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# A unified characterization of certain operators related to deformed oscillator algebras via $d$-orthogonality

## Charactérisation unifiée de certains opérateurs en relation avec l'algèbre des oscillateurs déformée via la d-orthogonalité

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This paper is dedicated to Professor Youssef Ben Cheikh
on the occasion of his seventieth birthday.

**ABSTRACT.** In the present work, we are interested in the linear operators of the form $S = T(a_+)R(a_-)$, where $a_-$ and $a_+$ are the annihilation and creation operators, respectively defined in irreducible representation of a deformed oscillator algebra and $T$, $R$ are analytic functions. We characterize all real sequences $(x_k)_{k\geq 0}$ and functions $T$ for which the matrix elements associated to the operator $S$ are expressed in terms of polynomial sets on the discrete variable $x_k$ and we show when the considered polynomial sets are $d$-orthogonal. The analytic function $R$, in most specific cases is expressed in terms of exponential or $q$-exponential functions. As a consequence, several known results are recovered and extended, including those related to the Heisenberg-Weyl algebra, and $q$-deformed oscillator algebras. Explicit realizations are given in terms of Meixner and Charlier-type $d$-orthogonal polynomials, together with their $q$-analogues.
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## 1. Background material and preliminary results

In the literature of mathematical physics, deformed oscillator algebras play a fundamental role and have found numerous applications in quantum optics, special functions, and the theory of orthogonal polynomials. Various deformations and realizations have been introduced in order to describe nonclassical quantum systems and to uncover new algebraic structures. A particularly fruitful connection has emerged between deformed oscillator algebras and families of orthogonal polynomials. In this context, matrix elements of operators expressed in terms of creation and annihilation operators often give rise to classical or generalized orthogonal polynomial systems [10,14,15,17,18]. More recently, this correspondence has been extended to the framework of $d$-orthogonal polynomials, which generalize the standard notion of orthogonality.

Before presenting the main problem, we recall the notion of $d$-orthogonality as introduced in [20,23].

Let $\mathcal{P}$ be the space of polynomials with complex coefficients. A polynomial sequence $(P_n)_{n\geq 0}$ is called a polynomial set (PS, for shorter) if $\deg(P_n) = n$. The corresponding dual sequence of $(P_n)_{n\geq 0}$ denoted by $(u_n)_{n\geq 0}$ is defined as

$$u_n(P_m) = \delta_{nm},\ n, m = 0, 1, \dots \quad \delta_{nm} \quad \text{being the Kronecker delta.}$$

For a positive integer $d$, a PS $(P_n)_{n\geq 0}$ is called $d$-orthogonal ($d$-OPS, for shorter) with respect to the linear $d$-dimensional functional vector $\mathcal{U} = (u_0, u_1, \ldots, u_{d-1})$, if it satisfies the following *vector orthogonality relations*

$$\begin{cases} u_i(P_m P_n) = 0, & n \geq md + i + 1, \\ u_i(P_m P_n) \neq 0, & n = md + i, \end{cases}$$

for each integer $i \in \{0, 1, \ldots, d-1\}$.

For $d = 1$, the notion reduces to the classical concept of orthogonality.

A fundamental characterization of $d$-orthogonality was established by P. Maroni [20] as follows:

**Theorem 1.1.** *A PS $(P_n)_{n\geq 0}$ is a $d$-OPS if and only if it satisfies a recurrence relation of order $d+1$ of the type*

$$xP_n(x) = \sum_{i=0}^{d+1} c_{i,n+1} P_{n+1-i}(x), \quad \textit{where} \quad c_{0,n+1} c_{d+1,n+1} \neq 0. \tag{1.1}$$

By convention $P_{-n} = 0,\ n \geq 1$. The result for $d = 1$ reduces to the so-called Favard Theorem.

The next subsections are devoted to a review of some foundational results that we need in the remainder of the paper.

### 1.1. *Deformed oscillator algebras and basic properties*

Recall that a DOA (deformed oscillator algebra) is an algebra generated by the operators $1, a_-, a_+$, and $N$. These operators play the role of the identity, annihilation, creation, and number operators, respectively. The algebra is further characterized by a structure function $\phi$. These operators satisfy the following defining relations (see [6,9]).

$$a_- a_+ = \phi(N+1), \quad a_+ a_- = \phi(N), \quad [N, a_+] = a_+, \quad [N, a_-] = -a_-, \tag{1.2}$$

$\big(\phi$ is analytic, strictly increasing in the subset of non-negative integers and $\phi(0) = 0$, $[\ ,\ ]$ is the Lie bracket given as $[x, y] := xy - yx\big)$.

The Fock representation of such DOA in a Hilbert space with orthonormal basis $(e_n)_{n\geq 0}$ is given by

$$a_- e_n = \sqrt{\phi(n)} e_{n-1}, \quad a_+ e_n = \sqrt{\phi(n+1)} e_{n+1}, \quad N e_n = n e_n. \tag{1.3}$$

Note that $1 e_n = e_n$ and the vacuum vector $e_0$ satisfies the property $a_- e_0 = 0$.

In what follows, we will adopt for simplicity the standard Dirac notation : symbol $\big(|n\rangle\big)_{n\geq 0}$ stands for an orthonormal basis $\big(e_n\big)_{n\geq 0}$ in a Hilbert space. With this notation, we have for an operator $S$, the associated matrix elements are denoted as $\langle e_k | S e_n \rangle := \langle k | S | n \rangle$.

- Any state in the Hilbert space denoted as $|\psi\rangle$ has the decomposition

$$|\psi\rangle = \sum_{n=0}^{\infty} \langle n|\psi\rangle |n\rangle,$$

where $\langle n|\psi\rangle$ means the scalar product of the states $|\psi\rangle$ and $|n\rangle$.

 

The Hermitian conjugate $K^*$ (if it exists) of an operator $K$ is defined as

$$\langle\varphi|K|\psi\rangle = \langle K^*|\varphi|\psi\rangle = \overline{\langle\psi|K^*|\varphi\rangle},$$

$\psi$ and $\varphi$ are two states and $\overline{\langle ...\rangle}$ means the complex conjugation.

• By induction on $m$, we show by means of (1.3) that the action by powers of the operators $a_-$ and $a_+$ on the basis $(|n\rangle)_{n\geq 0}$ are given, for every integer $m \geq 0$ by

$$a_+^m|n\rangle = \sqrt{\frac{\phi(m+n)!}{\phi(n)!}}|n+m\rangle, \;\; a_-^m|n\rangle = \sqrt{\frac{\phi(n)!}{\phi(n-m)!}}|n-m\rangle, \; n \geq m, \tag{1.4}$$

where $\phi(0)! = 1, \;\; \phi(n)! = \phi(1)\cdots\phi(n)$.

As a consequence

$$|m\rangle = \frac{1}{\sqrt{\phi(m)!}}a_+^m|0\rangle, \quad \text{and} \quad a_-^m|n\rangle = 0, \; n < m. \tag{1.5}$$

Let $f(z) = \sum_{m=0}^{\infty} \alpha_m z^m$ be an analytic function defined in a neighborhood of zero.

• According to (1.4) and (1.5), the action of the operator $f(a_-)$ on every state $|n\rangle$ is well defined. In fact, we have

$$f(a_-)|n\rangle = \sum_{m=0}^{\infty} \alpha_m a_-^m|n\rangle = \sum_{m=0}^{n} \alpha_m \sqrt{\frac{\phi(n)!}{\phi(n-m)!}}|n-m\rangle. \tag{1.6}$$

Henceforth, the operator $f(a_-)$ will be defined on the subspace $V = \text{span}(|n\rangle)_{n\geq 0}$ consisting of all finite linear combination of the states $(|n\rangle)_{n\geq 0}$.

• If $g$ is also an analytic function defined in a neighborhood of zero, then the operator $(fg)(a_-)$ defined on the subspace $V = \text{span}(|n\rangle)_{n\geq 0}$ is given by: $(fg)(a_-) = f(a_-)g(a_-) = g(a_-)f(a_-)$. Hence, if $f(0) \neq 0$,

$$f(a_-) : V \longrightarrow V \quad \text{is invertible} \quad \text{and} \;\; f(a_-)^{-1} = \frac{1}{f}(a_-). \tag{1.7}$$

• In our work we will assume that the series

$$\sum_{m\geq 0} \alpha_m \sqrt{\phi(m+n)!} \quad \text{is absolutely convergent for every } n \geq 0, \tag{1.8}$$

which ensures the existence of $f(a_+)|n\rangle$.Thus, according to (1.4)

$$f(a_+)|n\rangle = \sum_{m=0}^{\infty} \alpha_m \sqrt{\frac{\phi(m+n)!}{\phi(n)!}}|n+m\rangle. \tag{1.9}$$

## 1.2. *Coherent states and matrix elements of an operator*

The concept of coherent states has several applications, mainly in quantum mechanics and applied mathematics [22]. Additionally, it can be presented as an algebraic tool which will be exploited in order to establish basic properties of some PSs by means of their generating functions.

 

Let $z$ be a complex number and $e_\phi$ the function defined as

$$e_\phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{\phi(n)!}, \qquad |z| < \lim_{n\to\infty} \sup \sqrt[n]{\phi(n)!}. \tag{1.10}$$

We denote by $|z\rangle$ the corresponding coherent state to the structure function $\phi$ defined as (see[9]),

$$|z\rangle := e_\phi(za_+)|0\rangle, \tag{1.11}$$

which is equivalent by means of (1.5) and (1.10) to

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\phi(n)!}}|n\rangle. \tag{1.12}$$

Its expansion coefficients are $\langle n|z\rangle = \dfrac{z^n}{\sqrt{\phi(n)!}}$.

If $z, z'$ are two coherent states, the inner product and the norm are given, respectively by

$$\langle z|z'\rangle = e_\phi\big(\overline{z}z'\big), \;\; \big|\big||z\rangle\big|\big| = \sqrt{e_\phi\big(|z|^2\big)}.$$

The state $|z\rangle$ can be looked upon as an eigenstate of the operators $a_-$ and $f(a_-)$. Indeed, we have

$$a_-|z\rangle = z|z\rangle, \quad f(a_-)|z\rangle = f(z)|z\rangle, \tag{1.13}$$

$f$ being an analytic function in a neighborhood of zero and $|z| < r$, where $r$ is the radius of convergence of $f$.

According to (1.12), $\langle k|S|z\rangle$ is a formal generating function of the matrix elements $\psi_{n,k} = \langle k|S|n\rangle$ associated to an operator $S$. Indeed, we have

$$\langle k|S|z\rangle = \sum_{n=0}^{+\infty} \psi_{n,k} \frac{z^n}{\sqrt{\phi(n)!}}. \tag{1.14}$$

## 1.3. *Lowering operator of a polynomial set*

We denote by $\wedge^{(-1)}$ the set of linear operators $\sigma$ defined on $\mathcal{P}$ such that for every polynomial $P$, $\deg(\sigma(P)) = \deg(P) - 1$ and $\sigma(c) = 0$ if $c \in \mathbb{C}$.

• Let $(P_n)_{n\geq 0}$ be a PS. A linear operator $\sigma \in \wedge^{(-1)}$ is called the lowering operator of $\big(P_n\big)_{n\geq 0}$ $\big($we say $\big(P_n\big)_{n\geq 0}$ is $\sigma$-Appell$\big)$ if

$$\sigma P_n(x) = P_{n-1}(x).$$

• Let $G(x,z) = \sum_{n=0}^{\infty} P_n(x) z^n$ be a generating function of $\big(P_n\big)_{n\geq 0}$. We have

$$\big(P_n\big)_{n\geq 0} \text{ is } \sigma\text{-Appell} \iff \sigma_x G(x,z) = zG(x,z).$$

$\sigma_x$ means that $\sigma$ acts on the $x$ variable.

• A polynomial set $\left(B_n\right)_{n\geq 0}$ is called the sequence of basic polynomials for $\sigma \in \wedge^{(-1)}$ if and only if for all positive integer $n$, we have

$$B_0(x) = 1,\ B_n(0) = \delta_{n,0} \ \text{ and } \ \sigma B_n(x) = B_{n-1}(x).$$

• It was shown in [3] that every polynomial set is $\sigma$-Appell ($\sigma \in \wedge^{(-1)}$) and that every lowering operator possesses a unique basic sequence of polynomials. Furthermore, the following equivalence holds:

$$\left(B_n\right)_{n\geq 0} \text{ is the basic sequence of } \sigma \iff \begin{cases} \sigma_x G_0(x,z) = zG_0(x,z), \\ G_0(0,z) = 1, \end{cases} \tag{1.15}$$

where $G_0(x,z) = \sum_{n=0}^{\infty} B_n(x) z^n$ is a generating function of $\left(B_n\right)_{n\geq 0}$.

• If $\left(P_n\right)_{n\geq 0}$ is $\sigma$-Appell and $G(x,z) = \sum_{n=0}^{\infty} P_n(x) z^n$, then we have

$$G(x,z) = A(z)G_0(x,z) \quad \text{or equivalently} \quad P_n = A(\sigma)B_n,\ n = 0,1,\ldots \tag{1.16}$$

where $A$ is a formal power series such that $A(0) \neq 0$. The operator $A(\sigma)$ is called the transfer operator of $(P_n)_{n\geq 0}$.

### 1.4. *Hypergeometric and basic hypergeometric functions*

In the sequel, we need some results from the $q$-analysis. For more details, the reader can consult [11,19].

• The hypergeometric series is denoted and defined by

$${}_rF_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}\middle| z\right) = \sum_{m=0}^{+\infty} \frac{(a_1)_m (a_2)_m \cdots (a_r)_m}{(b_1)_m (b_2)_m \cdots (b_s)_m} \frac{z^m}{m!},$$

where $r$ et $s$ are nonnegative integers, $z$ is a complex number, $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m := a(a+1)\cdots(a+m-1),\ \ (a)_0 = 1.$$

• The basic hypergeometric series is denoted and defined by

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}\middle| q; z\right) := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} z^n,$$

with $\binom{n}{2} = n(n-1)/2$, $(a;q)_n$ stands for the $q$-shifted factorial

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1,2,\ldots. \end{cases}$$

and $(a_1, a_2, \ldots, a_r; q)_n = \prod_{i=1}^{r} (a_i;q)_n$.

- The infinite $q$-shifted factorial defined by

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n) \quad \text{satisfies the property} \quad (z;q)_k = \frac{(z;q)_\infty}{(zq^k;q)_\infty}, \quad \text{with} \quad |q|<1. \tag{1.17}$$

- The little and the big $q$-exponential functions denoted, respectively as $e_q(z)$ and $E_q(z)$ are defined by

$$e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}, \ |z|<1, \quad E_q(z) = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}z^n, \ z\in\mathbb{C}, \tag{1.18}$$

- When $|q|<1$,

$$e_q(z) = \frac{1}{(z;q)_\infty}, \ \ |z|<1, \quad E_q(z) = \big(-z;q\big)_\infty, \ z\in\mathbb{C}. \tag{1.19}$$

- In addition,

$$e_q(z)E_q(-z) = 1, \quad e_{q^{-1}}(z) = E_q(-qz), \tag{1.20}$$

and the $q$-binomial formula is stated as

$$E_q(-xz)e_q(z) = \frac{(xz;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(x;q)_n}{(q;q)_n}z^n, \ \ |q|<1. \tag{1.21}$$

### 1.5. *Motivation examples*

In addition to their applications in several areas of mathematics, such as approximation theory and graph theory, over the past three decades [1,2,4,5,8,23], multiple and $d$-orthogonality also arise in Lie algebras, where many authors have constructed explicit examples of polynomial systems [7,12,13,24,26]. Most of the examples presented in these works appear as matrix elements of operators constructed from annihilation and creation operators and expressed as products of exponential or basic $q$-exponential functions. Moreover, several properties, characterization results and explicit realizations of the resulting polynomial systems are derived directly from their definitions as operator matrix elements. We quote, for instance.

**Example 1.** Let $(P_n)_{n\geq 0}$ be the monic $d$-OPS satisfying the $(d+1)$-order recurrence relation,

$$P_{n+1}(x) = (x+\beta_0-n)P_n(x) + \sum_{i=1}^{d}\beta_i n(n-1)\cdots(n-i+1)P_{n-i}(x), \tag{1.22}$$

where the coefficients $\beta_i$ are complex with $\beta_d \neq 0$, and with initial conditions $P_0(x)=1, \ P_n(x)=0$, $n<0$.

It has been shown in [4] that:

- $(P_n)_{n\geq 0}$ is the unique $d$-OPS satisfying the following *difference Appell property:*

$$\Delta P_n(x) := P_n(x+1) - P_n(x) = nP_{n-1}(x). \tag{1.23}$$

- $(P_n)_{n\geq 0}$ is a solution of the $(d+1)$-order difference operator equation:

$$\left(\sum_{i=2}^{d+1}\beta_{i-1}\Delta^i + x + \beta_0 - (n-1)\Delta - n\right)P_n(x) = 0. \tag{1.24}$$

Note that the PS $(P_n)_{n\geq 0}$ was the first example introduced as an explicit realization of a Lie algebra via a $d$-OPS. More precisely, it was shown in [24] that $P_n(k) = \dfrac{\alpha^n\sqrt{n!}}{\psi_{0,k}}\psi_{n,k}$, where $\psi_{n,k} := \langle k|S|n\rangle$ are the matrix elements of the operator

$$S = e^{\alpha a_+}e^{Q(a_-)},\ \ \alpha \in \mathbb{C}\backslash\{0\},\ \deg(Q) = d. \tag{1.25}$$

Here, $a_-$ and $a_+$ denote generators of the Heisenberg-Weyl (H-W) algebra which is characterized by the structure function $\phi(x) = x$.

• When $\alpha = 1$ and $Q(z) = \mu z^d$, the corresponding polynomial set reduces to $P_n(k) = C_n(k; \mu; d)$ which was introduced in [4] and is generated by

$$\sum_{n=0}^{\infty} C_n(k;\mu;d)\frac{z^n}{n!} = \big(1+z\big)^k e^{\mu z^d},\ \mu \neq 0. \tag{1.26}$$

For $d = 1$, these polynomials reduce to the classical Charlier polynomials $C_n(k;\mu)$ .

**Example 2.** The $q$-deformed algebra $su_q(1,1)$, generated by the elements $1, J_-,\ J_+, J_0$ is associated with the structure function $\phi$ given by

$$\phi(x) = \frac{q^{-(x+\frac{\beta-3}{2})}}{(1-q)^2}(1-q^x)(1-q^{\beta+x-1}),\ \ 0<q<1,\ \beta>0.$$

This algebra was considered by the first author, who also introduced the operator (see [26]),

$$S = E_q(J_+)\prod_{i=1}^{r} e_q(a_i J_-),$$

where $r$ is a positive integer and $a_1, a_2, \ldots, a_r$ are nonzero complex parameters.

It has been shown, after an appropriate change of parameters, that the PS $M_n(q^{-k}; q^{\beta-1}, c, d; q)$ generated by

$$\sum_{n=0}^{\infty} M_n(q^{-k}; q^{\beta-1}; c; d; q)\frac{z^n}{(q;q)_n} = \frac{1}{(z^r;q^r)_\infty}\,{}_1\phi_1\left(\begin{matrix} q^{-k} \\ q^{\beta}\end{matrix}\middle| q; \frac{-qz}{c}\right),$$

arises as matrix elements of the operator $S$.

Moreover, this PS is $d$-orthogonal, with $d = 2r-1$, and in the particular case $d = 1$ (that is $r = 1$), it reduces to the classical $q$-Meixner polynomials $M_n(x;\beta;c;q)$ (see [19]).

## 2. Characterization problem and main result

One of the central problems connected with a given DOA, as discussed by the authors in [24] is

*"Under which choice of the operator $S$ we obtain matrix elements $\psi_{n,k} := \langle k|S|n\rangle$ expressible in terms of d-OPS".*

In the present work, we are interested in the characterization of linear operators of the form

$$S = T(a_+)R(a_-),$$

where $a_+$ and $a_-$ denote the creation and annihilation operators of a DOA and $T$ and $R$ are suitable analytic functions. The main problem addressed in this paper is to determine under which conditions the matrix elements of $S$ can be expressed in terms of $d$-orthogonal polynomial sets on discrete variables.

Our approach provides a unified framework that encompasses several known cases as particular examples. In particular, we recover and extend results associated with the H-W algebra and the $su(1,1)$ algebra, as well as their $q$-deformed counterparts. The analysis leads to explicit realizations involving $d$-orthogonal Meixner and Charlier polynomials and their $q$-analogues, together with their generating functions and lowering operators.

*Note that a family of polynomials $(P_n)_{n\geq 0}$ is a polynomial set in the variable $x_k$ with each polynomial of degree $n$ if and only if the transformed family obtained by multiplying each polynomial by a nonzero real factor and applying a real affine change of the variable $x_k$ is also a polynomial set in the same variable.*

### 2.1. *Main result*

The following fundamental lemma is the main result of the work.

**Lemma 2.1.** *Let $T$ and $R$ be analytic functions in a neighborhood of zero with $T(0) = R(0) = 1$, and let $(x_k)_{k\geq 0}$ be a sequence of pairwise distinct real numbers, and let $\psi_{n,k} := \langle k|S|n\rangle$ be the matrix elements associated to the operator $S = T(a_+)R(a_-)$. If $T$ satisfies (1.8), then we have equivalence between the following statements.*

*(1) $\psi_{n,k} = \psi_{0,k}P_n\big(x_k\big)$, where $P_n(x_k)$ is a PS in the argument $x_k$.*

*(2) One of the following assertions holds.*

*(i)* $x_k = k,\quad T(z) = e^{\alpha z}$,

*(ii)* $x_k = q^k,\quad T(z) = e_q(\alpha z)$,

*where $\alpha$ is a non-zero complex number, $q$ a real number such that $|q| \neq 0, 1$.*

**Remark 2.2.** *Replacing $q$ by $q^{-1}$, then according to (1.20), assertion (ii) is equivalent to*

$$x_k = q^{-k},\quad T(z) = E_q(\alpha z),\quad \alpha \neq 0. \tag{2.1}$$

**Proof of Lemma 2.1.** <u>Step 1.</u> Let $\varphi_{n,k} = \langle k|S_0|n\rangle$ be the matrix elements of the operator $S_0 = T(a_+)$. Assume that

$$\varphi_{n,k} = \varphi_{0,k}Q_n(x_k)\ , \tag{2.2}$$

where $Q_n(x_k)$ is a polynomial of degree $n$ in the argument $x_k$.

Writing $T$ in the form $T(z) = \sum_{m=0}^{+\infty} \alpha_m z^m$, with $\alpha_0 = 1$. We thus get according to (1.8) and (1.9)

$$\varphi_{n,k} = \sum_{m=0}^{+\infty} \alpha_m \langle k|a_+^m|n\rangle = \sum_{m=0}^{+\infty} \alpha_m \sqrt{\frac{\phi(m+n)!}{\phi(n)!}}\langle k|m+n\rangle.$$

Taking into account the fact that $\big(|n\rangle\big)_{n\geq 0}$ is orthonormal, then we readily obtain

$$\varphi_{n,k}=0,\ \ 0\leq k\leq n-1\ \text{ and }\ \varphi_{n,k}=\alpha_{k-n}\sqrt{\frac{\phi(k)!}{\phi(n)!}},\ \ k\geq n. \tag{2.3}$$

Now we will show that $\alpha_k\neq 0$ and $\varphi_{0,k}\neq 0$, $k\geq 0$.

Taking $n=1$ in (2.2) and (2.3), then $n=0$ in (2.3) we get

$$\varphi_{1,k}=Q_1(x_k)\varphi_{0,k}=\alpha_{k-1}\sqrt{\frac{\phi(k)!}{\phi(1)}},\ \text{ and }\ \varphi_{0,k}=\alpha_k\sqrt{\phi(k)!}\,, \tag{2.4}$$

which leads to $\alpha_{k-1}=\alpha_k\sqrt{\phi(1)}Q_1(x_k)$ for $k\geq 1$. Since $\alpha_0=1$, thus $\alpha_k\neq 0$ and also $\varphi_{0,k}\neq 0$.

Consequently, (2.2) and (2.3) are equivalent to

$$Q_n(x_k)=0,\ \ 0\leq k\leq n-1\ \text{ and }\ Q_n(x_k)=\frac{\alpha_{k-n}}{\alpha_k\sqrt{\phi(n)!}},\ k\geq n. \tag{2.5}$$

According to (2.5), $x_0,x_1,\ldots,x_{n-1}$ are zeros of $Q_n$. Since $\deg(Q_n)=n$, and $x_0,\ldots,x_{n-1}$ are pairwise distinct, then there exist non-zero complex numbers $r_n$ such that

$$Q_n(x)=r_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}). \tag{2.6}$$

By virtue of (2.5) we successively obtain for each $k\geq n$,

$$\begin{cases}\dfrac{\alpha_{k-n}}{\alpha_k}=\sqrt{\phi(n)!}Q_n(x_k)\\[2ex]\dfrac{\alpha_{k-n}}{\alpha_{k+1}}=\sqrt{\phi(n+1)!}Q_{n+1}(x_{k+1}).\end{cases} \tag{2.7}$$

Therefore, we get from (2.7), after some easy computations,

$$\frac{\alpha_k}{\alpha_{k+1}}=\sqrt{\phi(n+1)}\frac{Q_{n+1}(x_{k+1})}{Q_n(x_k)},\ \ k\geq n. \tag{2.8}$$

Since $\dfrac{\alpha_k}{\alpha_{k+1}}$ is independent of $n$, then taking $n=1$ and $n=2$ in (2.6),(2.8), gives

$$\sqrt{\frac{\phi(2)}{\phi(3)}}=\frac{r_1r_3}{r_2^2}q,\ \text{ where }\ q=\frac{x_{k+1}-x_2}{x_k-x_1}\ \text{ is real.} \tag{2.9}$$

*Case 1.* $q=1$. By virtue of (2.9) there exist two real numbers $c_1,\ c_2$ such that $x_k=c_1k+c_2$ with $c_1\neq 0$. Hence, without loss of generality, it can be assumed that $c_1=1,\ c_2=0$. Taking $n=1$ in (2.7), we thus get

$$x_k=k,\quad \frac{\alpha_{k-1}}{\alpha_k}=\frac{1}{\alpha}(x_k-x_0)=\frac{1}{\alpha}k,\quad \text{where}\quad \alpha=\frac{1}{\sqrt{\phi(1)}r_1},$$

which leads to $\alpha_k=\dfrac{\alpha^k}{k!}$ and $T(z)=e^{\alpha z}$ (because $T(0)=1$).

 

*Case 2.* $q \neq 1$. We obtain from (2.9), $x_k = c_1 q^k + c_2$, with $c_1 \neq 0$ and $q \neq 0, -1$ because the $x_k,\ k \geq 0$ are pairwise distinct.

Consequently, we get successively according to (2.7) after taking $c_1 = 1,\ c_2 = 0$,

$$x_k = q^k, \quad \frac{\alpha_{k-1}}{\alpha_k} = \frac{1}{\alpha}(x_0 - x_k) = \frac{1}{\alpha}(1 - q^k), \qquad \alpha_k = \frac{\alpha^k}{(q;q)_k}, \qquad T(z) = e_q\big(\alpha z\big), \tag{2.10}$$

where $\alpha = \dfrac{-1}{\sqrt{\phi(1)} r_1}$.

<u>Step 2</u>. Since $S = S_0 R(a_-)$, then by virtue of (1.13) we obtain

$$\langle k|S_0 R(a_-)|z\rangle = R(z)\langle k|S_0|z\rangle, \tag{2.11}$$

where

$$\langle k|S_0|z\rangle = \sum_{n=0}^{+\infty} \varphi_{n,k} \frac{z^n}{\sqrt{\phi(n)!}}. \tag{2.12}$$

Writing $R$ in the form $R(z) = \sum_{n=0}^{\infty} h_n z^n$, with $h_0 = 1$, we get from (1.14), (2.11) and (2.12) after identifying the coefficients of $z^n$,

$$\psi_{0,k} = \varphi_{0,k} \neq 0, \;\; \psi_{n,k} = \frac{1}{\sqrt{\phi(n)!}} \varphi_{n,k} + \sum_{i=1}^{n} \frac{h_i \varphi_{n-i,k}}{\sqrt{\phi(n-i)!}}, n \geq 0. \tag{2.13}$$

Dividing both sides of (2.13) by $\psi_{0,k} = \varphi_{0,k}$, we get

$$\frac{\psi_{n,k}}{\psi_{0,k}} = \frac{1}{\varphi_{0,k}\sqrt{\phi(n)!}} \varphi_{n,k} + \sum_{i=1}^{n} \frac{h_i \varphi_{n-i,k}}{\varphi_{0,k}\sqrt{\phi(n-i)!}},$$

or equivalently,

$$P_n(x_k) = \frac{1}{\sqrt{\phi(n)!}} Q_n(x_k) + \sum_{i=1}^{n} \frac{h_i}{h_0} \frac{1}{\sqrt{\phi(n-i)!}} Q_{n-i}(x_k). \tag{2.14}$$

Hence $P_n(x_k)$ is a polynomial of degree $n$ in the argument $x_k$ if and only if $Q_n(x_k)$ is a polynomial of degree $n$ in the argument $x_k$. Then, according to step 1, $T$ is one of the functions $e^{\alpha z},\ e_q(\alpha z)$ with $\alpha \neq 0$.

Conversely, assume at first that $S = S_0 R(a_-)$, where $S_0 = e^{\alpha a_+}$, $\alpha \neq 0$. According to (1.8) and (1.9) we have

$$\varphi_{n,k} = \langle k|S_0|n\rangle = \sum_{m=0}^{+\infty} \frac{\alpha^m}{m!} \langle k|a_+^m|n\rangle = \sum_{m=0}^{+\infty} \frac{\alpha^m}{m!} \sqrt{\frac{\phi(m+n)!}{\phi(n)!}} \langle k|n+m\rangle.$$

Hence

$$\varphi_{n,k} = 0,\ k \leq n-1 \text{ and } \varphi_{n,k} = \frac{\alpha^{k-n}}{(k-n)!} \sqrt{\frac{\phi(k)!}{\phi(n)!}}, \quad k \geq n.$$

So we get

$$\frac{\varphi_{n,k}}{\varphi_{0,k}} = Q_n(k) = \frac{1}{\alpha^n\sqrt{\phi(n)!}}k(k-1)\cdots(k-n+1). \tag{2.15}$$

On account of (2.14) and (2.15), $P_n(k)$ is a polynomial of degree $n$ in the argument $k$.

Similarly, if $S_0(z) = e_q(\alpha z)$, $\alpha \neq 0$, we have

$$\varphi_{n,k} = \langle k|S|n\rangle = \sum_{m=0}^{+\infty} \frac{\alpha^m}{(q;q)_m}\langle k|a_+^m|n\rangle = \sum_{m=0}^{+\infty} \frac{\alpha^m}{(q;q)_m}\sqrt{\frac{\phi(m+n)!}{\phi(n)!}}\langle k|n+m\rangle.$$

Therefore,

$$\varphi_{n,k} = 0,\ 0 \le k \le n-1, \text{ and } \varphi_{n,k} = \frac{\alpha^{k-n}}{(q;q)_{k-n}}\sqrt{\frac{\phi(k)!}{\phi(n)!}}, \quad k \ge n,$$

which leads to

$$\frac{\varphi_{n,k}}{\varphi_{0,k}} = Q_n(q^k) = \frac{1}{\alpha^n\sqrt{\phi(n)!}}(1-q^k)(1-q^{k-1})\cdots(1-q^{k-n+1}). \tag{2.16}$$

Consequently, $P_n(q^k)$ is a polynomial of degree $n$ in the argument $q^k$ according to (2.14) and (2.16).

Now, we state the following lemma which will provide a unified framework in order to express in terms of $T$ and $R$, the generating function $\langle k|S|z\rangle$ of the matrix elements $\psi_{n,k}$ associated to the operator $S$.

**Lemma 2.3.** *Under the hypothesis (1.8), the generating function $\langle k|S|z\rangle$ of the matrix elements $\psi_{n,k}$ associated to the operator $S = T(a_+)R(a_-)$ is expressed as*

$$\langle k|S|z\rangle = \sqrt{\phi(k)!}R(z)v_k(z), \tag{2.17}$$

*where $v_k(z)$ are the coefficients in the expansion of $T(a_+)e_\phi(za_+)$ in powers of $a_+$ given by*

$$T(a_+)e_\phi(za_+) = \sum_{m=0}^{\infty} v_m(z)a_+^m. \tag{2.18}$$

**Proof.** On account of (1.5), (1.11) and (1.13) we have

$$\begin{aligned}\langle k|S|z\rangle &= \langle k|T(a_+)R(a_-)|z\rangle = R(z)\langle k|T(a_+)e_\phi(za_+)|0\rangle \\ &= R(z)\sum_{m=0}^{\infty} v_m(z)\langle k|a_+^m|0\rangle = R(z)\sum_{m=0}^{\infty}\sqrt{\phi(m)!}v_m(z)\langle k|m\rangle \\ &= \sqrt{\phi(k)!}R(z)v_k(z).\end{aligned}$$

## 3. The Lie algebra su(1,1)

Let $\beta > 0$ and $\phi(x) = x(x+\beta-1)$ be the structure function associated to the Lie algebra $su(1,1)$ spanned by the elements $1, J_-, J_+, J_0$ satisfying commutation relations (see [14,18]),

$$[J_-, J_+] = 2J_0, \quad [J_0, J_-] = -J_-, \quad [J_0, J_+] = J_+. \tag{3.1}$$

 

• The Fock representation in a Hilbert space with an orthonormal basis $\big(|n\rangle\big)_{n\geq 0}$ is given by

$$J_-|n\rangle = \sqrt{n(n+\beta-1)}|n-1\rangle, \quad J_+|n\rangle = \sqrt{(n+1)(n+\beta)}|n+1\rangle, \quad J_0|n\rangle = \big(n+\frac{\beta}{2}\big)|n\rangle. \quad (3.2)$$

In this representation, we have the following conjugation relations

$$J_0^* = J_0, \quad J_-^* = J_+, \; J_+^* = J_-. \quad (3.3)$$

• The action by powers of the operators $J_-$ and $J_+$ on the state $|n\rangle$ are given as

$$J_-^m|n\rangle = \gamma_{n,m}|n-m\rangle, \; n\geq m \; \text{ and } \; J_+^m|n\rangle = \gamma_{n+m,n}|n+m\rangle, \quad (3.4)$$

where $\gamma_{n,m} = \sqrt{\dfrac{n!(\beta)_n}{(n-m)!(\beta)_{n-m}}}$.

• For every $z \in \mathbb{C}$, the function $e_\phi$ and the corresponding coherent state $|z\rangle$ are expressed as

$$e_\phi(z) = {}_0F_1\left(\begin{matrix} - \\ \beta \end{matrix}\middle| z\right), \quad \text{and} \quad |z\rangle = {}_0F_1\left(\begin{matrix} - \\ \beta \end{matrix}\middle| zJ_+\right)|0\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!(\beta)_n}}|n\rangle. \quad (3.5)$$

• It is worth to mention that the absolutely convergent of the series

$\sum_{m\geq 0}\sqrt{(m+n)!(\beta)_{m+n}}\dfrac{\alpha^m}{m!}$ is satisfied when $|\alpha| < 1$.

Hence, on account of Lemma 2.1, we will be interested in the operator

$$S = e^{\alpha J_+}R(J_-), \quad \text{with} \quad 0 < |\alpha| < 1, \; R(0) = 1, \quad (3.6)$$

defined on the subspace $V = \text{span}(|n\rangle)_{n\geq 0}$.

• Next, we require the following commutation relations involving an analytic function $R$ defined in a neighborhood of zero. In fact, according to (3.1), we have $J_-J_+ = J_+J_- + 2J_0$. By induction, we get for every integer $m \geq 1$,

$$J_-^m J_+ = J_+J_-^m + 2mJ_-^{m-1}J_0 - m(m-1)J_-^{m-1}.$$

Therefore, the relation

$$R(J_-)J_+ = J_+R(J_-) + 2R'(J_-)J_0 - J_-R''(J_-) \quad (3.7)$$

holds in the subspace $V = \text{span}(|n\rangle)_{n\geq 0}$.

Proceeding analogously, we show that

$$J_0R(J_-) = R(J_-)J_0 - J_-R'(J_-). \quad (3.8)$$

• Other commutation relations can be proven by the Baker-Campbell-Hausdorff (BCH) formula (see [21]) stated as

$$e^{\widehat{A}}\widehat{B}e^{-\widehat{A}} = \widehat{B} + [\widehat{A},\widehat{B}] + \frac{1}{2!}[\widehat{A},[\widehat{A},\widehat{B}]] + \frac{1}{3!}[\widehat{A},[\widehat{A},[\widehat{A},\widehat{B}]]] + \cdots \quad (3.9)$$

valid for generic operators $\widehat{A}, \widehat{B}, \widehat{C}$ of $su(1,1)$.

For the particular case, $\widehat{A} = \alpha J_+$ and $\widehat{B} = J_-$, we have $[\widehat{A}, [\widehat{A}, \widehat{B}]] = 0$. Therefore, we get according to (3.1) and (3.9),

$$e^{\alpha J_+} J_- = (J_- - 2\alpha J_0 + \alpha^2 J_+) e^{\alpha J_+}. \tag{3.10}$$

By the same manner,

$$J_0 e^{\alpha J_+} = e^{\alpha J_+}(\alpha J_+ + J_0). \tag{3.11}$$

### 3.1. *Generating function and lowering operator*

Firstly, let us express a generating function of the polynomials $P_n(k) = \dfrac{\psi_{n,k}}{\psi_{0,k}}$ in terms of an hypergeometric function. Indeed, with the help of the Brenke generating function of Laguerre PS (see [19]),

$$e^{\alpha t}\, {}_0F_1\left(\begin{matrix} - \\ \beta \end{matrix}\middle| zt\right) = \sum_{m=0}^{\infty} {}_1F_1\left(\begin{matrix} -m \\ \beta \end{matrix}\middle| \frac{-z}{\alpha}\right) \frac{\alpha^m t^m}{m!},$$

we get according to (3.5)

$$T(J_+) e_\phi(zJ_+) = e^{\alpha J_+}\, {}_0F_1\left(\begin{matrix} - \\ \beta \end{matrix}\middle| zJ_+\right) = \sum_{m=0}^{\infty} \frac{\alpha^m}{m!}\, {}_1F_1\left(\begin{matrix} -m \\ \beta \end{matrix}\middle| \frac{-z}{\alpha}\right) J_+^m.$$

By virtue of (2.17) and (2.18), we obtain successively

$$v_k(z) = \frac{\alpha^k}{k!}\, {}_1F_1\left(\begin{matrix} -k \\ \beta \end{matrix}\middle| \frac{-z}{\alpha}\right), \quad \langle k|S|z\rangle = \alpha^k R(z) \sqrt{\frac{(\beta)_k}{k!}}\, {}_1F_1\left(\begin{matrix} -k \\ \beta \end{matrix}\middle| \frac{-z}{\alpha}\right).$$

It follows from (2.11) that the matrix elements $\psi_{n,k}$ are generated by

$$\sum_{n=0}^{\infty} \frac{\psi_{n,k}}{\sqrt{n!(\beta)_n}} z^n = R(z)\alpha^k \sqrt{\frac{(\beta)_k}{k!}}\, {}_1F_1\left(\begin{matrix} -k \\ \beta \end{matrix}\middle| \frac{-z}{\alpha}\right). \tag{3.12}$$

Since $R(0) = 1$, then according to (3.12) we get, $\psi_{0,k} = \alpha^k \sqrt{\dfrac{(\beta)_k}{k!}}$. Dividing by $\psi_{0,k}$, leads to the following.

**Proposition 3.1.** *The polynomials* $P_n(k) = \dfrac{\psi_{n,k}}{\psi_{0,k}}$ *are generated by*

$$\sum_{n=0}^{\infty} \frac{P_n(k)}{\sqrt{n!(\beta)_n}} z^n = R(z)\, {}_1F_1\left(\begin{matrix} -k \\ \beta \end{matrix}\middle| \frac{-z}{\alpha}\right). \tag{3.13}$$

In order to calculate the lowering operator of the PS $\left(P_n\right)_{n\geq 0}$, we consider the matrix elements of $SJ_-$. We have

$$\langle k|SJ_-|n\rangle = \sqrt{n(n+\beta-1)}\,\psi_{n-1,k}. \tag{3.14}$$

By virtue of (3.10), the operator $SJ_-$ is expressed as

$$SJ_- = \left(e^{\alpha J_+}J_-\right)R(J_-) = \left(J_- - 2\alpha J_0 + \alpha^2 J_+\right)e^{\alpha J_+}R(J_-) = J_-S - 2\alpha J_0 S + \alpha^2 J_+ S. \quad (3.15)$$

Therefore, we get from (3.2) and (3.3),

$$\begin{aligned}\langle k|SJ_-|n\rangle =&\langle k|J_-S|n\rangle - 2\alpha\langle k|J_0S|n\rangle + \alpha^2)\langle k|J_+S|n\rangle.\\ &= \sqrt{(k+1)(k+\beta)}\psi_{n,k+1} - 2\alpha\big(k+\frac{\beta}{2}\big)\psi_{n,k} + \alpha^2\sqrt{k(k+\beta-1)}\psi_{n,k-1}. \quad (3.16)\end{aligned}$$

Comparing (3.14) and (3.16), then dividing by $\psi_{0,k}$ we get

$$\begin{aligned}\sqrt{n(n+\beta-1)}P_{n-1}(k) &= \alpha(k+\beta)P_n(k+1) - \alpha(2k+\beta)P_n(k) + \alpha k P_n(k-1)\\ &= \alpha k\big(P_n(k+1) - 2P_n(k) + P_n(k-1)\big) + \alpha\beta\big(P_n(k+1) - P_n(k)\big)\\ &= \alpha\big(k\tau\Delta^2 + \beta\Delta\big)P_n(k),\end{aligned}$$

where $\tau f(x) = f(x-1)$ and $\Delta$ is the difference operator defined in (1.23).

Let $\hat{P}_n(k) = \dfrac{\alpha^n}{\sqrt{n!(\beta)_n}}P_n(k)$, then we get

**Proposition 3.2.** *The operator $\sigma_\beta$ defined by*

$$\sigma_\beta = x\tau\Delta^2 + \beta\Delta$$

*satisfies the following equality*

$$\sigma_\beta \hat{P}_n(k) = \hat{P}_{n-1}(k).$$

That is $\sigma_\beta$ is the lowering operator of $(\hat{P}_n)_{n\geq 0}$.

### 3.2. $d$-orthogonal Meixner polynomials

In this subsection, we shall determine the analytic function $R$ for which $\big(P_n\big)_{n\geq 0}$ generated by (3.13) is $d$-OPS. Indeed, we have

**Proposition 3.3.** *The PS defined by $P_n(k) = \dfrac{\psi_{n,k}}{\psi_{0,k}}$ generated by (3.13) is $d$-orthogonal if and only if*

$$S = e^{\alpha J_+}e^{Q(J_-)},$$

*where $d = 2r-1$, $Q$ is a polynomial of degree $r$.*

*In this case, $\big(P_n\big)_{n\geq 0}$ is generated by*

$$\sum_{n=0}^{\infty} \frac{P_n(k)}{\sqrt{n!(\beta)_n}} z^n = e^{Q(z)}\,{}_1F_1\left(\begin{matrix}-k\\ \beta\end{matrix}\middle|\frac{-z}{\alpha}\right). \quad (3.17)$$

**Proof.** On the one hand we have according to (3.2) and (3.3),

$$\langle k|J_0S|n\rangle = \langle J_0k|S|n\rangle = \big(k+\frac{\beta}{2}\big)\psi_{n,k}. \quad (3.18)$$

On the other hand, in view of (3.7),(3.8) and (3.11), the operator $J_0S$ is transformed in the following manner

$$\begin{aligned}
J_0S &= \Big(J_0e^{\alpha J_+}\Big)R(J_-) = \alpha e^{\alpha J_+}\Big(J_+R(J_-)\Big) + e^{\alpha J_+}\Big(J_0R(J_-)\Big)\\
&= \alpha e^{\alpha J_+}\Big(R(J_-)J_+ - 2R'(J_-)J_0 + J_-R''(J_-)\Big) + e^{\alpha J_+}\Big(R(J_-)J_0 - J_-R'(J_-)\Big)\\
&= \alpha e^{\alpha J_+}R(J_-)J_+ - 2\alpha e^{\alpha J_+}R'(J_-)J_0 + \alpha e^{\alpha J_+}J_-R''(J_-) + e^{\alpha J_+}R(J_-)J_0 - e^{\alpha J_+}J_-R'(J_-)\\
&= \alpha SJ_+ - 2\alpha S\frac{R'}{R}(J_-)J_0 + \alpha SJ_-\frac{R''}{R}(J_-) + SJ_0 - SJ_-\frac{R'}{R}(J_-)\\
&= \alpha SJ_+ + S\Big(1-2\alpha\frac{R'}{R}(J_-)\Big)J_0 + SJ_-\Big(\alpha\frac{R''}{R}(J_-) - \frac{R'}{R}(J_-)\Big). \qquad (3.19)
\end{aligned}$$

Consider the power series expansions of the following analytic functions.

$$1-2\alpha\frac{R'(z)}{R(z)} = \sum_{i=0}^{\infty}\alpha_i z^i, \quad z\Big(\alpha\frac{R''(z)}{R(z)} - \frac{R'(z)}{R(z)}\Big) = \sum_{i=1}^{\infty}\beta_i z^i, \qquad (3.20)$$

where $\alpha_i$ and $\beta_i$ are complex numbers.

Combining, (3.18), (3.19) and (3.20), then, we obtain with the help of (3.2) and (3.4),

$$\begin{aligned}
\big(k+\frac{\beta}{2}\big)\psi_{n,k} &= \alpha\langle k|SJ_+|n\rangle + \langle k|S\sum_{i=0}^{\infty}\alpha_i J_-^i J_0|n\rangle + \langle k|S\sum_{i=1}^{\infty}\beta_i J_-^i|n\rangle\\
&= \alpha\langle k|SJ_+|n\rangle + \sum_{i=0}^{\infty}\alpha_i\langle k|SJ_-^iJ_0|n\rangle + \sum_{i=1}^{\infty}\beta_i\langle k|SJ_-^i|n\rangle\\
&= \alpha\sqrt{(n+1)(n+\beta)}\psi_{n+1,k} + \sum_{i=0}^{n}\gamma_{n,i}\alpha_i\big(n+\beta/2\big)\psi_{n-i,k} + \sum_{i=1}^{n}\gamma_{n,i}\beta_i\psi_{n-i,k}.
\end{aligned}$$

Hence, we obtain after dividing by $\psi_{0,k}$

$$\big(k-\alpha_0 n+\beta(1-\alpha_0)/2\big)P_n(k) = \alpha\sqrt{(n+1)(n+\beta)}P_{n+1}(k) + \sum_{i=1}^{n}\gamma_{ni}\Big(\alpha_i\big(n+\beta/2\big)+\beta_i\Big)P_{n-i}(k). \qquad (3.21)$$

Therefore, according to (1.1) the polynomials $P_n(k)$ are $d$-orthogonal if and only if for every non-negative integers $n, i$, such that $n \geq i \geq d+1$,

$$\alpha_d\big(n+\beta/2\big)+\beta_d \neq 0 \quad \text{and} \quad \alpha_i\big(n+\beta/2\big)+\beta_i = 0. \qquad (3.22)$$

From (3.22) we easily get $\alpha_i = \beta_i = 0$, for every $i \geq d+1$. Then, by virtue of (3.20), $\frac{R'}{R}$ is a polynomial of degree $r-1$, with $r \leq d+1$. Hence, we obtain from (3.20),

$$\alpha_{r-1} \neq 0, \;\; \alpha_i = 0, \; i \geq r, \quad R(z) = e^{Q(z)} \;\text{ with }\; \deg(Q) = r. \qquad (3.23)$$

It follows that $\deg\left(z\Big(\alpha\frac{R''(z)}{R(z)} - \frac{R'(z)}{R(z)}\Big)\right) = \deg\Big(z\big(\alpha\big(Q'^2(z)+Q''(z)\big)-Q'(z)\big)\Big) = 2r-1$, which implies according to (3.20),

$$\beta_{2r-1} \neq 0, \beta_i = 0, \; i \geq 2r. \qquad (3.24)$$

Since $2r-1 \geq r$, we get by means of (3.23),

$$\alpha_i = 0, \ i \geq 2r-1, \tag{3.25}$$

In view of the above results, the recurrence relation (3.21) becomes

$$\big(k-\alpha_0 n+\beta(1-\alpha_0)/2\big)P_n(k) = \alpha\sqrt{(n+1)(n+\beta)}P_{n+1}(k)+\sum_{i=1}^{2r-1}\gamma_{ni}\Big(\alpha_i\big(n+\beta/2\big)+\beta_i\big)\Big)P_{n-i}(k),$$

with, $\gamma_{n,2r-1}\Big(\alpha_{2r-1}\big(n+\beta/2\big)+\beta_{2r-1}\big) = \gamma_{n,2r-1}\beta_{2r-1} \neq 0$, which leads to $d = 2r-1$ and finishes the proof.

In connection with Proposition 3.3, when $Q(z) = z^r$ and $\alpha = \dfrac{c}{c-1}$, we recognize the $d$-OPS $M_n(.;\beta;c;d)$ introduced in [7] and generated by

$$\sum_{n=0}^{\infty} M_n(k;\beta;c;d)\frac{z^n}{n!} = e^{z^r}\ {}_1F_1\left(\begin{matrix}-k\\ \beta\end{matrix}\middle|\frac{1-c}{c}z\right).$$

When $d = r = 1$, the polynomials $M_n(k;\beta;c;d)$ are reduced to the Meixner polynomials of the first kind $M_n(.;\beta;c)$.

The polynomials $M_n(k;\beta;c;d)$ are called $d$-OPS of Meixner type.

It has been shown in [7] that the $d$-orthogonality relations are expressed in terms of hypergeometric functions reduced to the orthogonality relations of $M_n(k;\beta;c;d)$.

## 4. A model of $q$-harmonic oscillator algebra $\mathcal{A}_q$

This section is devoted to the $q$-harmonic oscillator algebra $\mathcal{A}_q$ characterized by its structure function $\phi$ given by $\phi(x) = [x]_q$, where $[x]_q := \dfrac{1-q^x}{1-q}$, $0 < q < 1$ (see [10]).

$\mathcal{A}_q$ is spanned by $1, A_-, A_+, A_0$ satisfying the commutation relations

$$[A_-, A_+] = q^{A_0}, \quad [A_0, A_-] = -A_-, \quad [A_0, A_+] = A_+.$$

• The Fock representation of the $q$-harmonic oscillator algebra $\mathcal{A}_q$ in a Hilbert space with an orthonormal basis $\big(|n\rangle\big)_{n\geq 0}$ is given by

$$A_-|n\rangle = \sqrt{[n]_q}|n-1\rangle, \quad A_+|n\rangle = \sqrt{[n+1]_q}|n+1\rangle, \quad A_0|n\rangle = n|n\rangle.$$

• The action by powers of $A_+$ on the vacuum vector $|0\rangle$ is given by

$$A_+^n|0\rangle = \sqrt{[n]_q!}|n\rangle.$$

• The corresponding $q$-coherent state is defined as

$$|z\rangle := e_\phi(zA_+)|0\rangle = e_q((1-q)zA_+)|0\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]_q!}}|n\rangle.$$

 

• Note that for $n\geq 0$, $\sum_{m\geq 0}\sqrt{[m+n]_q!}\frac{\alpha^m}{(q;q)_m}$ is absolutely convergent for every non-zero complex number $\alpha$, which ensures the existence of $E_q(\alpha A_+)|n\rangle$.

Then, hypothesis (1.8) is satisfied (in this case $T(A_+)=E_q(\alpha A_+)$) and by virtue of Lemma 2.1, we will be interested in the characterization of the operator $S$ defined by

$$S=E_q(\alpha A_+)R(A_-),\quad \text{with}\quad R(0)=1,\quad \alpha\neq 0,$$

for which the polynomials $P_n(q^{-k})=\frac{\psi_{n,k}}{\psi_{0,k}}$ are $d$-orthogonal in the argument $q^{-k}$.

### 4.1. $D_q$-Appell d-orthogonal polynomial sets

Let us begin with a generating function of $P_n(q^{-k})$. By the use of the $q$-binomial formula (1.21), we readily get

$$T(A_+)e_\phi(zA_+)=E_q(\alpha A_+)e_q\big((1-q)zA_+\big)=\sum_{m=0}^{\infty}\frac{z^m\big(-\alpha/(1-q)z;q\big)_m}{[m]_q!}A_+^m.$$

Since the expansion coefficients are given by $v_k(z)=\frac{z^k\big(-\alpha/(1-q)z;q\big)_k}{[k]_q!}$, then by virtue of Lemma 2.3, the generating function $\langle k|S|z\rangle$ of the matrix elements $\psi_{n,k}$ is given by

$$\langle k|S|z\rangle=R(z)\frac{z^k\big(-\alpha/(1-q)z;q\big)_k}{\sqrt{[k]_q!}}.\tag{4.1}$$

It is easy to see that

$$\begin{aligned}z^k\big(-\alpha/((1-q)z);q\big)_k&=\prod_{i=0}^{k-1}\left(z+\frac{\alpha}{1-q}q^i\right)=\alpha^k(1-q)^{-k}q^{\binom{k}{2}}\prod_{i=0}^{k-1}\left(1+\frac{q(1-q)q^{-k}z}{\alpha}q^i\right)\\&=\alpha^k(1-q)^{-k}q^{\binom{k}{2}}\left(\frac{q(1-q)}{\alpha}q^{-k}z;q\right)_k=\alpha^k(1-q)^{-k}q^{\binom{k}{2}}\frac{\left(\frac{q(1-q)}{\alpha}q^{-k}z;q\right)_\infty}{\left(\frac{q(1-q)}{\alpha}z;q\right)_\infty}\\&=\alpha^k(1-q)^{-k}q^{\binom{k}{2}}e_q\left(\frac{q(1-q)}{\alpha}z\right)E_q\left(\frac{q(q-1)}{\alpha}q^{-k}z\right).\end{aligned}$$

Since the transformation $(z\rightarrow\alpha z)$ will not change the nature of the polynomials $P_n(q^{-k})$, then without loss of generality, we take $\alpha=q(1-q)$. Thus, dividing by $\psi_{0,k}=\frac{\alpha^k(1-q)^{-k}q^{k(k-1)/2}}{\sqrt{[k]_q!}}$, obtained when $z=0$, we get according to (1.14) and (4.1) the following

**Proposition 4.1.** *The polynomials* $P_n(q^{-k}):=\frac{\psi_{n,k}}{\psi_{0,k}}$ *associated to the operator* $S=E_q\big(q(1-q)A_+\big)R(A_-)$ *are generated by*

$$\sum_{n=0}^{\infty}P_n(q^{-k})\frac{z^n}{\sqrt{[n]_q!}}=R(z)e_q(z)E_q(-q^{-k}z).\tag{4.2}$$

**Remark 4.2.** *Let us recall that in [25], we have been interested in the following Problem:*

*Find all PSs which are at the same time $d$-orthogonal and $D_q$-Appell, where $D_q$ is the $q$-difference operator defined by $D_q f(x) = \dfrac{f(qx) - f(x)}{(q-1)x}$.*

The first author demonstrated that, upon replacing $q$ by $q^{-1}$ [25,Theorem 1.1] the following:

A PS $\big(Q_n\big)_{n\geq 0}$ is $D_{1/q}$-Appell if and only if it is generated by

$$\sum_{n=0}^{\infty} Q_n(q^{-k}) \frac{z^n}{(q;q)_n} = H(z) E_q(-q^{-k} z), \tag{4.3}$$

where $H$ is an analytic function, with $H(0) \neq 0$.

Moreover $\big(Q_n\big)_{n\geq 0}$ is also $d$-OPS if and only if

$$H(z) = \prod_{i=0}^{d} e_q(c_i z), \ \ c_i \in \mathbb{C}\backslash\{0\}, \ 0 \leq i \leq d. \tag{4.4}$$

A comparison of equations (4.2)-(4.4) shows that the polynomials are $d$-orthogonal if and only if

$$R(z) = \prod_{i=1}^{d} e_q(c_i z).$$

**Proposition 4.3.** *The polynomials $P_n(q^{-k}) = \dfrac{\psi_{n,k}}{\psi_{0,k}}$ generated by (4.2) are $d$-orthogonal if and only if*

$$S = E_q\big(q(1-q)A_+\big) \prod_{i=1}^{d} e_q(c_i A_-), \quad c_i \in \mathbb{C}\backslash\{0\}, \ 1 \leq i \leq d. \tag{4.5}$$

*Additionally, the function $G(q^{-k}, z)$ defined by*

$$G(q^{-k}, z) = E_q(-q^{-k} z) e_q(z) \prod_{i=1}^{d} e_q(c_i z), \tag{4.6}$$

*is a generating function of the polynomials $P_n(q^{-k})$.*

### 4.2. $d$-orthogonal $q$-Charlier polynomials

Let $\mu$ be a non-zero complex number , and $c_j = (\mu/q)^{1/d} \omega^j, \ j = 1, ..., d$, with $\omega = e^{2i\pi/d}$.

Since $1 - (\mu/q) z^d = \prod_{j=1}^{d} \big(1 - (\mu/q)^{1/d} \omega^j z\big)$, we get the obvious identity :

$$\big((\mu/q) z^d; q^d\big)_n = \big((\mu/q)^{1/d} \omega z, (\mu/q)^{1/d} \omega^2 z, ..., (\mu/q)^{1/d} \omega^d z; q\big)_n.$$

 

Letting $n \to \infty$, we obtain

$$\prod_{j=1}^{d} e_q((\mu/q)^{1/d}\omega^j z) = e_{q^d}\big(\mu z^d/q\big).$$

It is natural by virtue of (4.6) to consider the $d$-OPS $\big(C_n(q^{-k};\mu;d;q)\big)_{n\geq 0}$ defined and generated by

$$\sum_{n=0}^{\infty} \frac{(\mu/q)^n}{(q;q)_n} C_n(q^{-k};\mu,d;q)z^n = e_q(z)e_{q^d}\big(\mu z^d/q\big)E_q(-q^{-k}z). \tag{4.7}$$

Under the above assumptions, the operator $S$ related to the $d$-OPS $C_n(q^{-k};\mu,d;q)$ is given by

$$S = E_q(q(1-q)A_+)e_{q^d}\big(\mu A_-^d/q\big). \tag{4.8}$$

When $d=1$, the polynomials $C_n(q^{-k};\mu,d;q)$ are reduced to the $q$-Charlier polynomials $C_n(q^{-k};\mu;q)$ introduced in [10] as matrix elements of the operator $S = E_q(q(1-q)A_+)e_q\big(\mu A_-/q\big)$ and $C_n(q^{-k};\mu;d;q)$ are called $d$-OPS of $q$-Charlier type.

### 4.3. *Link with the H-W algebra*

Note that when $q \uparrow 1$ the above $q$-oscillator algebra contracts to the H-W algebra.

Let $\mu' = (-1)^d \mu q(1-q^d)$, then when $q \uparrow 1$, the operator (4.8) contracts to $S = e^{a_+}e^{\mu a_-^d}$ stated in (1.25), ($\alpha = 1$ and $Q(z) = \mu z^d$). Additionally, replacing $z$ by $-z$ in (4.7), we get the following limit relations connected with the $d$-OPS $\big(C_n(.;\mu;d)\big)_{n\geq 0}$ of Charlier type.

$$\lim_{q\to 1} \frac{e_q(-z)}{e_q(-q^{-k}z)} e_{q^d}((1-q^d)\mu z^d) = (1+z)^k e^{\mu z^d},$$

$$\lim_{q\to 1} \big((-1)^d d(1-q)\big)^{-n} C_n(q^k;\mu',d,q) = C_n(k;\mu;d).$$

Hence, we meet again the generating function (1.26) of the $d$-OPS $\big(C_n(.;\mu;d)\big)_{n\geq 0}$.

## 5. Conclusion

In conclusion, this work has adressed a characterization problem related to DOAs, aiming to determine, under certain assumptions, pairwise distinct real sequences and operators of the form $S = T(a_+)R(a_-)$, where $T$ and $R$ are analytic functions, and $a_-$ and $a_+$ denote the annihilation and creation operators, respectively, such that the matrix elements $\langle k|S|n\rangle$ associated with $S$ can be expressed in terms of $d$-OPSs on the discrete variable $x_k$.

From one perspective, we have established the following necessary condition that must be satisfied by $x_k$ and $T$: $\psi_{n,k} = \psi_{0,k}P_n(x_k)$, where $P_n(x_k)$ is a polynomial set of degree $n$ on the discrete variable $x_k$ if and only if $x_k = k$ or $x_k = q^{\pm k}$ and $T(z) = e^{\alpha z}$, $e_q(\alpha z)$ or $E_q(\alpha z)$ with $\alpha \neq 0,\ R(0) \neq 0$. Furthermore, a summary table showing that, in most cases, the analytic function $R$ can also be expressed in terms of exponential or basic exponential functions whenever $P_n(x_k)$ forms a $d$-OPS.

| The DOA | The structure function $\phi(x)$ | The operator $S$ | The associated $d$-OPS |
|---|---|---|---|
| $su_q(1,1)$ | $q^{-(x+\beta-3/2)}\dfrac{(1-q^x)(1-q^{\beta+x-1})}{(1-q)^2}$ | $E_q(J_+)e_{q^r}(J_-^r)$<br>$d=2r-1$ | $M_n(q^{-k};\beta;q;d)$<br>$q$-Meixner type |
| $su(1,1)$ | $x(x+\beta-1)$ | $e^{J_+}e^{J_-^r}$<br>$d=2r-1$ | $M_n(k;\beta;q;d)$<br>Meixner type |
| $\mathcal{A}_q$ | $\dfrac{q^x-1}{q-1}$ | $E_q\big(cA_+\big)e_{q^d}\big(\mu A_-^d/q\big)$<br>$c=q(1-q),\ \mu\neq 0$ | $C_n(q^{-k};\mu;d;q)$<br>$q$-Charlier type |
| H-W algebra | $x$ | $e^{a_+}e^{\mu a_-^d}$<br>$\mu\neq 0$ | $C_n(k;\mu;d;q)$<br>Charlier type |

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