(0, r) \mapsto F_{x_0}(x) = \exp_{f(y)}^M \left(\sum_{i=k+1}^N z_i X_i \right) \in M.$$

This corresponds to the coordinate vector fields

$$X_i := F_{x_0*} \left(\frac{\partial}{\partial z_i} \right) \text{ and } X_a := F_{x_0*} \left(\frac{\partial}{\partial y_a} \right) \text{ for all } a = 1, \dots, k \text{ and } i = k+1, \dots, N.$$

By construction, we have $X_\alpha|_{x_0} = E_\alpha$, for all $\alpha = 1, \dots, N$ and for $a = 1, \dots, k$ and for $i = k+1, \dots, N$, we have $\nabla_{X_a} X_i|_{x_0} \in T_p \Sigma$ so that

$$\nabla_{X_a} X_i|_{x_0} = \nabla_{X_i} X_a|_{x_0} = \nabla_{X_j} X_i|_{x_0} = 0. \quad (2.2)$$

Moreover it's well known that, in this local parametrization, the distance function ρ_Σ defined above satisfies

$$\rho_\Sigma(x) = |z|. \quad (2.3)$$

In the following, we derive the expansion of the metric induced by the parametrization F defined above. Then for $x \in Q_r$ and $\alpha, \beta = 1, \dots, N$, we define

$$g_{\alpha\beta}(x) = \langle X_\alpha, X_\beta \rangle,$$

to be the components of the metric g .

2.2. The Metric

By Taylor expansion we have :

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} + \sum_{\gamma=1}^N X_\gamma \langle X_\alpha, X_\beta \rangle x_\gamma + \frac{1}{2} \sum_{\gamma, \mu=1}^N X_\gamma X_\mu \langle X_\alpha, X_\beta \rangle x_\mu x_\gamma + O(|x|^3).$$

Next, we define

$$B_{\alpha\beta}^{\gamma\mu} = X_\gamma X_\mu \langle X_\alpha, X_\beta \rangle.$$

By (2.1) and (2.2), we have

$$X_\gamma \langle X_\alpha, X_\beta \rangle = 0.$$

Then we obtain

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} + \frac{1}{2} \sum_{\gamma, \mu=1}^N B_{\alpha\beta}^{\gamma\mu} x_\mu x_\gamma + O(|x|^3) \quad \text{and} \quad g^{\alpha\beta}(x) = \delta_{\alpha\beta} - \frac{1}{2} \sum_{\gamma, \mu=1}^N B_{\alpha\beta}^{\gamma\mu} x_\gamma x_\mu + O(|x|^3). \quad (2.4)$$

Therefore the matrix g can be rewritten in the form $g = I_N + H$, where I_N is the N -dimensional identity matrix. Then as $H \rightarrow 0$, we have

$$\sqrt{|g|} = \sqrt{|I_N + H|} = 1 + \frac{\text{tr}(H)}{2} + \frac{(\text{tr}(H))^2}{4} - \frac{\text{tr}(H^2)}{4} + O(H^3).$$

A straightforward computation yields

$$\sqrt{|g|}(x) = 1 + \frac{1}{4} \sum_{\gamma, \mu, \alpha=1}^N B_{\alpha\alpha}^{\gamma\mu} x_\mu x_\gamma + O(|x|^3). \quad (2.5)$$

Lemma 2.1. *We have*

$$\frac{1}{4} \sum_{\alpha \gamma=1}^N B_{\alpha\alpha}^{\gamma\gamma} = -\frac{1}{6} R_g - \frac{1}{6} S_g - W_g, \quad (2.6)$$

where R_g and S_g are the scalar curvatures of M and Σ respectively and

$$W_g = \sum_{i=k+1}^N \sum_{a=1}^k R_{iaia}.$$

Proof. Splitting, we have

$$\sum_{\gamma=1}^N \sum_{\alpha=1}^N B_{\alpha\alpha}^{\gamma\gamma} = \sum_{i=1}^k \sum_{j=1}^k B_{ii}^{jj} + \sum_{a=k+1}^N \sum_{b=k+1}^N B_{aa}^{bb} + \sum_{a=k+1}^N \sum_{i=1}^k B_{aa}^{ii} + \sum_{a=k+1}^N \sum_{i=1}^k B_{ii}^{aa}. \quad (2.7)$$

By (2.2) and [Gray [10], Lemma 9.19], we have

$$\begin{aligned} \sum_{i=1}^k \sum_{j=1}^k B_{ii}^{jj} &= \sum_{i=1}^k \sum_{j=1}^k X_i X_i \langle X_j, X_j \rangle = \sum_{i=1}^k \sum_{j=1}^k 2 \langle \nabla_{X_i X_i}^2 X_j, X_j \rangle + \sum_{i=1}^k \sum_{j=1}^k 2 \langle \nabla_{X_i} X_j, \nabla_{X_i} X_j \rangle \\ &= -\frac{2}{3} \sum_{i=1}^k \sum_{j=1}^k R_{ijij} = -\frac{2}{3} R_g + \frac{2}{3} S_g. \end{aligned} \quad (2.8)$$

By (2.1) and the Gauss equation, see [B. O’neill [14], page10], we get:

$$\begin{aligned} \sum_{a=k+1}^N \sum_{b=k+1}^N B_{aa}^{bb} &= \sum_{a=k+1}^N \sum_{b=k+1}^N X_a X_a \langle X_b, X_b \rangle \\ &= 2 \sum_{a=k+1}^N \sum_{b=k+1}^N \langle \nabla_{X_a X_a}^2 X_b, X_b \rangle + 2 \sum_{a=k+1}^N \sum_{b=k+1}^N \langle \nabla_{X_a} X_b, \nabla_{X_a} X_b \rangle \\ &= -\frac{4}{3} \sum_{a=k+1}^N \sum_{b=k+1}^N R_{abab}^M = -\frac{4}{3} \sum_{a=k+1}^N \sum_{b=k+1}^N R_{abab}^\Sigma = -\frac{4}{3} S_g. \end{aligned} \quad (2.9)$$

By (2.2) and [Gray [10], Lemma 9.17 and Lemma 9.19], we obtain

$$\begin{aligned} \sum_{i=k+1}^N \sum_{a=1}^k (B_{aa}^{ii} + B_{ii}^{aa}) &= \sum_{i=k+1}^N \sum_{a=1}^k X_i X_i \langle X_a, X_a \rangle + \sum_{i=k+1}^N \sum_{a=1}^k X_a X_a \langle X_i, X_i \rangle \\ &= 2 \sum_{i=k+1}^N \sum_{a=1}^k \langle \nabla_{X_i X_i}^2 X_a, X_a \rangle + 2 \sum_{i=k+1}^N \sum_{a=1}^k \langle \nabla_{X_a X_a}^2 X_i, X_i \rangle \\ &\quad + 2 \sum_{i=k+1}^N \sum_{a=1}^k \langle \nabla_{X_i} X_a, \nabla_{X_i} X_a \rangle + 2 \sum_{i=k+1}^N \sum_{a=1}^k \langle \nabla_{X_a} X_i, \nabla_{X_a} X_i \rangle \\ &= 2 \sum_{i=k+1}^N \sum_{a=1}^k \langle \nabla_{X_i X_i}^2 X_a, X_a \rangle + 2 \sum_{i=k+1}^N \sum_{a=1}^k \langle \nabla_{X_a X_a}^2 X_i, X_i \rangle = -4 \sum_{i=k+1}^N \sum_{a=1}^k R_{iaia} \\ &= -4W_g. \end{aligned} \quad (2.10)$$

The result follows immediately from (2.7), (2.8), (2.9) and (2.10). This then ends the proof.

Lemma 2.2. *We have*

$$\sum_{\gamma\mu=1}^N \sum_{a=1}^k \sum_{i=k+1}^N B_{ai}^{\gamma\mu} = \frac{1}{3} \sum_{ijl=k+1}^N \sum_{a=1}^k R_{laji} + \frac{1}{3} \sum_{abc=1}^k \sum_{i=k+1}^N R_{abci}. \quad (2.11)$$

Proof. By definition, we have

$$\begin{aligned} \sum_{\gamma\mu=1}^N B_{ai}^{\gamma\mu} &= \sum_{\gamma\mu=1}^N X_\gamma X_\mu \langle X_a, X_i \rangle = \sum_{jl=k+1}^N X_l X_j \langle X_a, X_i \rangle \\ &+ \sum_{bc=1}^k X_c X_b \langle X_a, X_i \rangle + 2 \sum_{j=k+1}^N \sum_{b=1}^k X_j X_b \langle X_a, X_i \rangle. \end{aligned}$$

By (2.1) and (2.2), we have

$$\begin{aligned} \sum_{jl=k+1}^N X_l X_j \langle X_a, X_i \rangle &= \sum_{jl=k+1}^N \langle \nabla_{X_l X_j}^2 X_a, X_i \rangle + \sum_{jl=k+1}^N \langle \nabla_{X_l X_j}^2 X_i, X_a \rangle + 2 \sum_{jl=k+1}^N \langle \nabla_{X_l} X_a, \nabla_{X_j} X_i \rangle \\ &= - \sum_{jl=k+1}^N R_{laji} - \frac{1}{3} \sum_{jl=k+1}^N (R_{ijla} + R_{ilja}) \\ &= \sum_{jl=k+1}^N \frac{2}{3} R_{laji} - \sum_{jl=k+1}^N \frac{1}{3} R_{ilja} = \sum_{jl=k+1}^N \frac{1}{3} R_{ilja}, \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} \sum_{bc=1}^k X_c X_b \langle X_a, X_i \rangle &= \sum_{bc=1}^k \langle \nabla_{X_c X_b}^2 X_a, X_i \rangle + \sum_{bc=1}^k \langle \nabla_{X_c X_b}^2 X_i, X_a \rangle + 2 \sum_{bc=1}^k \langle \nabla_{X_c} X_a, \nabla_{X_b} X_i \rangle \\ &= \sum_{bc=1}^k \langle \nabla_{X_c X_b}^2 X_a, X_i \rangle + \sum_{bc=1}^k \langle \nabla_{X_c X_b}^2 X_i, X_a \rangle \\ &= - \sum_{bc=1}^k \frac{1}{3} (R_{abci} - R_{acbi}) + \sum_{bc=1}^k R_{ciba}. \end{aligned}$$

Hence, we obtain

$$\sum_{a,b,c=1}^k \sum_{i=k+1}^N X_c X_b \langle X_a, X_i \rangle = \sum_{a,b,c=1}^k \sum_{i=k+1}^N \left(\frac{2}{3} R_{abci} - R_{ciba} \right). \quad (2.13)$$

By (2.12), and (2.13), we finally obtain:

$$\sum_{\gamma\mu=1}^N \sum_{a=1}^k \sum_{i=k+1}^N B_{ai}^{\gamma\mu} = \frac{1}{3} \sum_{ijl=k+1}^N \sum_{a=1}^k R_{laji} + \frac{1}{3} \sum_{abc=1}^k \sum_{i=k+1}^N R_{abci}.$$

This then ends the proof.

3. Existence Result in domains

Remark 3.1. *The key analytical step in the proof of Theorem 1.1 is the strict inequality*

$$c^* < \beta^*,$$

where c^* is the mountain-pass level of J on $H^1(M)$ and β^* is the mountain-pass level of the limit functional Π on $\mathcal{D}^{1,2}(\mathbb{R}^N)$. This inequality plays a double role: on one hand, it restores the Palais–Smale compactness for J at level c^* (see Lemma 3.5 below); on the other hand, its proof (carried out in Section 5 via explicit test-function computations in Fermi coordinates) is precisely where condition (1.3) on the curvature enters. Thus, condition (1.3) is not merely a sufficient condition imposed for convenience: it is the exact geometric threshold that makes the comparison $c^* < \beta^*$ achievable through the concentrating test-function construction.

We recall that

$$\beta^* = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N), u \geq 0, u \neq 0} \max_{t \geq 0} \Pi(tu) \quad \text{and} \quad c^* := \inf_{u \in H^1(M), u \geq 0, u \neq 0} \max_{t \geq 0} J(tu), \quad (3.1)$$

with

$$\Pi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2_{s_1}^*} \int_{\mathbb{R}^N} |z|^{-s_1} |u|^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^N} |z|^{-s_2} |u|^{2_{s_2}^*} dx, \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$$

and

$$J(u) = \frac{1}{2} \int_M |\nabla_g u|^2 dv_g + \frac{1}{2} \int_M h(x) u^2 dv_g - \frac{\lambda}{2_{s_1}^*} \int_M \frac{|u|^{2_{s_1}^*}}{\rho_{\Sigma}^{s_1}(x)} dv_g - \frac{1}{2_{s_2}^*} \int_M \frac{|u|^{2_{s_2}^*}}{\rho_{\Sigma}^{s_2}(x)} dv_g, \quad \forall u \in H^1(M).$$

The aim of this section is to prove the following compactness result.

Proposition 3.2. *Let $N \geq 4$, $0 \leq s_2 < s_1 < 2$, $\lambda > 0$ and M be a compact Riemannian manifold of dimension N . Consider $\Sigma \subset M$ a closed submanifold of dimension $1 \leq k \leq N - 2$. Let h be a continuous function such that the linear operator $-\Delta_g + h$ is coercive. We assume that*

$$c^* < \beta^*.$$

Then there exists a positive function $u \in H^1(M)$ solution of the Euler-Lagrange equation

$$-\Delta_g u + hu = \lambda \rho_{\Sigma}^{-s_1} u^{2_{s_1}^* - 1} + \rho_{\Sigma}^{-s_2} u^{2_{s_2}^* - 1} \quad \text{in } (M, g). \quad (3.2)$$

The proof of this result is divided into various preliminaries results. We start by the following so-called mountain pass lemma due to Ambrosetti and Robinowitz, see [6].

Lemma 3.3. (Mountain Pass Lemma)

Let $(X, \|\cdot\|_X)$ be a Banach space and $\Psi : X \rightarrow \mathbb{R}$ a functional of class $\mathcal{C}^1$. We assume that

1. $\Psi(0) = 0$;
2. *there exist $\lambda_0, r > 0$ such that $\Psi(u) \geq \lambda_0$ for all $u \in X$ satisfying $\|u\|_X = r$;*
3. *there exists $u_0 \in X$ such that*

$$\lim_{t \rightarrow +\infty} \sup \Psi(tu_0) < 0.$$

Consider $t_0 > 0$ sufficiently large such that $\|t_0 u_0\|_X > r$ and $\Psi(t_0 u_0) < 0$. Define $\beta \leq \lambda_0$ as

$$\beta := \inf_{\gamma \in \mathcal{P}} \sup_{t \in [0,1]} \Psi(\gamma(t)),$$

where $\mathcal{P} = \{\gamma \in \mathcal{C}^0([0,1]; X) \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = t_0 u_0\}$. Then, there exists a sequence $(u_n)_n \subset X$ such that $\Psi(u_n) \rightarrow \beta$ and $\Psi'(u_n) \rightarrow 0$ strongly in X' . Moreover, we have

$$\beta \leq \sup_{t \geq 0} \Psi(tu_0).$$

Lemma 3.4. Let (M, g) be a compact Riemannian manifold of dimension $N \geq 4$, Σ a submanifold of M of dimension $1 \leq k \leq N - 2$ and h a continuous function such that the linear operator $-\Delta_g + h$ is coercive. Let $u_0 \in H^1(M) \setminus \{0\}$. Then there exists c_0 a positive constant depending on u_0 and $(u_n)_n \subset H^1(M)$ a Palais-Smale sequence for J at level c_0 . Moreover

$$c_0 \leq \sup_{t \geq 0} J(tu_0).$$

Proof. We let $t \in \mathbb{R}$. Recall that for all $u \in H^1(M)$, we have

$$J(tu) := \frac{t^2}{2} \int_M (|\nabla_g u|^2 + hu^2) dv_g - \lambda \frac{|t|^{2_{s_1}^*}}{2_{s_1}^*} \int_M \rho_{\Sigma}^{-s_1} |u|^{2_{s_1}^*} dv_g - \frac{|t|^{2_{s_2}^*}}{2_{s_2}^*} \int_M \rho_{\Sigma}^{-s_2} |u|^{2_{s_2}^*} dv_g$$

Then $\Psi \in \mathcal{C}^1(H^1(M), \mathbb{R})$. Since $0 \leq s_2 < s_1 < 2$ and the fact that the function $s \mapsto 2_s^* := \frac{2(N-s)}{N-2}$ is decreasing, we have

$$\lim_{t \rightarrow \infty} J(tu) = -\infty.$$

Moreover, using the fact that $2_{s_1}^*, 2_{s_2}^* > 2$, then there exists positive numbers λ_0, r such that

$$\inf_{\|u\|_{H^1(M)}=r} J(u) \geq \lambda_0.$$

Therefore by the Mountain pass Lemma 3.3, we get the desired result.

Lemma 3.5. Let $\alpha < \beta^*$ and $(u_n)_n \subset H^1(M)$ be a Palais-Smale sequence for J at level α . Then, up to a subsequence, there exists $u \in H^1(M)$ such that

$$\begin{cases} u_n \rightarrow u & \text{in } H^1(M) \\ J(u) = \alpha \\ J'(u) = 0. \end{cases}$$

Proof. Let $(u_n)_n \subset H^1(M)$ be a Palais-Smale sequence for J at level α :

$$J(u_n) = \frac{1}{2} \int_M |\nabla u_n|^2 dv_g + \frac{1}{2} \int_M h u_n^2 dv_g - \frac{\lambda}{2_{s_1}^*} \int_M \rho_{\Sigma}^{-s_1} |u_n|^{2_{s_1}^*} dv_g - \frac{1}{2_{s_2}^*} \int_M \rho_{\Sigma}^{-s_2} |u_n|^{2_{s_2}^*} dv_g = \alpha + o_n(1) \quad (3.3)$$

and

$$\begin{aligned} \langle J'(u_n), \varphi \rangle &= \int_M \nabla u_n \nabla \varphi dv_g + \int_M h u_n \varphi dv_g - \lambda \int_M \rho_{\Sigma}^{-s_1} |u_n|^{2_{s_1}^* - 2} u_n \varphi dv_g \\ &\quad - \frac{1}{2_{s_2}^*} \int_M \rho_{\Sigma}^{-s_2} |u_n|^{2_{s_2}^* - 2} u_n \varphi dv_g = o_n(1), \end{aligned} \quad (3.4)$$

for all $\varphi \in \mathcal{C}_c^\infty(M)$ as $n \rightarrow \infty$. Combining (3.3) and (3.4), we obtain

$$\alpha = \left(\frac{1}{2} - \frac{1}{2_{s_1}^*} \right) \int_M |\nabla u_n|^2 dv_g + \left(\frac{1}{2} - \frac{1}{2_{s_1}^*} \right) \int_M h u_n^2 dv_g + \left(\frac{1}{2_{s_1}^*} - \frac{1}{2_{s_2}^*} \right) \int_M \rho_{\Sigma}^{-s_2} |u_n|^{2_{s_2}^*} dv_g + o_n(1). \quad (3.5)$$

Since both $\frac{1}{2_{s_1}^*} - \frac{1}{2_{s_2}^*}$ and $\frac{1}{2} - \frac{1}{2_{s_2}^*}$ are positive and the operator $-\Delta_g + h$ is coercive, we obtain

$$\frac{\alpha}{\left(\frac{1}{2} - \frac{1}{2_{s_1}^*} \right)} + o(1) \geq \int_M |\nabla u_n|^2 dv_g + \int_M h u_n^2 dv_g \geq c \|u_n\|_{H^1(M)},$$

for some positive constant c independent of n . Consequently, up to a subsequence, there exists $u \in H^1(M)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(M), \quad u_n \rightarrow u \quad \text{a.e. in } M \quad \text{and} \quad u_n \rightarrow u \quad \text{in } L^2(M). \quad (3.6)$$

Set the defect $v_n := u_n - u$. By Brezis–Lieb lemma and the weak convergence in (3.6), we have:

$$\int_M |\nabla u_n|^2 dv_g = \int_M |\nabla u|^2 dv_g + \int_M |\nabla v_n|^2 dv_g + o(1)$$

and

$$\int_M \rho_{\Sigma}^{-s_i} |u_n|^{2_{s_i}^*} dv_g = \int_M \rho_{\Sigma}^{-s_i} |u|^{2_{s_i}^*} dv_g + \int_M \rho_{\Sigma}^{-s_i} |v_n|^{2_{s_i}^*} dv_g + o_n(1) \quad \text{for all } i = 1, 2.$$

Therefore

$$J(u_n) = J(u) + J_{\text{mod}}(v_n) + o_n(1),$$

with

$$J_{\text{mod}}(v) := \frac{1}{2} \int_M |\nabla v|^2 dv_g - \frac{\lambda}{2_{s_1}^*} \int_M \rho_{\Sigma}^{-s_1} |v|^{2_{s_1}^*} dv_g - \frac{1}{2_{s_2}^*} \int_M \rho_{\Sigma}^{-s_2} |v|^{2_{s_2}^*} dv_g.$$

Moreover

$$J'(u) = 0 \quad \text{and} \quad J'_{\text{mod}}(v_n) = o(1) \quad \text{in } (H^1(M))'. \quad (3.7)$$

Define the concentration functions

$$0 < r \longmapsto Q_{n,i}(x, r) = \int_{F_x(Q_r) \cap M} \rho_{\Sigma}^{-s_i} |v_n|^{2_{s_i}^*} dv_g, \quad \text{for all } i = 1, 2.$$

We have three alternatives.

Alternative 1: Vanishing: Let $r > 0$. We have

$$\lim_{n \rightarrow +\infty} \sup_{x \in \Sigma} Q_{n,i}(x, r) = 0, \quad \text{for } i = 1, 2.$$

Next, we define

$$\Sigma_r := \{x \in M : \rho_\Sigma(x) < r\}.$$

Then we can cover Σ_r by a finite number of open sets $(F_{x_j}(Q_r))_{1 \leq j \leq m}$. Therefore

$$\int_M \rho_\Sigma^{-s_i} |v_n|^{2_{s_i}^*} dv_g = \int_{\Sigma_r} \rho_\Sigma^{-s_i} |v_n|^{2_{s_i}^*} dv_g + \int_{M \setminus \Sigma_r} \rho_\Sigma^{-s_i} |v_n|^{2_{s_i}^*} dv_g.$$

Since in $M \setminus \Sigma_r$, the distance function ρ_Σ is bounded and $v_n \rightarrow 0$ strongly in $L^{2_{s_i}^*}(M)$, we obtain

$$\begin{aligned} \int_M \rho_\Sigma^{-s_i} |v_n|^{2_{s_i}^*} dv_g &= \int_{\Sigma_r} \rho_\Sigma^{-s_i} |v_n|^{2_{s_i}^*} dv_g + \int_{M \setminus \Sigma_r} \rho_\Sigma^{-s_i} |v_n|^{2_{s_i}^*} dv_g \\ &= \sum_{j=1}^m \int_{F_{x_j}(Q_r)} \rho_\Sigma^{-s_i} |v_n|^{2_{s_i}^*} dv_g + o_n(1) \leq m \sup_{x \in \Sigma} Q_{n,i}(x, r) + o_n(1) = o_n(1). \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} \int_M \rho_\Sigma^{-s_i} |v_n|^{2_{s_i}^*} dv_g = 0 \quad \text{for all } i = 1, 2.$$

By this, (3.6) and the second identity in (3.7), we obtain

$$\int_M (|\nabla v_n|^2 + h v_n^2) dv_g = \lambda \int_M \rho_\Sigma^{-s_1} |v_n|^{2_{s_1}^*} dv_g + \int_M \rho_\Sigma^{-s_2} |v_n|^{2_{s_2}^*} dv_g = o_n(1).$$

Moreover, since the operator $-\Delta_g + h$ is coercive, we immediately obtain that

$$v_n \rightarrow u \text{ strongly in } H^1(M).$$

Alternative 2: Compactness. There exists $x_0 \in \Sigma$ and $r_0 > 0$:

$$v_n \rightarrow v \neq 0 \quad \text{in } L^{2_i^*}(F_{x_0}(Q_{r_0}), \rho_\Sigma^{-s_i} dx).$$

Hence $v_n \rightarrow v \neq 0$ in $H^1(M)$ thanks to local compactness and the Euler–Lagrange residual in (3.7). Thus we are already done.

Alternative 3: Concentration. Then there exists $\varepsilon_0 > 0$, a sequence of points $(y_n)_n \subset M$, and $r > 0$ such that:

$$\liminf_{n \rightarrow \infty} Q_{n,1}(y_n, r) \geq \varepsilon_0 \quad \text{or} \quad \liminf_{n \rightarrow \infty} Q_{n,2}(y_n, r) \geq \varepsilon_0. \quad (3.8)$$

Since M is compact then (up to a subsequence) $y_n \rightarrow y_0$ in M . It is clear that $y_0 \in \Sigma$. Indeed, otherwise $\rho_\Sigma(y_0) \geq r_0 > 0$ for some $r_0 > 0$, then $\rho_\Sigma^{-s_i}$ is bounded on $F_{y_0}(Q_{r_0/2})$, so the usual Rellich compactness yields strong $L^{2_{s_i}^*}(F_{y_0}(Q_{r_0/2}), \rho_\Sigma^{-s_i} dx)$ convergence of v_n . Therefore $Q_{n,i}(y_n, r) \rightarrow 0$, contradicting (3.8). Hence $y_0 \in \Gamma$. In the sequel, we assume that concentration occurs. Without any loss of generality, we may assume that:

$$\liminf_{n \rightarrow \infty} Q_{n,1}(y_n, r) \geq \varepsilon_0 \quad (\text{The argument is the same if we assume that } \liminf_{n \rightarrow \infty} Q_{n,2}(y_n, r) \geq \varepsilon_0).$$

Let $R \geq 1$ and $\eta \in (0, \varepsilon_0/2)$ fixed. Since the map $r \mapsto Q_{n,1}(y_n, r)$ is continuous, there exists $r_n > 0$ such that

$$\int_{F_{y_n}(Q_{Rr_n})} \rho_{\Gamma}^{-s_1} |v_n|^{2_{s_1}^*} dx = \eta. \quad (3.9)$$

Define the rescaled functions :

$$w_n(\xi) = r_n^{\frac{N-2}{2}} v_n(F_{y_n}(r_n\xi)) \chi\left(\frac{\xi}{R}\right),$$

where $\chi \in C_c^\infty(Q_2)$ is a cut-off function with $\chi \equiv 1$ on Q_1 . In Fermi coordinates along Γ (near y_n), we have

$$\rho_{\Gamma}(F_{y_n}(r_n\xi)) = r_n|z| \quad \text{and} \quad dx = r_n^N(1 + o_n(1))d\xi \quad (3.10)$$

uniformly for $|\xi| \leq 2R$ with $\xi = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, see for instance Section 2. Combining (3.9) and (3.10), we obtain

$$\int_{Q_R} |z|^{-s_1} |w_n|^{2_{s_1}^*} dx = \int_{F_{y_n}(Q_{Rr_n})} \rho_{\Sigma}^{-s_1} |v_n|^{2_{s_1}^*} dv_g + o_n(1) = \eta + o_n(1).$$

In particular the sequence $(w_n)_n$ is bounded in $\mathcal{D}^{1,2}(Q_R)$. Consequently

$$w_n \rightharpoonup w \quad \text{in } \mathcal{D}^{1,2}(Q_R), \quad w_n \rightarrow w \quad \text{in } L_{\text{loc}}^q(\mathbb{R}^N) \text{ for } 1 \leq q < 2_0^* = \frac{2N}{N-2}, \text{ and } w_n \rightarrow w \text{ a.e. in } Q_R.$$

Moreover the sequence $\left(|z|^{-s_1} |w_n|^{2_{s_1}^*}\right)_n$ is uniformly integrable on Q_R , we use Vitali's theorem to obtain

$$0 < \eta = \lim_{n \rightarrow \infty} \int_{Q_R} |z|^{-s_1} |w_n|^{2_{s_1}^*} dx = \int_{Q_R} |z|^{-s_1} |w|^{2_{s_1}^*} dx.$$

Therefore $w \not\equiv 0$. Taking the limit as $R \rightarrow \infty$, we obtain $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Next, we let $\varphi \in C_c^\infty(\mathbb{R}^N)$ and define the rescaled test functions

$$\varphi_n(x) = r_n^{\frac{2-N}{2}} \varphi\left(\frac{F_{y_n}^{-1}(x)}{r_n}\right) \chi\left(\frac{F_{y_n}^{-1}(x)}{R}\right).$$

By the change of variables formula $x = F_{y_n}(r_n\xi)$, (3.10), and letting $n \rightarrow \infty$ then $R \rightarrow \infty$, the metric errors vanish by Dominated Convergence Theorem and the smallness of the cut-off gradient region, we get

$$\int_M \nabla v_n \cdot \nabla \varphi_n dv_g = \int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi dx + o_n(1),$$

and

$$\int_M \rho_{\Sigma}^{-s_i} |v_n|^{2_{s_i}^*-2} v_n \varphi_n dv_g = \int_{\mathbb{R}^N} |z|^{-s_i} |w|^{2_{s_i}^*-2} w \varphi dx + o_n(1), \quad \forall i = 1, 2.$$

Combining these two identities and taking φ_n as a test function in the second identity of (3.7), we obtain:

$$\int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi dx = \lambda \int_{\mathbb{R}^N} |z|^{-s_1} |w|^{2_{s_1}^*-2} w \varphi dx + \int_{\mathbb{R}^N} |z|^{-s_2} |w|^{2_{s_2}^*-2} w \varphi dx. \quad (3.11)$$

We recall that the functional $\Pi : \mathcal{D}^{1,2}(\mathbb{R}^N) \longrightarrow \mathbb{R}$ is given by

$$w \longmapsto \Pi(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 dx - \frac{\lambda}{2_{s_1}^*} \int_{\mathbb{R}^N} |z|^{-s_1} |w|^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^N} |z|^{-s_2} |w|^{2_{s_2}^*} dx,$$

whose mountain-pass level is β^* . Hence thanks to (3.11), w is a nontrivial weak solution of the Euler–Lagrange equation for the functional Π .

Next, we claim that

$$\liminf_{n \rightarrow \infty} J_{\text{mod}}(v_n) \geq \beta^*$$

with β^* as in (3.1). To prove the claim, we fix $\varepsilon > 0$. For $R > 1$, we set $\chi_R(x) = \chi\left(\frac{x}{R}\right)$ and define a localized bubble as

$$\tilde{v}_{n,R}(x) = r_n^{\frac{2-N}{2}} \chi_R \left(\frac{F_{y_n}^{-1}(x)}{r_n} \right) w \left(\frac{F_{y_n}^{-1}(x)}{r_n} \right),$$

where the cut-off function χ is defined as above. By change of variable formula, we have

$$\int_M |\nabla \tilde{v}_{n,R}|^2 dx = \int_{Q_{2R}} g^{ij}(\xi) \partial_i (\tilde{v}_{n,R} \circ F_{y_n})(\xi) \partial_j (\tilde{v}_{n,R} \circ F_{y_n})(\xi) \sqrt{|g|}(\xi) d\xi.$$

Using

$$(\Phi_R w)(\xi) = r_n^{\frac{N-2}{2}} \tilde{v}_{n,R}(F_{y_n}(\xi)),$$

we have

$$\nabla_x (\Phi_R w)(x) = r_n^{\frac{N-2}{2}} D F_{y_n}(x)^\top \nabla_x \tilde{v}_{n,R}(F_{y_n}(\xi)), \quad \text{and} \quad D F_{y_n}(\xi) = r_n (I_n + O(r_n |\xi|)).$$

Therefore

$$\int_M |\nabla \tilde{v}_{n,R}|^2 dv_g = \int_{Q_{2R}} |\nabla \Phi_R w|^2 d\xi + o_n(1) \int_{Q_{2R}} (|\nabla (\Phi_R w)|^2 + R^{-2} |\Phi_R w|^2) d\xi.$$

Now $\Phi_R \equiv 1$ on Q_R , $\nabla \Phi_R$ is supported in $A_R := Q_{2R} \setminus Q_R$ and $|\nabla \Phi_R| \leq \frac{C}{R}$ for some positive constant C . By Hardy's inequality on A_R , we have

$$\int_{A_R} w^2 d\xi \leq R^2 \int_{A_R} |\nabla w|^2 d\xi.$$

Thus

$$\left| \int_M |\nabla \tilde{v}_{n,R}|^2 dv_g - \int_{\mathbb{R}^N} |\nabla (\Phi_R w)|^2 d\xi \right| \leq o_n(1) + C \int_{\mathbb{R}^N \setminus Q_R} |\nabla w|^2 d\xi. \quad (3.12)$$

By change of variable formula, (3.10), we have

$$\int_M \rho_\Sigma^{-s_1} |\tilde{v}_{n,R}|^{2_{s_1}^*} dv_g = \int_{Q_{2R}} |z|^{-s_1} |\Phi_R w|^{2_{s_1}^*} d\xi + o_n(1) \int_{Q_{2R}} |z|^{-s_1} |\Phi_R w|^{2_{s_1}^*} d\xi.$$

Therefore

$$\left| \int_M \rho_{\Sigma}^{-s_1} |\tilde{v}_{n,R}|^{2_{s_1}^*} dv_g - \int_{Q_{2R}} |z|^{-s_1} |\Phi_R w|^{2_{s_1}^*} d\xi \right| \leq o_n(1) + \int_{\mathbb{R}^N \setminus Q_R} |z|^{-s_1} |w|^{2_{s_1}^*} d\xi. \quad (3.13)$$

The same computations give

$$\left| \int_M \rho_{\Sigma}^{-s_2} |\tilde{v}_{n,R}|^{2_{s_2}^*} dv_g - \int_{Q_{2R}} |z|^{-s_2} |\Phi_R w|^{2_{s_2}^*} d\xi \right| \leq o_n(1) + \int_{\mathbb{R}^N \setminus Q_R} |z|^{-s_2} |w|^{2_{s_2}^*} d\xi. \quad (3.14)$$

From (3.12), (3.13), (3.14) and the definition of J_{mod} and Π , we obtain

$$\begin{aligned} |J_{mod}(\tilde{v}_{n,R}) - \Pi(\Phi_R w)| &\leq o_n(1) + C \int_{\mathbb{R}^N \setminus Q_R} |\nabla w|^2 d\xi - \frac{\lambda}{2_{s_1}^*} \int_{\mathbb{R}^N \setminus Q_R} |z|^{-s_1} |w|^{2_{s_1}^*} d\xi - \frac{1}{2_{s_2}^*} \\ &\int_{\mathbb{R}^N \setminus Q_R} |z|^{-s_2} |w|^{2_{s_2}^*} d\xi. \end{aligned}$$

Since $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $|z|^{-s_i} |w|^{2_{s_i}^*} \in L^1(\mathbb{R}^N)$ thanks to the limiting equation and Hardy-Sobolev, the three tail integrals go to zero as R tend to ∞ . Hence

$$J_{mod}(\tilde{v}_{n,R}) - \Pi(\Phi_R w) = o_R(1) + o_n(1).$$

Now for each R , define the Nehari scaling $t_{n,R} > 0$ by

$$E_{n,R}(t) := \langle J'_{mod}(t\tilde{v}_{n,R}), t\tilde{v}_{n,R} \rangle = 0 \quad \text{at } t = t_{n,R}.$$

Since the map $t \mapsto E_{n,R}(t)$ is continuous, $E_{n,R}(0) = 0$, and $E'_{n,R}(0) > 0$, while $E_{n,R}(t) \rightarrow -\infty$ as $t \rightarrow \infty$, such a $t_{n,R}$ exists and is unique. By the convergence of coefficients under blow-up, we have

$$t_{n,R} \longrightarrow t_R > 0,$$

where t_R is the Nehari multiplier for Π and the truncated profile $\Psi_R w$. Furthermore, the maximal value satisfies

$$\begin{aligned} \max_{t>0} J_{mod}(t\tilde{v}_{n,R}) &= J_{mod}(t_{n,R}\tilde{v}_{n,R}) \\ &\geq \Pi(t_R \Psi_R w) - \varepsilon, \quad \text{for all large } n. \end{aligned}$$

After choosing R so large and then n large, we get $o_R(1) < \varepsilon/2$ and $o_n(1) < \varepsilon/2$. We obtain the desired estimate.

Since $\Psi_R w \longrightarrow w$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and in the weighted $L^{2_{s_i}^*}(\mathbb{R}^N, |z|^{-s_i} dx)$ spaces as $R \rightarrow \infty$, the Nehari multiplier $t_R \rightarrow t_*$ where t_* maximizes $\Pi(tw)$, and

$$J_{\mathbb{R}^N}(t_R \Psi_R w) \longrightarrow \max_{t>0} J_{\mathbb{R}^N}(tw) \geq \beta^*.$$

Combining the previous displays and letting first $n \rightarrow \infty$, then $R \rightarrow \infty$, and finally $\varepsilon \rightarrow 0$, yields the localized lower bound

$$\liminf_{n \rightarrow \infty} \max_{t>0} J_{mod}(t\tilde{v}_{n,R}) \geq \beta^*.$$

It remains to relate $J_{\text{mod}}(v_n)$ to $\max_t J_{\text{mod}}(t\tilde{v}_{n,R})$. For R fixed, choose n -dependent cut-off $\chi_{n,R} \in C_c^\infty(M)$ such that $\chi_{n,R} \equiv 1$ in $F_{y_n}(Q_{Rr_n})$ and $\chi_{n,R} \equiv 0$ in $M \setminus F_{y_n}(Q_{2Rr_n})$. Consider the split

$$v_n = \chi_{n,R} v_n + (1 - \chi_{n,R}) v_n = Z_{n,R} + W_{n,R},$$

where we have set $Z_{n,R} := \chi_{n,R} v_n$ and $W_{n,R} := (1 - \chi_{n,R}) v_n$. By construction, for any $\delta > 0$, there exist $n_0 \in \mathbb{N}$ and R_0 such that for all $R \geq R_0$,

$$|J_{\text{mod}}(Z_{n,R}) - J_{\text{mod}}(v_n)| \leq \delta \quad \text{for all } n \geq n_0,$$

and the interaction terms between $Z_{n,R}$ and $W_{n,R}$ are $o_n(1)$ because the supports are separated and the weights are locally integrable. Comparing $Z_{n,R}$ with $\tilde{v}_{n,R}$, we get:

$$\liminf_{n \rightarrow \infty} J_{\text{mod}}(v_n) \geq \liminf_{n \rightarrow \infty} J_{\text{mod}}(Z_{n,R}) - \delta \geq \liminf_{n \rightarrow \infty} \max_{t > 0} J_{\text{mod}}(t\tilde{v}_{n,R}) - \delta \geq \beta^* - \delta.$$

Since $\delta > 0$ is arbitrary, we conclude

$$\liminf_{n \rightarrow \infty} J_{\text{mod}}(v_n) \geq \beta^*,$$

which prove the claim. Recall that

$$\alpha = \lim_n J(u_n) = J(u) + \lim_n J_{\text{mod}}(v_n).$$

If a nontrivial bubble occurs, then $\lim_n J_{\text{mod}}(v_n) \geq \beta^* + o(1)$, so that $\alpha \geq J(u) + \beta^* \geq \beta^*$. This contradicts the hypothesis $\alpha < \beta^*$. Thus, no nontrivial bubble can appear, and necessarily $v_n \rightarrow 0$ (so $u_n \rightarrow u$) in $H_0^1(\Omega)$.

Lemma 3.6. *Let Ω be a bounded domain of $\mathbb{R}^N$, Γ a closed curve included in Ω and h a continuous function such that the linear operator $-\Delta + h$ is coercive. Let $u_0 \in H_0^1(\Omega) \setminus \{0\}$. Then there exists c_0 a positive constant depending on u_0 and $(u_n)_n \subset H_0^1(\Omega)$ a Palais-Smale sequence for J at level c_0 . Moreover*

$$c_0 \leq \sup_{t \geq 0} J(tu_0).$$

Proof. We let $t \in \mathbb{R}$. Recall that for all $u \in H_0^1(\Omega)$, we have

$$J(tu) := \frac{t^2}{2} \int_{\Omega} (|\nabla u|^2 + hu^2) dx - \lambda \frac{|t|^{2_{s_1}^*}}{2_{s_1}^*} \int_{\Omega} \rho_{\Gamma}^{-s_1} |u|^{2_{s_1}^*} dx - \frac{|t|^{2_{s_2}^*}}{2_{s_2}^*} \int_{\Omega} \rho_{\Gamma}^{-s_2} |u|^{2_{s_2}^*} dx$$

Then $\Psi \in \mathcal{C}^1(H_0^1(\Omega), \mathbb{R})$. Since $0 \leq s_2 < s_1 < 2$ and the fact that the function $s \mapsto 2_s^* := \frac{2(N-s)}{N-2}$ is decreasing, we have

$$\lim_{t \rightarrow \infty} J(tu) = -\infty.$$

Moreover, using the fact that $2_{s_1}^*, 2_{s_2}^* > 2$, then there exists positive numbers λ_0, r such that

$$\inf_{\|u\|_{H_0^1(\Omega)}=r} J(u) \geq \lambda_0.$$

Therefore by the Mountain pass Lemma 3.3, we get the desired result.

Proof. of Proposition 3.5

Let $u_0 \in H_0^1(\Omega)$ be a non-negative, non-vanishing function such that

$$\sup_{t \geq 0} J(tu_0) < \beta^*.$$

Then by Lemma 3.6, there exists $c_0 > 0$ depending on u_0 and a Palais-Smale sequence $(u_n)_n \subset H_0^1(\Omega)$ for Ψ at level c_0 such that

$$c_0 \leq \sup_{t \geq 0} J(tu_0) < \beta^*.$$

By Lemma 3.5, there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that, up to a subsequence,

$$u_n \longrightarrow u \quad \text{strongly in } H_0^1(\Omega) \text{ as } n \rightarrow \infty \text{ and } J'(u) = 0.$$

The last equality corresponds exactly to the Euler-Lagrange equation (3.2). This then ends the proof.

4. Some preliminary results

We let $0 < s_2 < s_1 < 2$, $\lambda > 0$ and consider the problem

$$\begin{cases} -\Delta u = \lambda \frac{u^{2_{s_1}^*-1}(x)}{|z|^{s_1}} + \frac{u^{2_{s_2}^*-1}}{|z|^{s_2}} & \text{in } \mathbb{R}^N \\ u(x) > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (4.1)$$

where $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and for $i = 1, 2$, we have

$$2_{s_i}^* = \frac{2(N - s_i)}{N - 2}$$

are two critical Hardy-Sobolev exponents. Then equation (4.1) admits a positive solution $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ depending only on $|y|$ and $|z|$. In other words, there exists a function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$w(x) = \theta(|y|, |z|). \quad (4.2)$$

Moreover there exists c_1, c_2 and C positive constants satisfying

$$\frac{c_1}{1 + |x|^{N-2}} \leq w(x) \leq \frac{c_2}{1 + |x|^{N-2}}, \quad \text{and} \quad |\nabla w(x)| \leq \frac{C}{1 + |x|^{N-1}}, \quad \forall x \in \mathbb{R}^N, \quad (4.3)$$

see Diatta-Thiam [[3], Section 2]. The energy functional $\Pi : \mathcal{D}^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ associated to the Euler-Lagrange equation (4.1) is given by

$$\Pi(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2_{s_1}^*} \int_{\mathbb{R}^N} |z|^{-s_1} |u|^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^N} |z|^{-s_2} |u|^{2_{s_2}^*} dx. \quad (4.4)$$

Then we define

$$\beta^* := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N), u \geq 0, u \neq 0} \max_{t \geq 0} \Pi(tu).$$

We have proved previously that, we recover compactness provided

$$c^* < \beta^*.$$

We are now in position to prove the following results that will play an important role in the sequel.

Lemma 4.1. *Let $N \geq 4$. Then we have*

$$\begin{aligned} \sum_{ab=1}^k \int_{Q_{r/\varepsilon}} g^{ab}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \frac{\partial w}{\partial y_b}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{\mathbb{R}^N} |\nabla_y w|^2 dy - \varepsilon^2 \frac{R_g(x_0)}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla_y w|^2 dx \\ &+ \varepsilon^2 S_g(x_0) \left(\frac{4}{3k^2} \int_{\mathbb{R}^N} |y|^2 |\nabla_y w|^2 dx - \frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla_y w|^2 dx \right) \\ &+ \varepsilon^2 W_g(x_0) \left(\frac{1}{k(N-k)} \int_{\mathbb{R}^N} |z|^2 |\nabla_y w|^2 dx - \frac{1}{N} \int_{\mathbb{R}^N} |x|^2 |\nabla_y w|^2 dx \right) + o(\varepsilon^2) \quad \text{for } N \geq 5, \end{aligned}$$

and

$$\begin{aligned} \sum_{ab=1}^k \int_{Q_{r/\varepsilon}} g^{ab}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \frac{\partial w}{\partial y_b}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{\mathbb{R}^N} |\nabla_y w|^2 dy - \varepsilon^2 \frac{R_g(x_0)}{6N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_y w|^2 dx \\ &+ \varepsilon^2 S_g(x_0) \left(\frac{4}{3k^2} \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_y w|^2 dx - \frac{1}{6N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_y w|^2 dx \right) \\ &+ \varepsilon^2 W_g(x_0) \left(\frac{1}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_y w|^2 dx - \frac{1}{N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_y w|^2 dx \right) + O(\varepsilon^2) \quad \text{for } N = 4. \end{aligned}$$

Proof. By (2.4) and (2.5), we have :

$$\begin{aligned} \sum_{ab=1}^k \int_{Q_{r/\varepsilon}} g^{ab}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \frac{\partial w}{\partial y_b}(x) \sqrt{|g|}(\varepsilon x) dx \\ = \sum_{ab=1}^k \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 \frac{y_a y_b}{|y|^2} \left(\delta_{ab} - \frac{\varepsilon^2}{2} \sum_{\gamma\mu=1}^N B_{\alpha\beta}^{\mu\nu} x_\gamma x_\mu + O(\varepsilon^3 |x|^3) \right) \\ \times \left(1 + \frac{\varepsilon^2}{4} \sum_{\gamma\mu\alpha=1}^N B_{\alpha\alpha}^{\gamma\mu} x_\mu x_\gamma + O(\varepsilon^3 |x|^3) \right) dx. \end{aligned} \quad (4.5)$$

Therefore by (4.2) we have

$$\begin{aligned} \sum_{ab=1}^k \int_{Q_{r/\varepsilon}} g^{ab}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \frac{\partial w}{\partial y_b}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 dy + \frac{\varepsilon^2}{4} \sum_{\alpha\gamma\mu=1}^N B_{\alpha\alpha}^{\gamma\mu} \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 x_\gamma x_\mu dy \\ &- \frac{\varepsilon^2}{2} \sum_{ab=1}^k \sum_{\gamma\mu=1}^N B_{ab}^{\mu\gamma} \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 \frac{y_a y_b}{|y|^2} x_\gamma x_\mu dx + O \left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_y w|^2 \right). \end{aligned} \quad (4.6)$$

Since the domain $Q_{r/\varepsilon}$ is symmetric with respect to the directions z and y , then we can prove that

$$\sum_{\alpha\gamma\mu=1}^N B_{\alpha\alpha}^{\mu\gamma} \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 x_\gamma x_\mu dx = \frac{1}{N} \sum_{\alpha\gamma=1}^N B_{\alpha\alpha}^{\gamma\gamma} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_y w|^2 dx \quad (4.7)$$

and

$$\begin{aligned} \sum_{ab=1}^k \sum_{\gamma\mu=1}^N B_{ab}^{\mu\gamma} \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 \frac{y_a y_b}{|y|^2} x_\gamma x_\mu dx &= \frac{2}{k^2} \sum_{ab=1}^k B_{aa}^{bb} \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_y w|^2 dx \\ &+ \frac{1}{k(N-k)} \sum_{a=1}^k \sum_{i=k+1}^N B_{aa}^{ii} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_y w|^2 dx. \end{aligned} \quad (4.8)$$

Then combining (4.6), (4.7) and (4.8) we obtain

$$\begin{aligned} \sum_{ab=1}^k \int_{Q_{r/\varepsilon}} g^{ab}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \frac{\partial w}{\partial y_b}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 dy + \frac{\varepsilon^2}{4N} \sum_{\alpha\gamma=1}^N B_{\alpha\alpha}^{\gamma\gamma} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_y w|^2 dx \\ &- \frac{\varepsilon^2}{k^2} \sum_{ab=1}^k B_{aa}^{bb} \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_y w|^2 dx - \frac{\varepsilon^2}{2k(N-k)} \sum_{a=1}^k \sum_{i=k+1}^N B_{aa}^{ii} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_y w|^2 dx + O\left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_y w|^2\right). \end{aligned}$$

Therefore by (2.9) and (2.10), we get

$$\begin{aligned} \sum_{ab=1}^k \int_{Q_{r/\varepsilon}} g^{ab}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \frac{\partial w}{\partial y_b}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 dy + \frac{\varepsilon^2}{N} \left(-\frac{R_g}{6} - \frac{S_g}{6} - W_g\right) \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_y w|^2 dx \\ &+ \varepsilon^2 \frac{4}{3k^2} S_g \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_y w|^2 dx + \frac{\varepsilon^2}{k(N-k)} W_g \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_y w|^2 dx + O\left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_y w|^2\right). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{ab=1}^k \int_{Q_{r/\varepsilon}} g^{ab}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \frac{\partial w}{\partial y_b}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{Q_{r/\varepsilon}} |\nabla_y w|^2 dx + \frac{\varepsilon^2}{N} \left(-\frac{R_g}{6} - \frac{S_g}{6} - W_g\right) \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_y w|^2 dx \\ &+ \varepsilon^2 \frac{4}{3k^2} S_g \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_y w|^2 dx + \frac{\varepsilon^2}{k(N-k)} W_g \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_y w|^2 dx + O\left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_y w|^2 dx\right). \end{aligned} \quad (4.9)$$

Recall that the function w satisfies

$$|\nabla w| \leq \frac{C}{1 + |x|^{N-1}} \quad \text{in } \mathbb{R}^N,$$

for some positive constant C . Therefore, we have

$$\int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |\nabla_y w|^2 dx + \varepsilon^2 \int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx + \varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_y w|^2 dx = o(\varepsilon^2) \quad \text{for } N \geq 5 \quad (4.10)$$

and

$$\int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |\nabla_y w|^2 dx + \varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_y w|^2 dx = O(\varepsilon^2) \quad \text{for } N = 4. \quad (4.11)$$

Then the result follows from (4.9), (4.10) and (4.11). This then ends the proof.

Lemma 4.2. *Let $N \geq 4$. Then, we have*

$$\begin{aligned} \sum_{ij=k+1}^N \int_{Q_{r/\varepsilon}} g^{ij}(\varepsilon x) \frac{\partial w}{\partial z_i}(x) \frac{\partial w}{\partial z_j}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{\mathbb{R}^N} |\nabla_z w|^2 dy \\ &- \varepsilon^2 R_g(x_0) \left(\frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla_z w|^2 dx - \frac{2}{3(N-k)^2} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx \right) \\ &- \varepsilon^2 S_g(x_0) \left(\frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla_z w|^2 dx + \frac{2}{3(N-k)^2} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx \right) \\ &- \varepsilon^2 W_g(x_0) \left(\frac{1}{N} \int_{\mathbb{R}^N} |x|^2 |\nabla_z w|^2 dx - \frac{1}{k(N-k)} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx \right) + o(\varepsilon^2) \quad \text{for } N \geq 5 \end{aligned}$$

and

$$\begin{aligned} \sum_{ij=k+1}^N \int_{Q_{r/\varepsilon}} g^{ij}(\varepsilon x) \frac{\partial w}{\partial z_i}(x) \frac{\partial w}{\partial z_j}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{\mathbb{R}^N} |\nabla_z w|^2 dy \\ &- \varepsilon^2 R_g(x_0) \left(\frac{1}{6N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_z w|^2 dx - \frac{2}{3(N-k)^2} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \right) \\ &- \varepsilon^2 S_g(x_0) \left(\frac{1}{6N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_z w|^2 dx + \frac{2}{3(N-k)^2} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \right) \\ &- \varepsilon^2 W_g(x_0) \left(\frac{1}{N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_z w|^2 dx - \frac{1}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \right) + O(\varepsilon^2) \quad \text{for } N = 4. \end{aligned}$$

Proof. By (2.4) and (2.5), we have

$$\sum_{ij=k+1}^N \int_{Q_{r/\varepsilon}} g^{ij}(\varepsilon x) \frac{\partial w}{\partial z_i}(x) \frac{\partial w}{\partial z_j}(x) \sqrt{|g|}(\varepsilon x) dx = \sum_{ij=k+1}^N \int_{Q_{r/\varepsilon}} g^{ij}(\varepsilon x) |\nabla_z w|^2 \frac{z_i z_j}{|z|^2} \sqrt{|g|}(\varepsilon x) dx$$

$$= \sum_{ij=k+1}^N \int_{Q_{r/\varepsilon}} |\nabla_z w|^2 \frac{z_i z_j}{|z|^2} \left(\delta_{ij} - \frac{\varepsilon^2}{2} \sum_{\gamma\mu=1}^N B_{ij}^{\mu\nu} x_\gamma x_\mu + O(\varepsilon^3 |x|^3) \right) \times \left(1 + \frac{\varepsilon^2}{4} \sum_{\gamma\mu\alpha=1}^N B_{\alpha\alpha}^{\gamma\mu} x_\mu x_\gamma + O(\varepsilon^3 |x|^3) \right) dx. \quad (4.12)$$

Therefore using (4.2), we get

$$\begin{aligned} \sum_{ij=k+1}^N \int_{Q_{r/\varepsilon}} g^{ij}(\varepsilon x) \frac{\partial w}{\partial z_i}(x) \frac{\partial w}{\partial z_j}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{Q_{r/\varepsilon}} |\nabla_z w|^2 dy + \frac{\varepsilon^2}{4} \sum_{\alpha\gamma\mu=1}^N B_{\alpha\alpha}^{\gamma\mu} \int_{Q_{r/\varepsilon}} |\nabla_z w|^2 x_\gamma x_\mu dy \\ &\quad - \frac{\varepsilon^2}{2} \sum_{ij=k+1}^N \sum_{\gamma\mu=1}^N B_{ij}^{\mu\gamma} \int_{Q_{r/\varepsilon}} |\nabla_z w|^2 \frac{z_i z_j}{|z|^2} x_\gamma x_\mu dx + O \left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_z w|^2 \right). \end{aligned} \quad (4.13)$$

Using again the symmetry of the domain $Q_{r/\varepsilon}$ with respect to the directions $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{N-k}$ and (4.2), it easy follows that

$$\sum_{\alpha\gamma\mu=1}^N B_{\alpha\alpha}^{\gamma\mu} \int_{Q_{r/\varepsilon}} |\nabla_z w|^2 x_\gamma x_\mu dx = \frac{1}{N} \sum_{\alpha\gamma=1}^N B_{\alpha\alpha}^{\gamma\gamma} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_z w|^2 dx \quad (4.14)$$

and

$$\begin{aligned} \sum_{ij=k+1}^N \sum_{\gamma\mu=1}^N B_{ij}^{\mu\gamma} \int_{Q_{r/\varepsilon}} |\nabla_z w|^2 \frac{z_i z_j}{|z|^2} x_\gamma x_\mu dx &= \frac{2}{(N-k)^2} \sum_{ij=k+1}^N B_{ii}^{jj} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \\ &\quad + \frac{1}{k(N-k)} \sum_{a=1}^k \sum_{i=k+1}^N B_{ii}^{aa} \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_z w|^2 dx. \end{aligned} \quad (4.15)$$

Then combining (4.13), (4.14) and (4.15) we obtain

$$\begin{aligned} \sum_{ij=k+1}^k \int_{Q_{r/\varepsilon}} g^{ij}(\varepsilon x) \frac{\partial w}{\partial z_i}(x) \frac{\partial w}{\partial z_j}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{Q_{r/\varepsilon}} |\nabla_z w|^2 dy + \frac{\varepsilon^2}{4N} \sum_{\alpha\gamma=1}^N B_{\alpha\alpha}^{\gamma\gamma} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_z w|^2 dx \\ &\quad - \frac{\varepsilon^2}{(N-k)^2} \sum_{ij=k+1}^N B_{ii}^{jj} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx - \frac{\varepsilon^2}{2k(N-k)} \sum_{a=1}^k \sum_{i=k+1}^N B_{ii}^{aa} \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_z w|^2 dx \\ &\quad + O \left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_z w|^2 \right). \end{aligned}$$

Therefore by (2.6), (2.8) and (2.10), we get

$$\begin{aligned} \sum_{ij=k+1}^k \int_{Q_{r/\varepsilon}} g^{ij}(\varepsilon x) \frac{\partial w}{\partial z_i}(x) \frac{\partial w}{\partial z_j}(x) \sqrt{|g|}(\varepsilon x) dx &= \int_{Q_{r/\varepsilon}} |\nabla_z w|^2 dy - \frac{\varepsilon^2}{N} \left(\frac{R_g}{6} + \frac{S_g}{6} + W_g \right) \int_{Q_{r/\varepsilon}} |x|^2 |\nabla_z w|^2 dx \\ &\quad + \frac{2\varepsilon^2}{3(N-k)^2} (R_g - S_g) \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx + \frac{\varepsilon^2}{k(N-k)} W_g \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_z w|^2 dx \end{aligned}$$

$$+ O \left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_z w|^2 \right). \quad (4.16)$$

Using (4.3) it easy follows that

$$\int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |\nabla_y w|^2 dx + \varepsilon^2 \int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx + \varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_y w|^2 dx = o(\varepsilon^2) \quad \text{for } N \geq 5 \quad (4.17)$$

and

$$\int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |\nabla_y w|^2 dx + \varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |\nabla_y w|^2 dx = O(\varepsilon^2) \quad \text{for } N = 4. \quad (4.18)$$

Then the result follows from (4.16), (4.17) and (4.18). This then ends the proof.

Lemma 4.3. *Let $N \geq 4$. Then we have*

$$\sum_{a=1}^k \sum_{i=k+1}^N \int_{Q_{r/\varepsilon}} g^{ai}(\varepsilon x) \frac{\partial w}{\partial y_i}(x) \frac{\partial w}{\partial z_a}(x) \sqrt{|g|}(\varepsilon x) dx = o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. We have

$$\begin{aligned} \sum_{a=1}^k \sum_{i=k+1}^N \int_{Q_{r/\varepsilon}} g^{ai}(\varepsilon x) \frac{\partial w}{\partial y_i}(x) \frac{\partial w}{\partial z_a}(x) \sqrt{|g|}(\varepsilon x) dx &= \sum_{a=1}^k \sum_{i=k+1}^N \int_{Q_{r/\varepsilon}} |\nabla_y w| |\nabla_z w| \frac{y_i}{|y|} \frac{z_a}{|z|} g^{ai}(\varepsilon x) \sqrt{|g|}(\varepsilon x) dx \\ &= \sum_{a=1}^k \sum_{i=k+1}^N \int_{Q_{r/\varepsilon}} |\nabla_y w| |\nabla_z w| \frac{y_i z_a}{|y| |z|} \left(-\frac{\varepsilon^2}{2} \sum_{\gamma\mu=1}^N B_{ai}^{\mu\gamma} x_\gamma x_\mu + O(\varepsilon^3 |x|^3) \right) \\ &= \times \left(1 + \frac{\varepsilon^2}{4} \sum_{\gamma\mu\alpha=1}^N B_{\alpha\alpha}^{\gamma\mu} x_\mu x_\gamma + O(\varepsilon^3 |x|^3) \right). \\ &= -\frac{\varepsilon^2}{2} \sum_{\gamma\mu=1}^N \sum_{a=1}^k \sum_{i=k+1}^N B_{ai}^{\mu\gamma} \int_{Q_{\frac{r}{\varepsilon}}} |\nabla_y w| |\nabla_z w| \frac{y_a}{|y|} \frac{z_i}{|z|} x_\gamma x_\mu dx + O \left(\varepsilon^3 \int_{Q_{\frac{r}{\varepsilon}}} |x|^3 |\nabla_y w| |\nabla_z w| \right) \\ &= -\frac{\varepsilon^2}{k(N-k)} \sum_{a=1}^k \sum_{i=k+1}^N B_{ai}^{ai} \int_{Q_{\frac{r}{\varepsilon}}} |\nabla_y w| |\nabla_z w| |y| |z| dx + O \left(\varepsilon^3 \int_{Q_{\frac{r}{\varepsilon}}} |x|^3 |\nabla w|^2 dx \right). \end{aligned}$$

Thanks to the Bianchi identities and the definition of B_{ai}^{ai} in Section 2, we obtain

$$B_{ai}^{ai} = 0.$$

Therefore

$$\sum_{a=1}^k \sum_{i=k+1}^N \int_{Q_{r/\varepsilon}} g^{ai}(\varepsilon x) \frac{\partial w}{\partial y_i}(x) \frac{\partial w}{\partial z_a}(x) \sqrt{|g|}(\varepsilon x) dx = O \left(\varepsilon^3 \int_{Q_{\frac{r}{\varepsilon}}} |x|^3 |\nabla w|^2 dx \right).$$

Moreover using polar coordinates and the second estimate in (4.3), we obtain

$$\varepsilon^3 \int_{Q_{\frac{r}{\varepsilon}}} |x|^3 |\nabla w|^2 dx = o(\varepsilon^2), \quad \text{for } N \geq 4.$$

This then ends the proof.

5. Construction of Test function and expansion of $J(u_\varepsilon)$

We consider (M, g) a compact Riemannian manifold of dimension $N \geq 3$ and Σ a totally geodesic submanifold of M of dimension $1 \leq k \leq N - 2$. We define the functional $J : H^1(M) \setminus \{0\} \rightarrow \mathbb{R}$ by :

$$J(u) = \frac{1}{2} \int_M |\nabla_g u|^2 dv_g + \frac{1}{2} \int_M h u^2 dv_g - \frac{1}{2_{s_1}^*} \int_M \rho_{\Sigma}^{-s_1} |u|^{2_{s_1}^*} dv_g - \frac{1}{2_{s_2}^*} \int_M \rho_{\Sigma}^{-s_2} |u|^{2_{s_2}^*} dv_g. \quad (5.1)$$

Our aim in this section is to construct a family of test functions $(u_\varepsilon)_\varepsilon \subset H^1(M)$ and provide an expansion of $J(u_\varepsilon)$ as $\varepsilon \rightarrow 0$. For that, we consider a cut-off function $\eta \in C_c^\infty(F_{x_0}(Q_{2r}))$ satisfying

$$0 \leq \eta \leq 1 \quad \text{and} \quad \eta \equiv 1 \quad \text{in } F_{x_0}(Q_r). \quad (5.2)$$

Then for $\varepsilon > 0$, we define the test function $u_\varepsilon : M \rightarrow \mathbb{R}$ by :

$$u_\varepsilon(X) = \varepsilon^{\frac{2-N}{2}} \eta(F_{x_0}^{-1}(X)) w\left(\frac{F_{x_0}^{-1}(X)}{\varepsilon}\right). \quad (5.3)$$

In particular, for every $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, we have

$$u_\varepsilon(F_{x_0}(x)) = \varepsilon^{\frac{2-N}{2}} \eta(x) \theta\left(\frac{|y|}{\varepsilon}, \frac{|z|}{\varepsilon}\right). \quad (5.4)$$

5.1. Initial insights

We start by some important results.

Proposition 5.1. *We let $N \geq 4$. Then, as $\varepsilon \rightarrow 0$, we have*

$$\begin{aligned} \int_M |\nabla_g u_\varepsilon|^2 dv_g + \int_M h u_\varepsilon^2 dv_g &= \int_{\mathbb{R}^N} |\nabla w|^2 dx - \varepsilon^2 R_g(x_0) \left[\frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{2}{3(N-k)^2} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx \right] \\ &- \varepsilon^2 S_g(x_0) \left(\frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx + \frac{2}{3(N-k)^2} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx - \frac{4}{3k^2} \int_{\mathbb{R}^N} |y|^2 |\nabla_y w|^2 dx \right) \\ &- \varepsilon^2 W_g(x_0) \left(\frac{1}{N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{1}{k(N-k)} \int_{\mathbb{R}^N} |z|^2 |\nabla w|^2 dx \right) + h(x_0) \varepsilon^2 \int_{\mathbb{R}^N} w^2 dx + o(\varepsilon^2) \quad \text{for } N \geq 5. \end{aligned}$$

For $N = 4$, we have

$$\begin{aligned}
& \int_M |\nabla_g u_\varepsilon|^2 dv_g + \int_M h u_\varepsilon^2 dv_g = \int_{\mathbb{R}^N} |nw|^2 dx - \varepsilon^2 R_g(x_0) \left[\frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{2}{3(N-k)^2} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \right] \\
& - \varepsilon^2 S_g(x_0) \left(\frac{1}{6N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx + \frac{2}{3(N-k)^2} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx - \frac{4}{3k^2} \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_y w|^2 dx \right) \\
& - \varepsilon^2 W_g(x_0) \left(\frac{1}{N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx - \frac{1}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 dx \right) + h(x_0) \varepsilon^2 \int_{Q_{r/\varepsilon}} w^2 dx + O(\varepsilon^2).
\end{aligned}$$

Proof. By (5.2) and (5.3), we have

$$\int_M |\nabla_g u_\varepsilon|^2 dv_g = \int_{F_{x_0}(Q_{2r})} |\nabla_g u_\varepsilon|^2 dv_g = \varepsilon^{2-N} \int_{F_{x_0}(Q_r)} \left| \nabla_g \left(w \left(\frac{F_{x_0}^{-1}(X)}{\varepsilon} \right) \right) \right|^2 dv_g = + \int_{F_{x_0}(Q_{2r}) \setminus F_{x_0}(Q_r)} |\nabla_g u_\varepsilon|^2 dv_g. \quad (5.5)$$

By change of variable formula, we have

$$\begin{aligned}
& \varepsilon^{2-N} \int_{F_{x_0}(Q_r)} \left| \nabla_g \left(w \left(\frac{F_{x_0}^{-1}(X)}{\varepsilon} \right) \right) \right|^2 dv_g \\
& = \sum_{\alpha, \beta=1}^N \int_{Q_{r/\varepsilon}} g^{\alpha\beta}(\varepsilon x) \frac{\partial w}{\partial x_\alpha}(x) \cdot \frac{\partial w}{\partial x_\beta}(x) \sqrt{|g|}(\varepsilon x) dx \\
& = \sum_{a, b=1}^k \int_{Q_{r/\varepsilon}} g^{ab}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \cdot \frac{\partial w}{\partial y_b}(x) \sqrt{|g|}(\varepsilon x) dx + \sum_{i, j=k+1}^N \int_{Q_{r/\varepsilon}} g^{ij}(\varepsilon x) \frac{\partial w}{\partial z_i}(x) \cdot \frac{\partial w}{\partial z_j}(x) \sqrt{|g|}(\varepsilon x) dx \\
& + 2 \sum_{i=k+1}^k \sum_{a=1}^k \int_{Q_{r/\varepsilon}} g^{ai}(\varepsilon x) \frac{\partial w}{\partial y_a}(x) \cdot \frac{\partial w}{\partial z_i}(x) \sqrt{|g|}(\varepsilon x) dx. \quad (5.6)
\end{aligned}$$

Then by Lemma 4.1, Lemma 4.2 and Lemma 4.3, we have

$$\begin{aligned}
& \int_{F_{x_0}(Q_r)} |\nabla_g u_\varepsilon|^2 dv_g = \int_{\mathbb{R}^N} |\nabla w|^2 dx - \varepsilon^2 R_g(x_0) \left[\frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{2}{3(N-k)^2} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx \right] \\
& - \varepsilon^2 S_g(x_0) \left(\frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx + \frac{2}{3(N-k)^2} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx - \frac{4}{3k^2} \int_{\mathbb{R}^N} |y|^2 |\nabla_y w|^2 dx \right) \\
& - \varepsilon^2 W_g(x_0) \left(\frac{1}{N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{1}{k(N-k)} \int_{\mathbb{R}^N} |z|^2 |\nabla w|^2 dx - \frac{1}{k(N-k)} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx \right) \\
& = +o(\varepsilon^2) \quad \text{for } N \geq 5 \quad (5.7)
\end{aligned}$$

and for $N = 4$, we have

$$\begin{aligned}
\int_{F_{x_0}(Q_r)} |\nabla_g u_\varepsilon|^2 dv_g &= \int_{\mathbb{R}^N} |\nabla w|^2 dx - \varepsilon^2 R_g(x_0) \left[\frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{2}{3(N-k)^2} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \right] \\
&\quad - \varepsilon^2 S_g(x_0) \left(\frac{1}{6N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx + \frac{2}{3(N-k)^2} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx - \frac{4}{3k^2} \int_{Q_{r/\varepsilon}} |y|^2 |\nabla_y w|^2 dx \right) \\
&\quad - \varepsilon^2 W_g(x_0) \left(\frac{1}{N} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx - \frac{1}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 dx - \frac{1}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \right) + O(\varepsilon^2). \quad (5.8)
\end{aligned}$$

Next, we have

$$\int_{F(Q_{2r}) \setminus F(Q_r)} |\nabla_g u_\varepsilon|^2 dv_g = \int_{F(Q_{2r}) \setminus F(Q_r)} w_\varepsilon^2 |\nabla \eta|^2 dv_g - \frac{1}{2} \int_{F(Q_{2r}) \setminus F(Q_r)} w_\varepsilon^2 \Delta \eta^2 dv_g.$$

Since $\Delta \eta^2 = 2 \nabla \eta \cdot \nabla \eta + 2 \eta \Delta \eta$, we have

$$\int_{F(Q_{2r}) \setminus F(Q_r)} |\nabla_g u_\varepsilon|^2 dv_g = - \int_{F(Q_{2r}) \setminus F(Q_r)} w_\varepsilon^2 \eta \Delta \eta dv_g = O \left(\int_{F(Q_{2r}) \setminus F(Q_r)} w_\varepsilon^2 dv_g \right).$$

By the change of variable $x = \frac{F(\tilde{x})}{\varepsilon}$, we have :

$$\int_{F(Q_{2r}) \setminus F(Q_r)} |\nabla_g u_\varepsilon|^2 dv_g = O \left(\varepsilon^2 \int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} w^2 dx \right)$$

Thanks to the second estimate in (4.3) and polar coordinates, we have

$$\int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} w^2 dx = \begin{cases} O(\varepsilon) & \text{for } N \geq 5 \\ O(\ln(\varepsilon)) & \text{for } N = 4. \end{cases}$$

Therefore

$$\int_{F(Q_{2r}) \setminus F(Q_r)} |\nabla_g u_\varepsilon|^2 dv_g = O \left(\varepsilon^2 \int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} w^2 dx \right) = o(\varepsilon^2) \quad \text{for } N \geq 4. \quad (5.9)$$

In the sequel, we compute

$$\int_M h u_\varepsilon^2 dv_g = \int_{F(Q_r)} h u_\varepsilon^2 dv_g + \int_{F(Q_{2r}) \setminus F(Q_r)} h u_\varepsilon^2 dv_g.$$

By change of variable formula, (2.5) and using the fact that the function h is continuous, we have

$$\int_{F(Q_r)} h u_\varepsilon^2 dv_g = \varepsilon^2 h(x_0) \int_{Q_{r/\varepsilon}} w^2(x) \sqrt{|g|}(\varepsilon x) dx = \varepsilon^2 h(x_0) \int_{Q_{r/\varepsilon}} w^2(x) dx + O \left(\varepsilon^4 \int_{Q_{r/\varepsilon}} |x|^2 w^2(x) dx \right).$$

We use (4.3) and polar coordinates to get

$$\varepsilon^2 \int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} w^2(x) dx + \varepsilon^4 \int_{Q_{r/\varepsilon}} |x|^2 w^2(x) dx = o(\varepsilon^2) \quad \text{for } N \geq 5 \quad \text{and} \quad \varepsilon^4 \int_{Q_{r/\varepsilon}} |x|^2 w^2(x) dx = O(\varepsilon^2) \quad \text{if } N = 4.$$

Moreover using the change of variables as before and (4.3), we get

$$\int_{F(Q_{2r}) \setminus F(Q_r)} h u_\varepsilon^2 dv_g = O \left(\varepsilon^2 \int_{Q_{r/2\varepsilon} \setminus Q_{r/\varepsilon}} w^2 dx \right) = O(\varepsilon^N).$$

Therefore we get

$$\int_M h u_\varepsilon^2 dv_g = \begin{cases} h(x_0) \varepsilon^2 \int_{\mathbb{R}^N} w^2 dx + o(\varepsilon^2) & \text{for } N \geq 5 \\ h(x_0) \varepsilon^2 \int_{Q_{r/\varepsilon}} w^2 dx + O(\varepsilon^2) & \text{for } N = 4. \end{cases} \quad (5.10)$$

Hence the result follows directly from this, (5.5), (5.7), (5.8), (5.9) and (5.10).

Proposition 5.2. *We have*

$$\int_M \rho_\Sigma^{-s} |u_\varepsilon|^{2^*_s} dv_g = \int_{\mathbb{R}^N} |z|^{-s} w^{2^*_s} dx - \frac{\varepsilon^2}{6N} (R_g + S_g + 6W_g) \int_{\mathbb{R}^N} |x|^2 |z|^{-s} w^{2^*_s} dx + o(\varepsilon^2).$$

Proof. By (5.2) and (5.3), we have

$$\int_M \rho_\Sigma^{-s} |u_\varepsilon|^{2^*_s} dv_g = \int_{F(Q_r)} \rho_\Sigma^{-s} |u_\varepsilon|^{2^*_s} dv_g + \int_{F(Q_{2r}) \setminus F(Q_r)} \rho_\Sigma^{-s} |u_\varepsilon|^{2^*_s} dv_g$$

By change of variable formula, (2.3), (4.2), (2.6) and (2.5), we have

$$\begin{aligned} \int_{F(Q_r)} \rho_\Sigma^{-s} |u_\varepsilon|^{2^*_s} dv_g &= \int_{Q_{r/\varepsilon}} |z|^{-s} w^{2^*_s} \sqrt{|g|}(\varepsilon x) dx \\ &= \int_{Q_{r/\varepsilon}} |z|^{-s} w^{2^*_s} dx + \frac{\varepsilon^2}{4} \sum_{\gamma \mu \alpha = 1}^N B_{\alpha\alpha}^{\gamma\mu} \int_{Q_{r/\varepsilon}} |z|^{-s} w^{2^*_s} x_\mu x_\gamma dx + O \left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |z|^{-s} w^{2^*_s} dx \right) \\ &= \int_{\mathbb{R}^N} |z|^{-s} w^{2^*_s} dx - \frac{\varepsilon^2}{6N} (R_g + S_g + 6W_g) \int_{\mathbb{R}^N} |x|^2 |z|^{-s} w^{2^*_s} dx + O(\rho(\varepsilon)), \end{aligned}$$

where

$$\rho(\varepsilon) = \int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |z|^{-s} w^{2^*_s} dx + \varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |z|^{-s} w^{2^*_s} dx + \varepsilon^2 \int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |x|^2 |z|^{-s} w^{2^*_s} dx.$$

By (4.3), we use polar coordinates to get

$$\rho(\varepsilon) = o(\varepsilon^2) \quad \text{for } N \geq 4,$$

as $\varepsilon \rightarrow 0$. Therefore

$$\int_{F(Q_r)} \rho_\Sigma^{-s} |u_\varepsilon|^{2^*_s} dv_g = \int_{Q_{r/\varepsilon}} |z|^{-s} w^{2^*_s} \sqrt{|g|}(\varepsilon x) dx$$

$$\begin{aligned}
&= \int_{Q_{r/\varepsilon}} |z|^{-s} w^{2_s^*} dx + \frac{\varepsilon^2}{4} \sum_{\gamma\mu\alpha=1}^N B_{\alpha\alpha}^{\gamma\mu} \int_{Q_{r/\varepsilon}} |z|^{-s} w^{2_s^*} x_\mu x_\gamma dx + O \left(\varepsilon^3 \int_{Q_{r/\varepsilon}} |x|^3 |z|^{-s} w^{2_s^*} dx \right) \\
&= \int_{\mathbb{R}^N} |z|^{-s} w^{2_s^*} dx - \frac{\varepsilon^2}{6N} (R_g + S_g + 6W_g) \int_{\mathbb{R}^N} |x|^2 |z|^{-s} w^{2_s^*} dx + O(\rho(\varepsilon)).
\end{aligned} \tag{5.11}$$

The result follows immediately from classical Taylor-Young expansion and (5.11). This then ends the proof.

5.2. Completion of Theorem 1.1

Now, we are in a position to prove the main result. For that, we let $N \geq 5$ and consider the functionals $(\Psi_i)_{i=1,2,3}$ defined, for $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, by :

$$\begin{aligned}
\Psi_1(v) &= \frac{1}{12N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{1}{3(N-k)^2} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx \\
&\quad - \frac{\lambda}{2_{s_1}^*} \frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |z|^{-s_1} w^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^N} |x|^2 |z|^{-s_2} w^{2_{s_2}^*} dx,
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\Psi_2(v) &= \frac{1}{12N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx + \frac{1}{3(N-k)^2} \int_{\mathbb{R}^N} |z|^2 |\nabla_z w|^2 dx \\
&\quad - \frac{2}{3k^2} \int_{\mathbb{R}^N} |y|^2 |\nabla_y w|^2 dx - \frac{\lambda}{2_{s_1}^*} \frac{1}{6N} \int_{\mathbb{R}^N} |x|^2 |z|^{-s_1} w^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^N} |x|^2 |z|^{-s_2} w^{2_{s_2}^*} dx
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}
\Psi_3(v) &= \frac{1}{2N} \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{1}{2k(N-k)} \int_{\mathbb{R}^N} |z|^2 |\nabla w|^2 dx \\
&\quad - \frac{\lambda}{2_{s_1}^*} \frac{1}{N} \int_{\mathbb{R}^N} |x|^2 |z|^{-s_1} w^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \frac{1}{N} \int_{\mathbb{R}^N} |x|^2 |z|^{-s_2} w^{2_{s_2}^*} dx.
\end{aligned} \tag{5.14}$$

For $N = 4$, we define

$$\begin{aligned}
\Psi_{1,\varepsilon}(v) &= \frac{1}{48} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx - \frac{1}{3(4-k)^2} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \\
&\quad - \frac{\lambda}{2_{s_1}^*} \frac{1}{24} \int_{\mathbb{R}^4} |x|^2 |z|^{-s_1} w^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^4} |x|^2 |z|^{-s_2} w^{2_{s_2}^*} dx,
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
\Psi_{2,\varepsilon}(v) &= \frac{1}{48} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx + \frac{1}{3(4-k)^2} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla_z w|^2 dx \\
&\quad - \frac{2}{3k^2} \int_{\mathbb{R}^4} |y|^2 |\nabla_y w|^2 dx - \frac{\lambda}{2_{s_1}^*} \frac{1}{24} \int_{\mathbb{R}^4} |x|^2 |z|^{-s_1} w^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^4} |x|^2 |z|^{-s_2} w^{2_{s_2}^*} dx
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
\Psi_{3,\varepsilon}(v) &= \frac{1}{8} \int_{Q_{r/\varepsilon}} |x|^2 |\nabla w|^2 dx - \frac{1}{2k(4-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 dx \\
&\quad - \frac{\lambda}{2_{s_1}^*} \frac{1}{4} \int_{\mathbb{R}^4} |x|^2 |z|^{-s_1} w^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \frac{1}{4} \int_{\mathbb{R}^4} |x|^2 |z|^{-s_2} w^{2_{s_2}^*} dx.
\end{aligned} \tag{5.17}$$

We recall that

$$\Pi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2_{s_1}^*} \int_{\mathbb{R}^N} |z|^{-s_1} |u|^{2_{s_1}^*} dx - \frac{1}{2_{s_2}^*} \int_{\mathbb{R}^N} |z|^{-s_2} |u|^{2_{s_2}^*} dx, \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

We let $t > 0$. By Proposition 5.1 and Proposition 5.2, we obtain

$$J(tu_\varepsilon) = \Pi(tw) + \begin{cases} \varepsilon^2 \frac{t^2}{2} h(x_0) \int_{\mathbb{R}^N} w^2 dx + \varepsilon^2 R_g(x_0) \Psi_1(tw) - \varepsilon^2 S_g(x_0) \Psi_2(tw) \\ - \varepsilon^2 W_g(x_0) \Psi_3(tw) + o(\varepsilon^2) & \text{for } N \geq 5 \\ \varepsilon^2 \frac{t^2}{2} h(x_0) \int_{Q_{r/\varepsilon}} w^2 dx + \varepsilon^2 R_g(x_0) \Psi_{1,\varepsilon}(tw) - \varepsilon^2 S_g(x_0) \Psi_{2,\varepsilon}(tw) \\ - \varepsilon^2 W_g(x_0) \Psi_{3,\varepsilon}(tw) + O(\varepsilon^2) & \text{for } N = 4. \end{cases}$$

Since the function $s \mapsto \frac{2(N-s)}{N-2}$ is decreasing and $s_2 < s_1$, then $J(tu_\varepsilon)$ has a unique maximum. Moreover, we have

$$\max_{t \geq 0} \Pi(tw) = \Pi(w) = \beta^*.$$

Therefore, the maximum of $J(tu_\varepsilon)$ occurs at $t_\varepsilon = 1 + o_\varepsilon(1)$. Next, setting

$$\mathcal{M}_\varepsilon(tw) = \begin{cases} R_g(x_0) \Psi_1(tw) + S_g(x_0) \Psi_2(tw) + W_g(x_0) \Psi_3(tw) - \frac{t^2}{2} h(x_0) \int_{\mathbb{R}^N} w^2 dx + o_\varepsilon(1) & \text{for } N \geq 5 \\ R_g(x_0) \Psi_{1,\varepsilon}(tw) + S_g(x_0) \Psi_{2,\varepsilon}(tw) + W_g(x_0) \Psi_{3,\varepsilon}(tw) - \frac{t^2}{2} h(x_0) \int_{Q_{r/\varepsilon}} w^2 dx + O_\varepsilon(1) & \text{for } N = 4, \end{cases}$$

we have $\mathcal{M}_\varepsilon(t_\varepsilon w) > 0$ thanks to the assumption (1.3), where the constants A_{N,s_1,s_2} , B_{N,s_1,s_2} and C_{N,s_1,s_2} in Theorem 1.1 are given by

$$A_{N,s_1,s_2} = \frac{\Psi_1(w)}{\int_{\mathbb{R}^N} w^2 dx}; \quad B_{N,s_1,s_2} = \frac{\Psi_2(w)}{\int_{\mathbb{R}^N} w^2 dx} \quad \text{and} \quad C_{N,s_1,s_2} = \frac{\Psi_3(w)}{\int_{\mathbb{R}^N} w^2 dx} \quad \text{if } N \geq 5$$

and for $N = 4$, we have

$$A_{4,s_1,s_2} = \lim_{\varepsilon \rightarrow 0} \frac{\Psi_{1,\varepsilon}(w)}{\int_{Q_{r/\varepsilon}} w^2 dx}; \quad B_{4,s_1,s_2} = \lim_{\varepsilon \rightarrow 0} \frac{\Psi_{2,\varepsilon}(w)}{\int_{Q_{r/\varepsilon}} w^2 dx} \quad \text{and} \quad C_{4,s_1,s_2} = \lim_{\varepsilon \rightarrow 0} \frac{\Psi_{3,\varepsilon}(w)}{\int_{Q_{r/\varepsilon}} w^2 dx}.$$

Therefore

$$\max_{t \geq 0} J(tu_\varepsilon) := J(t_\varepsilon u_\varepsilon) = \Pi(t_\varepsilon w) - \varepsilon^2 \mathcal{M}_\varepsilon(t_\varepsilon w) < \Pi(t_\varepsilon w) \leq \Pi(w) = \beta^*.$$

We thus get the desired result.

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References

- [1] T. Bartsch, S. Peng and Z. Zhang, *Existence and non-existence of solutions to elliptic equations related to the Caffarelli-Kohn-Nirenberg inequalities*, Calc. Var. Partial Differential Equations **30**(2007), no. 1, 113-136.

- [2] L. A. Caffarelli, R. V. Kohn, and L. Nirenberg, First order interpolation inequalities with weights, *Compositio Mathematica*, **53** (1984), no. 3, 259–275.
- [3] , A. Diatta, and E. H. A. Thiam, A nonlinear PDE with two Hardy-Sobolev critical exponents with one dimension singularity, To appear in the Journal of Mathematical Physics, Analysis and Geometry.
- [4] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. Part I, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), no. 2, 109–145.
- [5] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. Part II, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), no. 4, 223–283.
- [6] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), no. 4, 349–381.
- [7] N. Ghoussoub and X. S. Kang, Hardy–Sobolev critical elliptic equations with boundary singularities, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **21** (2004), no. 6, 767–793.
- [8] N. Ghoussoub and F. Robert, The effect of curvature on the best constant in the Hardy–Sobolev inequalities, *Geom. Funct. Anal.*, **16** (2006), no. 6, 1201–1245.
- [9] J.-L. Chern and C.-S. Lin, Minimizers of Caffarelli–Kohn–Nirenberg inequalities with the singularity on the boundary, *Arch. Ration. Mech. Anal.*, **197** (2010), no. 2, 401–432.
- [10] A. Gray, *Tubes*, Vol. 221. Springer Science and Business Media, 2003.
- [11] C. H. Hsia, C.S. Lin and H. Wadade, *Revisiting an idea of Brézis and Nirenberg*, Journal of Functional Analysis, 259(7)(2010), 1816-1849.
- [12] Y. Li and C Lin, *A nonlinear elliptic PDE with two Sobolev–Hardy critical exponents*, Archive for Rational Mechanics and Analysis 203.3 (2012): 943-968.
- [13] R. Musina, *Ground state solutions of a critical problem involving cylindrical weights*, Nonlinear Anal. 68 (2008), no. 12, 3972–3986.
- [14] B. O’Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure and Applied Mathematics, Academic Press, 1983.
- [15] E. H. A. Thiam and A. Diatta, Hardy–Sobolev critical equations with multiple singularities on compact manifolds, *to appear in the Journal of Mathematical Physics, Analysis and Geometry*.
- [16] E. H. A. THIAM, Mountain Pass Solutions for Nonlinear Elliptic PDEs with Curve Singularities on the Boundary, Preprint.
- [17] X. Zhong and W. Zou, *A nonlinear elliptic PDE with multiple Hardy-Sobolev critical exponents in R^N* , Journal of Differential Equations, 292(2021), 354-387.