Well-Posedness of the Maxwell-Boltzmann System in a Bianchi Type III Space-Time

Système de Maxwell-Boltzmann bien posé dans un espace-temps de type Bianchi III

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ABSTRACT. This paper investigates the Maxwell-Boltzmann system in the framework of a Bianchi type III space-time. We conduct a rigorous mathematical analysis of the system, emphasizing its structural properties, functional setting, and energy estimates. By employing a combination of the Faedo-Galerkin method and the standard iteration technique, we establish the local-in-time existence and uniqueness of a solution with good regularity. Our approach carefully integrates the geometric constraints imposed by the Bianchi type III background, ensuring a well-posed formulation of the problem. These results contribute to a broader study of the relativistic kinetic theory in anisotropic cosmological models.
KEYWORDS. Relativistic Boltzmann equation, Maxwell equation, existence and uniqueness, regularity.

1. Introduction

The Boltzmann equation and Maxwell's equations are among the most powerful mathematical tools used to study the evolution with collision plasmas. The Boltzmann equation describes the statistical dynamics of particles under the influence of collisions [6, 7]. When set in a magnetized context within curved space-time geometries, the equation acquires additional complexities, intertwining physics, geometry, and mathematical analysis. On the other hand, Maxwell's equations govern the bhavior of electromagnetic fields [15]. Coupling these two sets of equations is particularly relevant in systems where relativistic effects play a crucial role. This approach not only enables precise modeling of astrophysical plasmas and ionized media but also provides deeper insight into transport processes and energy interactions at relativistic speeds.

In this work, we focus on the Maxwell-Boltzmann system in a Bianchi type III space-time. Our primary objective is to establish the well-posedness of the system using rigorous mathematical techniques. This result is of great significance both mathematically and physically. From a mathematical perspective, it provides a rigorous proof of the existence and uniqueness of a solution (with a very good regularity) to this system, representing a substantial advancement in the analysis of partial differential equations. From a physical standpoint, it validates and enhances theoretical models describing relativistic plasmas and ionized environments. For instance, in relativistic jets around black holes and plasmas in neutron stars, this coupling is crucial for modeling interactions between charged particles and electromagnetic fields.

The choice of a Bianchi type III space-time is fundamental to this study, as it introduces a specific geometric structure that accounts for relativistic effects in a non-isotropic setting. This space-time serves as a generalization of both the Bianchi type I and the Robertson-Walker space-times. Such a feature

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is essential for modeling astrophysical environments where space-time symmetries are broken, such as regions near black holes or star-forming areas. By leveraging this geometric framework, the coupled Boltzmann-Maxwell system can be analyzed more precisely, yielding valuable insights into dynamic interactions in media where the geometry of space-time plays a significant role.

Several studies have explored the Boltzmann equation (see [1, 5, 6, 9, 12, 14, 16, 21, 22, 24, 26, 27] and the references therein), as well as its coupling with the Maxwell equation in other physical systems in various relativistic settings (see [4, 23, 25, 18, 20], among others). However, such studies have not yet been conducted in the framework of Bianchi type III space-time. In addition, in previous works ([1, 18, 22]) within the Bianchi type I framework, the methods employed for *a priori* estimates have not been sufficient to rigorously establish all necessary bounds. To address this limitation, we introduce an approach based on recurrence, which enables the rigorous derivation of all *a priori* estimates.

Our work extends this body of research by systematically analyzing the coupled Boltzmann-Maxwell system in Bianchi type III space-time, offering new insights into relativistic kinetic theory and electromagnetic field interactions in anisotropic cosmological models.

We structure the paper as follows: Section 2 describes the magnetized Boltzmann equation in the Bianchi type III background. Section 3 introduces the functional spaces and establishes the necessary energy estimates. Section 4 provides a detailed proof of existence and uniqueness, leveraging the Faedo-Galerkin method and iteration techniques. Finally, Section 5 concludes with a summary of findings and perspectives for future work.

2. Description of the magnetized Boltzmann equation in the Bianchi type III background

Throughout this work, lowercase Latin indices range from 1 to 3, while uppercase Latin indices either range from 1 to M or from 1 to $M-1$.

We use Greek indices $\alpha, \beta, \gamma, \dots$ ranging from 0 to 3, and Latin indices $i, j, k, \dots$ from 1 to 3. The Einstein summation convention applies: $A^\alpha B_\alpha = \sum_\alpha A^\alpha B_\alpha$.

2.1. *Background Space-Time and Unknown Functions*

We study the collisional evolution of a system of massive and charged particles in a time-oriented Bianchi type III space-time $(\mathbb{R}^4, g)$ with coordinates (t, x, y, z) and Lorentzian metric:

$$g = -dt^2 + a^2(t)dx^2 + b^2(t)e^{-2rx}dy^2 + c^2(t)dz^2, \tag{1}$$

where $a(t), b(t), c(t) > 0$ are given smooth functions. The only nonzero Christoffel symbols of the Levi-Civita connection are given by:

$$\begin{aligned} &\Gamma^0_{11} = \dot{a}a,\ \Gamma^0_{22} = \dot{b}be^{-2\alpha x},\ \Gamma^0_{33} = \dot{c}c,\ \Gamma^1_{01} = \frac{\dot{a}}{a}, \\ &\Gamma^2_{02} = \frac{\dot{b}}{b}, \Gamma^3_{03} = \frac{\dot{c}}{c},\ \Gamma^2_{21} = -r,\ \Gamma^1_{22} = \frac{r\, b^2}{a^2}e^{-2rx}. \end{aligned} \tag{2}$$

The dot notation represents differentiation with respect to t.

In all that follows, we will consider $r = 1$ in the expression of $g_{\alpha\beta}$.

We assume the existence of a constant $C > 0$ such that: $\left|\frac{\dot{a}}{a}\right| \leq C, \quad \left|\frac{\dot{b}}{b}\right| \leq C, \quad \left|\frac{\dot{c}}{c}\right| \leq C.$

Integrating these conditions over $[0, T]$ for $T > 0$ yields for all $t \in [0, T]$:

$$a(t) \leq a_0 e^{Ct},\ b(t) \leq b_0 e^{Ct},\ c(t) \leq c_0 e^{Ct}, \frac{1}{a(t)} \leq \frac{1}{a_0} e^{Ct},\ \frac{1}{b(t)} \leq \frac{1}{b_0} e^{Ct},\ \frac{1}{c(t)} \leq \frac{1}{c_0} e^{Ct}, \tag{3}$$

where $a_0 = a(0)$, $b_0 = b(0)$, and $c_0 = c(0)$. In the study of the Boltzmann equation taking as background the Bianchi type I spacetime, the boundedness assomptions above (3) have widely been used in literature to obtain regularity in the solution of the Boltzmann equation [2, 10, 11]. It is important to note that it is shown that the solution of the Einstein Equation verifies these conditions. (See relation (65) and remark 7 of page 19 in [2]).

According to Maxwell's equations, the relativistic charged particles generate an unknown electromagnetic field F, which in the homogeneous case is an antisymmetric and closed 2-form dependent only on t. We write:

$$F = (F^{0i}, F^{ij}), \tag{4}$$

where F^{0i} and F^{ij} correspond to the electric and magnetic field components, respectively. Indices are raised and lowered using the metric: $V^\alpha = g^{\alpha\beta} V_\beta, \quad V_\alpha = g_{\alpha\beta} V^\beta$.

The particles distribution function is a non-negative function:

$$f : T(\mathbb{R}^4) \simeq \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}_+, \quad (x^\alpha, p^\alpha) \mapsto f(x^\alpha, p^\alpha). \tag{5}$$

We define the spatial scalar product for two momenta $\bar{p} = (p^i)$ and $\bar{q} = (q^i)$ by:

$$\bar{p} \cdot \bar{q} = a^2 p^1 q^1 + b^2 e^{-2x} (p^2 q^2) + c^2 p^3 q^3. \tag{6}$$

For a beam of massive particles with rest mass normalized to unity, their motion is constrained to the future sheet of the mass shell or the mass hyperboloid, whose equation is:

$$g_{\alpha\beta} p^\alpha p^\beta = -1, \tag{7}$$

leading to:

$$p^0 = \sqrt{1 + a^2 (p^1)^2 + b^2 e^{-2x} (p^2)^2 + c^2 (p^3)^2}. \tag{8}$$

The choice of $p^0 > 0$ reflects the fact that particles propagate towards the future. Although the Bianchi type III metric exhibits explicit dependence on the spatial coordinate x, it remains geometrically homogeneous. The presence of x in the metric is a consequence of the choice of the basis of vectors (the Lie group) used to describe this specific spacetime, but the physical properties remain identical at every point in space. In this context, it is common to consider a distribution function that is invariant under the action of the symmetry group, as done in several works within the relativistic kinetic theory literature [13, 17, 25]. This assumption allows for an analytical simplification of the system and serves as a first step toward a full study of the model with spatial dependence.

2.2. *Maxwell's equations in Bianchi type III space-time*

The electromagnetic field F satisfies the system of Maxwell's equations which in the covariant form is written:

$$\nabla_\alpha F^{\alpha\beta} = 4\pi J^\beta, \tag{9}$$

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0. \tag{10}$$

where J^β denotes the Maxwell current, given by

$$J^\beta = \int\limits_{\mathbb{R}^3} \frac{1}{p^0} P^\beta f(t,\bar{p}) |g|^{\frac{1}{2}} d\bar{p} - eu^\beta. \tag{11}$$

Here, $abce^{-x} = |\det g|^{1/2}$ is the volume factor, $e = e(t,x) \geq 0$ is the charge density, and $u = (u^\beta)$ is a future-directed time-like unit vector satisfying $u^0 = 1$ and $u^i = 0$ for $i = 1,2,3$, indicating that particles are spatially at rest. The equation (10) expresses the closure condition $dF = 0$.

Since the identity $\nabla_\alpha \nabla_\beta F^{\alpha\beta} = 0$ holds generally, it follows from (9) that the current J^β obeys the conservation law:

$$\nabla_\beta J^\beta = 0. \tag{12}$$

Now, let us introduce the subgroup G of $O(3)$ defined by:

$$G = \left\{ N_{\varepsilon,\theta} \in O(3), \quad N_{\varepsilon,\theta} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \varepsilon, \theta \in \mathbb{R}, \quad \varepsilon^2 = 1 \right\}. \tag{13}$$

A function f on $\mathbb{R}^3$ is said to be invariant under G if:

$$f(Np) = f(p), \quad \forall N \in G, \quad \forall p \in \mathbb{R}^3. \tag{14}$$

Let f_0 be the initial condition for the Boltzmann equation. It has been shown in [**?**] that if f_0 is invariant under G, then the solution f remains invariant under G. We assume that the initial datum f_0 satisfies this invariance property.

As a consequence, we obtain:

$$I^i = \int\limits_{\mathbb{R}^3} \frac{p^i f(t,\bar{p})\, abce^{-2x}\, d\bar{p}}{p_0} = 0, \quad i = 1,2,3. \tag{15}$$

Since p^0 given by (8) is invariant under G, to prove (15), we choose specific elements $N_{-1,0}, N_{1,\pi} \in G$ to show that $I^1 = -I^1$, $I^2 = -I^2$, and $I^3 = -I^3$, leading to $I^i = 0, \quad i = 1,2,3$. From (11), since $u^i = 0$, it follows that:

$$J^i = 0, \quad i = 1,2,3. \tag{16}$$

Applying the standard formula for the covariant derivative of g to the left-hand side of equation (9), we obtain:

$$\begin{aligned} &\partial_\alpha F^{\alpha\beta} + \Gamma^\alpha_{\alpha\lambda} F^{\lambda\beta} + \Gamma^\beta_{\alpha\lambda} F^{\alpha\lambda} = 4\pi J^\beta, \\ &\Longleftrightarrow \partial_0 F^{0\beta} + \Gamma^\alpha_{\alpha 0} F^{0\beta} + \Gamma^\alpha_{\alpha k} F^{k\beta} + \Gamma^\beta_{\alpha 0} F^{\alpha 0} + \Gamma^\beta_{\alpha l} F^{\alpha l} = 4\pi J^\beta, \\ &\Longleftrightarrow \partial_0 F^{0\beta} + \Gamma^j_{j0} F^{0\beta} + \Gamma^2_{21} F^{1\beta} + \Gamma^\beta_{\alpha 0} F^{\alpha 0} + \Gamma^\beta_{\alpha l} F^{\alpha l} = 4\pi J^\beta. \end{aligned} \tag{17}$$

For $\beta = 0$, this simplifies to:

$$\partial_0 F^{00} + \Gamma^j_{j0} F^{00} + \Gamma^2_{21} F^{10} + \Gamma^0_{\alpha 0} F^{\alpha 0} + \Gamma^0_{\alpha l} F^{\alpha l} = 4\pi J^0. \tag{18}$$

Since the electromagnetic field tensor $F = (F^{0i}, F_{ij})$ is antisymmetric and taking into account the nonzero Christoffel symbols, Equation (18) reduces to:

$$\Gamma^2_{21} F^{10} = 4\pi J^0. \tag{19}$$

Thus, using equation (11), the charge density $e = e(t, x)$ is given by:

$$e(t,x) = \int_{\mathbb{R}^3} f(t, \bar{p}) abc e^{-2x} d\bar{p} + \frac{F^{10}}{4\pi}. \tag{20}$$

Applying the covariant derivative to equation (10) leads to:

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0 \Longleftrightarrow \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \tag{21}$$

and

$$\nabla_0 F_{ij} + \nabla_i F_{j0} + \nabla_j F_{0i} = 0 \Longleftrightarrow \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0. \tag{22}$$

Equation (21) is always satisfied since F depends only on time. Equation (22) simplifies to:

$$\partial_0 F_{ij} = 0,$$

which implies that $F_{ij}(t)$ are constants, i.e., there exist constant functions $\rho_{ki} \in \mathbb{R}$ such that:

$$F_{ij}(t) = F_{ij}(0) = \rho_{ki}. \tag{23}$$

From now on, we assume:

$$\rho_{12} = \rho_{13} = 0. \tag{24}$$

For $\beta = i$, equation (17) gives:

$$\begin{aligned} &\partial_0 F^{0i} + \Gamma^j_{j0} F^{0i} + \Gamma^2_{21} F^{1i} + \Gamma^i_{\alpha 0} F^{\alpha 0} + \Gamma^i_{\alpha l} F^{\alpha l} = 4\pi J^i, \\ &\Longleftrightarrow \partial_0 F^{0i} + \Gamma^j_{j0} F^{0i} + \Gamma^2_{21} F^{1i} + 2\Gamma^i_{i0} F^{i0} + \Gamma^2_{21} F^{21} + \Gamma^2_{12} F^{12} + \Gamma^1_{22} F^{22} = 4\pi J^i. \end{aligned} \tag{25}$$

Due to the antisymmetry of F, equation (25) simplifies to:

$$\partial_0 F^{0i} + \Gamma^j_{j0} F^{0i} + \Gamma^2_{21} F^{1i} = 4\pi J^i. \tag{26}$$

Using equation (16), equation (26) becomes:

$$\begin{aligned} &\partial_0 F^{0i} + \Gamma^j_{j0} F^{0i} + \Gamma^2_{21} F^{1i} = 0, \\ &\Longleftrightarrow \dot{F}^{0i} + \Gamma^j_{j0} F^{0i} + \Gamma^2_{21} F^{1i} = 0. \end{aligned} \tag{27}$$

Thus, the system becomes, using (24):

$$\begin{cases} \partial_0 F^{01} + \left(\dfrac{\dot a}{a} + \dfrac{\dot b}{b} + \dfrac{\dot c}{c}\right) F^{01} = 0, \\ \partial_0 F^{02} + \left(\dfrac{\dot a}{a} + \dfrac{\dot b}{b} + \dfrac{\dot c}{c}\right) F^{02} = 0, \\ \partial_0 F^{03} + \left(\dfrac{\dot a}{a} + \dfrac{\dot b}{b} + \dfrac{\dot c}{c}\right) F^{03} = 0. \end{cases} \tag{28}$$

Integrating over $[0, t]$, $t > 0$, yields:

$$F^{0i}(t) = \frac{a_0 b_0 c_0}{abc} F^{0i}(0), \quad i = 1, 2, 3. \tag{29}$$

Relations (20)-(23)-(29) give the expression of the Maxwell field. We therefore have to resolve the Boltzmann equation.

2.3. *The Boltzmann equation in* f

The relativistic Boltzmann equation describes the evolution of particles according to the principles of general relativity. By denoting by f the particle distribution function, this equation takes the form:

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} = Q(f, f), \tag{30}$$

with the force term P^α given by

$$P^\alpha = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + e p^\beta F^\alpha_{\ \beta}, \tag{31}$$

where the right-hand side $Q(f, f)$ is the collision operator, which we now describe.

Following the Lichnerowicz-Chernikov formalism, we consider an instantaneous, binary, and elastic collision framework. At any given point (t, x^i), collisions occur between two particles, modifying their momenta while preserving the total momentum.

Before the collision, let the momenta be p and q, and after the collision, let them be p' and q', satisfying the conservation relation:

$$p + q = p' + q'. \tag{32}$$

The collision operator is then expressed as:

$$Q(f, g) = Q^+(f, g) - Q^-(f, g), \tag{33}$$

where the gain and loss terms are given respectively by Q^+ and Q^-, with:

$$Q^+(f, g)(\bar p) = \int_{\mathbb{R}^3} \omega_q \int_{S^2} f(\bar p')g(\bar q')\sigma(t, a, b, c, \bar p, \bar q, \bar p', \bar q')\, d\omega, \tag{34}$$

$$Q^-(f, g)(\bar p) = \int_{\mathbb{R}^3} \omega_q \int_{S^2} f(\bar p)g(\bar q)\sigma(t, a, b, c, \bar p, \bar q, \bar p', \bar q')\, d\omega, \tag{35}$$

and

$$\omega_q = \frac{|detg|^{\frac{1}{2}} d\bar{q}}{q^0} = \frac{abce^{-2x} d\bar{q}}{q^0}.$$

Here:

- S^2 is the unit sphere in $\mathbb{R}^3$, with surface element $d\omega$.
- σ is the collision kernel, a non-negative, continuous function of its arguments, known as the cross-section of collisions. We assume it satisfies the boundedness and Lipschitz continuity conditions:

$$\begin{cases} 0 \leq \sigma(t, a, b, c, p, q, p', q') \leq C_1, \\ |\sigma|_{L^1(\mathbb{R}^3 \times S^2)} \in L^2(\mathbb{R}^3);\ \left(D^\beta{}_{\bar{p}}\sigma\right)\left(1 + |\bar{p}|^{\beta - 1}\right) \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times S^2),\ \beta \in \mathbb{N}^3,\ |\beta| \leq 3. \end{cases} \tag{36}$$

for some constant $C_1 > 0$, where $\|\bar{p}\| = \sqrt{\sum_{i=1}^3 (p^i)^2}$ is the Euclidean norm in $\mathbb{R}^3$.

Assumptions (36) are closely related to the "μ–N" regularity conditions introduced by Yvonne Choquet-Bruhat and Daniel Bancel in [3, 4], and applied in [10, 11, 19] and many other references in the literature. These are essentially technical conditions aimed at ensuring the existence and regularity of the solution to the Boltzmann Equation. Although their physical motivation is less direct, such assumptions help localize and control particle interactions, while guaranteeing that these interactions remain regular and do not introduce discontinuities in the gravitational field. A simple example of a collision kernel satisfying the assumptions (36) is given by

$$\sigma = k(t)\, e^{-|\bar{p}_1| - |\bar{q}_1| - |\bar{p}'_1| - |\bar{q}'_1|}, \tag{37}$$

where $k(t)$ is a continuous function of t.

The conservation law $p + q = p' + q'$ splits into:

$$p^0 + q^0 = p'^0 + q'^0, \tag{38}$$
$$p + q = p' + q'. \tag{39}$$

Equation (38) expresses, using (8), the conservation of the quantity $\tilde{e} = p^0 + q^0$

which is interpreted as the elementary energy of unit rest mass particles. To describe the transformation of momenta post-collision, following [21], we introduce:

$$\begin{cases} \bar{p'} = \bar{p} + d(\bar{p}, \bar{q}, \omega) \cdot \omega \\ \bar{q'} = \bar{q} - d(\bar{p}, \bar{q}, \omega) \cdot \omega \quad (\omega \in S^2), \end{cases} \tag{40}$$

where $d(\bar{p}, \bar{q}, \omega)$ is a real-valued function. By explicitly solving a quadratic equation derived from (38) as proceeded in [21], we obtain:

$$\begin{cases} d(\bar{p}, \bar{q}, w) = \dfrac{2\tilde{e}p^0 q^0 (w.(\tilde{\bar{q}} - \tilde{\bar{p}}))}{\tilde{e}^2 - \left[w.(\bar{p} + \bar{q})\right]^2}, \\ \dfrac{\partial(\bar{p'}, \bar{q'})}{\partial(\bar{p}, \bar{q})} = -\dfrac{p'^0 q'^0}{p^0 q^0}, \end{cases} \tag{41}$$

where $\tilde{\bar{p}} = \frac{\bar{p}}{p^0}$ and $(\cdot)$ denotes the inner product as defined earlier. The Jacobian determinant associated with the change of variables $(\bar{p}, \bar{q}) \mapsto (\bar{p}', \bar{q}')$ is given by the second equation in (41). This shows that the integrands in (34) and (35) depend only on $\bar{p}, \bar{q}, \omega$, and integrating over $\bar{q}$ and ω results in functions $Q^{+}(f,g)$ and $Q^{-}(f,g)$ depending solely on $\bar{p}$. For practical applications, we consider functions f on $\mathbb{R}^4$ that induce, for each time t, functions $f(t)$ on $\mathbb{R}^3$ defined as $f(t)(p) = f(t,p)$. Regarding the Boltzmann equation (30), since $f = f(t,p)$, it can be written as:

$$\frac{\partial f}{\partial t} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{Q(f,f)}{p^0}. \tag{42}$$

Using the Christoffel symbols and the electromagnetic tensor $(F_{\alpha\beta})$, we obtain from (31):

$$P^i = -\Gamma^i_{\lambda\mu}p^\lambda p^\mu - ep^0F^{i0} + ep^jg^{ii}F_{ij},\ i = 1,2,3. \tag{43}$$

Thus, the Boltzmann equation in the Bianchi type III framework takes the form:

$$\frac{\partial f}{\partial t} + \tilde{P}^i\frac{\partial f}{\partial p^i} = \frac{Q(f,f)}{p^0}, \tag{44}$$

where the modified force components $\tilde{P}^i = \frac{P^i}{p^0}$ are given explicitly as:

$$\tilde{P}^1 = -2\frac{\dot{a}}{a}p^1 - \frac{b^2e^{-2x}}{a^2}\frac{(p^2)^2}{p^0} + \left(F^{01} + \frac{g^{11}p^j\rho_{1j}}{p^0}\right)e(t,x), \tag{45a}$$

$$\tilde{P}^2 = -2\frac{\dot{b}}{b}p^2 + 2\frac{p^1p^2}{p^0} + \left(F^{02} + \frac{g^{22}p^j\rho_{2j}}{p^0}\right)e(t,x), \tag{45b}$$

$$\tilde{P}^3 = -2\frac{\dot{c}}{c}p^3 + \left(F^{03} + \frac{g^{33}p^j\rho_{3j}}{p^0}\right)e(t,x). \tag{45c}$$

The Maxwell-Boltzmann system is then given by expressions (20), (23), (29), and (44); with initial conditions:

$$f(0,\bar{p}) := f_0(\bar{p}), \quad F^{0i}(0) := E^i, \quad F^{ij} := F^{ij}(0) := \rho^{ij}, \quad i,j = 1,2,3. \tag{46}$$

It just remains to solve the Boltzmann equation.

3. Functional spaces and energy estimates

3.1. *Functional spaces for the Boltzmann equation*

We introduce the following functional spaces for the study of the Boltzmann equation.

Definitions

Let $T > 0$, $l \in \mathbb{N}$, and $d \in \mathbb{R}$ be given. The following spaces are defined:

a) The space $V^l_d(\mathbb{R}^3)$:

$$V^l_d(\mathbb{R}^3) = \left\{u : \mathbb{R}^3 \to \mathbb{R},\ (1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}u \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3),\ |\beta| \leq l\right\},$$

equipped with the norm: $\|u\|_{V^l_d(\mathbb{R}^3)} = \max_{|\beta|\leq l}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}u\right\|_{L^2(\mathbb{R}^3)}$.

b) The space $V_d^l(0,T,\mathbb{R}^3)$:

$$V_d^l(0,T,\mathbb{R}^3) = \left\{u \in C([0,T], C(\mathbb{R}^3)),\ (1+|\bar{p}|)^{d+|\beta|}\partial_{\bar{p}}^\beta u(t,\cdot) \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3),\ |\beta| \leq l\right\},$$

equipped with the norm: $\|u\|_{V_d^l(0,T,\mathbb{R}^3)} = \max_{|\beta|\leq l} \sup_{t\in[0,T]} \left\|(1+|\bar{p}|)^{d+|\beta|}\partial_{\bar{p}}^\beta u(t,\cdot)\right\|_{L^2(\mathbb{R}^3)}$.

c) The subspace $V_{d,r}^l(0,T,\mathbb{R}^3)$ of $V_d^l(0,T,\mathbb{R}^3)$, with $r > 0$, given by:
$V_{d,r}^l(0,T,\mathbb{R}^3) = \left\{u \in V_d^l(0,T,\mathbb{R}^3),\ u \geq 0 \text{ a.e.},\ \|u\|_{V_d^l(0,T,\mathbb{R}^3)} \leq r\right\}$.

d) The space $H_d^l(0,T,\mathbb{R}^3)$, in which $D_{\bar{p}}^\beta$ denotes derivatives in the sense of distributions:

$$H_d^l(0,T,\mathbb{R}^3) = \left\{u \in C([0,T], C(\mathbb{R}^3)),\ (1+|\bar{p}|)^{d+|\beta|}D_{\bar{p}}^\beta u(t,\cdot) \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3),\ |\beta| \leq l\right\},$$

where $D_{\bar{p}}^\beta$ denotes derivatives in the sense of distributions.

e) The corresponding bounded space:

$$H_{d,r}^l(0,T,\mathbb{R}^3) = \left\{u \in H_d^l(0,T,\mathbb{R}^3),\ u \geq 0 \text{ a.e.},\ \|u\|_{H_d^l(0,T,\mathbb{R}^3)} \leq r\right\}.$$

3.2. *Energy estimates*

We begin with the following lemma:

Lemma 3.1. *(See [8], [22]) Let $T > 0$, $l \in \mathbb{N}$, and $d \in \mathbb{R}$ be given. The following results hold:*

i) $H_d^l(0,T,\mathbb{R}^3)$ is a Banach space.

ii) $V_d^l(0,T,\mathbb{R}^3)$ is dense in $H_d^l(0,T,\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{H_d^l(0,T,\mathbb{R}^3)}$.

iii) By induction on the norm $\|\cdot\|_{H_d^l(0,T,\mathbb{R}^3)}$, the space $H_{d,r}^l(0,T,\mathbb{R}^3)$ is a complete metric subspace of $H_d^l(0,T,\mathbb{R}^3)$.

iv) $C([0,T];\mathbb{R}^{16})$ is a Banach space with respect to the norm $\|u\| = \sup_{t\in[0,T]} |u(t)|_{\mathbb{R}^{16}}$.

Remark 3.2. *The choice of the function f, solution of the Boltzmann equation, in the space $H_d^l(0,T,\mathbb{R}^3)$ with $l = 3$ and $d > \frac{5}{2}$ is justified as follows:*

We seek a continuously differentiable function $f = f(t,\bar{p})$ solving the Boltzmann equation, and the appropriate function space is $C_b^1(\mathbb{R}^3)$. This requires finding an integer l such that $H^l(\mathbb{R}^3) \hookrightarrow C_b^1(\mathbb{R}^3)$.

According to the Sobolev embedding theorem, we have $W_q^l \hookrightarrow C^k(\mathbb{R}^n)$ if $l > k + \frac{n}{q}$. In our case, $n = 3$, $q = 2$, and $k = 1$ (since $W_2^l = H^l$).

Therefore, we must choose l such that $l > 1 + \frac{3}{2} = \frac{5}{2}$. The smallest integer satisfying $l > \frac{5}{2}$ is $l = 3$, leading to

$$H_d^3(\mathbb{R}^3) \hookrightarrow H^l(\mathbb{R}^3) \hookrightarrow C_b^1(\mathbb{R}^3).$$

Lemma 3.3. *1. There exists a real number $T > 0$ such that:*

$$(\tilde{e})^2 - \left[w \cdot (\bar{p} + \bar{q})^2\right] > 2. \tag{47}$$

2. In this case, we have:

$$\begin{cases} (\tilde{e})^2 - \left[w \cdot (\bar{p} + \bar{q})^2\right] \geq (1 - \sum_i g_{ii})(\bar{p}^0)^2, \\ (\tilde{e})^2 - \left[w \cdot (\bar{p} + \bar{q})^2\right] \geq (1 - \sum_i g_{ii})(\bar{q}^0)^2, \\ (\tilde{e})^2 - \left[w \cdot (\bar{p} + \bar{q})^2\right] \geq (1 - \sum_i g_{ii})\bar{p}^0\bar{q}^0. \end{cases} \tag{48}$$

3. The function

$$(\bar{p}, \bar{q}, w) \mapsto D^{\beta}_{\bar{p}} d(\bar{p}, \bar{q}, w)$$

is bounded for $1 \leq |\beta| \leq 3$.

Proof. To derive the inequalities in (47) and (48), proceed as in [22]. For the boundedness of $D^{\beta}_{\bar{p}} d(\bar{p}, \bar{q}, w)$, see [1]. □

Remark 3.4. *Let $T > 0$. There exists a constant $C_1 = C(T) > 0$ such that:*

$$(1 + |\bar{p}|)^2 \leq C_1 (p^0)^2, \tag{49}$$

$$(1 + |\bar{p}|) \leq C_1 (1 + |\bar{p}'|)(1 + |\bar{p}'_*|), \tag{50}$$

$$(1 + |\bar{p}|)^2 \leq C_1 p^0 q^0 p'^0 q'^0. \tag{51}$$

Remark 3.5. *Let $(a_i)_{i=1}^n \in \mathbb{R}^n$. then we have:*

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq n \sum_{i=1}^{n} a_i^2.$$

Lemma 3.6. *There exists a real constant $C_3 = C_3(T) > 0$, such that for all $\beta \in \mathbb{N}^3$ with $0 \leq |\beta| \leq 3$,*

$$\left| (1 + |\bar{p}|)^{|\beta|} \partial^{\beta}_{\bar{p}} \left(\frac{1}{p^0}\right) \right| \leq C_3 \frac{1}{p^0}.$$

Proof. We proceed by induction on the order $|\beta|$. Define the induction hypothesis:

$$P(|\beta|): \quad \exists C > 0, \forall \beta \in \mathbb{N}^3, \quad 0 \leq |\beta| \leq 3, \quad \left| (1 + |\bar{p}|)^{|\beta|} \partial^{\beta}_{\bar{p}} \left(\frac{1}{p^0}\right) \right| \leq C \frac{1}{p^0}.$$

Base case: For $|\beta| = 0$, we have:

$$(1 + |\bar{p}|)^0 \cdot \frac{1}{p^0} = \frac{1}{p^0} \leq C \frac{1}{p^0},$$

which holds for any $C \geq 1$, proving $P(0)$.

Inductive step: Assume $P(k)$ holds for some $k \in \{0, 1, 2\}$ and prove $P(k+1)$.

For $|\beta| = 1$ and any $i_1 \in \{1, 2, 3\}$:

$$(1 + |\bar{p}|) \partial_{p^{i_1}} \left(\frac{1}{p^0}\right) = -(1 + |\bar{p}|) g_{i_1 i_1} p^{i_1} \frac{1}{(p^0)^3}.$$

Since $|g_{i_1i_1}p^{i_1}| \leq C(T)p^0$ and $\frac{1}{p^0} < 1$, we obtain:

$$\left|(1+|\bar{p}|)\partial_{p^{i_1}}\left(\frac{1}{p^0}\right)\right| \leq C\frac{1}{p^0}.$$

Thus, $P(1)$ holds.

For $|\beta| = 2$ and any $i_1, i_2 \in \{1,2,3\}$:

$$\partial_{p^{i_2}}\partial_{p^{i_1}}\left(\frac{1}{p^0}\right) = -g_{i_1i_1}\left[\delta_{i_1i_2}\frac{1}{(p^0)^3} - 3g_{i_2i_2}p^{i_2}p^{i_1}\frac{1}{(p^0)^6}\right].$$

Using $(1+|\bar{p}|)^2 \leq C_1(p^0)^2$ and $|g_{i_1i_1}p^{i_1}| \leq Cp^0$, we obtain:

$$\left|(1+|\bar{p}|)^2\partial^2_{p^{i_2}p^{i_1}}\left(\frac{1}{p^0}\right)\right| \leq C\frac{1}{p^0}.$$

Thus, $P(2)$ holds.

For $|\beta| = 3$ and any $i_1, i_2, i_3 \in \{1,2,3\}$:

$$\begin{aligned}\partial_{p^{i_3}}\partial^2_{p^{i_2}p^{i_1}}\left(\frac{1}{p^0}\right) = &\frac{3\delta_{i_1i_2}g_{i_1i_1}g_{i_3i_3}p^{i_3}}{(p_0)^5} \\ &+ \frac{3g_{i_1i_1}g_{i_2i_2}(\delta_{i_1i_3}p^{i_2} + \delta_{i_2i_3}p^{i_1})}{(p^0)^5} - \frac{3g_{i_1i_1}g_{i_2i_2}g_{i_3i_3}p^{i_1}p^{i_2}p^{i_3}}{(p^0)^7}.\end{aligned}$$

Using $(1+|\bar{p}|)^3 \leq C_1(p^0)^3$ and $|g_{i_ji_j}p^{i_j}| \leq C(T)p^0$, we conclude that:

$$\left|(1+|\bar{p}|)^3\partial^3_{p^{i_3}p^{i_2}p^{i_1}}\left(\frac{1}{p^0}\right)\right| \leq C\frac{1}{p^0}.$$

Thus, $P(3)$ holds, completing the induction. □

Proposition 3.7. *Let $d \in \left]\frac{5}{2}, +\infty\right[$ be a real number. If $f, g \in V^3_d(\mathbb{R}^3)$, then there exists a constant $C_2 = C_2(T) > 0$ such that:*

$$\|(1+|\bar{p}|)^d\frac{1}{p^0}Q^+(f,g)\|_{L^2(\mathbb{R}^3)} \leq C_2\|f\|_{V^3_d(\mathbb{R}^3)}\|g\|_{V^3_d(\mathbb{R}^3)}.$$

Proof. We have:

$$Q^+(f,g) = \int\limits_{\mathbb{R}^3}\frac{abce^{-2x}}{q^0}d\bar{q}\int\limits_{S^2}f(\bar{p}')g(\bar{q}')\sigma d\omega.$$

Thus,

$$\left\|\frac{(1+|\bar{p}|)^d}{p^0}Q^+(f,g)\right\|^2_{L^2(\Omega)} \leq C\int\limits_{\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d}}{(p^0)^2}\left[\int\limits_{\mathbb{R}^3\times S^2}\frac{f(\bar{p}')g(\bar{q}')}{q^0}\sigma d\omega d\bar{q}\right]^2 d\bar{p}.$$

Applying Cauchy-Schwarz inequality, we obtain:

$$\left\|\frac{(1+|\bar{p}|)^d}{p^0}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C\int\limits_{\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d}}{(p^0)^2}\left[\int\limits_{\mathbb{R}^3\times S^2}\sigma d\omega d\bar{q}\right] \times\left[\int\limits_{\mathbb{R}^3\times S^2}\frac{|f(\bar{p}')|^2|g(\bar{q}')|^2}{(q^0)^2}\sigma d\omega d\bar{q}\right]d\bar{p}.$$

Considering (36), we obtain:

$$\left\|\frac{(1+|\bar{p}|)^d}{p^0}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d}}{p^0q^0}|f(\bar{p}')|^2|g(\bar{q}')|^2d\bar{p}d\bar{q}.$$

Using Remark (3.4) and (41), we deduce:

$$\begin{aligned}\left\|(1+|\overline{p}|)^d\frac{1}{p_0}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} &\leq C\int\limits_{S^2}d\omega\int\limits_{\mathbb{R}^3}(1+|\overline{p}'|)^{2d}|f(\overline{p})|^2d\overline{p}' \\ &\quad\times\int\limits_{\mathbb{R}^3}(1+|\overline{q}'|)^{2d}|g(\overline{q}')|^2d\overline{q}', \\ &\leq C\|f\|^2_{L^2(\mathbb{R}^3)}\|g\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Thus,

$$\left\|(1+|\overline{p}|)^d\frac{1}{p_0}Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)} \leq C_2\|f\|_{V^3_d(\mathbb{R}^3)}\|g\|_{V^3_d(\mathbb{R}^3)}.$$

□

Proposition 3.8. *Let* $d\in\left]\frac{5}{2},\infty\right[$ *be a real number, and let* $f,g\in V^3_d(\mathbb{R}^3)$*. For all* $\beta\in\mathbb{N}^3$ *such that* $0\leq|\beta|\leq 3$*, if:*

$$\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}\left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{L^2(\mathbb{R}^3)} \leq C_2\|f\|_{V^3_d(\mathbb{R}^3)}\|g\|_{V^3_d(\mathbb{R}^3)},$$

then, for all $\beta'\in\mathbb{N}^3$ *such that* $|\beta'|=1$*, we have:*

$$\left\|(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{\bar{p}}\left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{L^2(\mathbb{R}^3)} \leq C_2\|f\|_{V^3_d(\mathbb{R}^3)}\|g\|_{V^3_d(\mathbb{R}^3)}.$$

Proof. For $\beta\in\mathbb{N}^3$ such that $|\beta|=0$, see the previous Proposition (3.7), which serves as the base case. For the inductive step, we perform the following multi-index change:

$$\alpha=\beta+\beta',\quad \text{where } |\alpha|\leq|\beta|+|\beta'|\leq 3.$$

Applying Leibniz's rule gives:

$$\partial^{\beta+\beta'}_{\bar{p}}\left(\frac{1}{p^0}Q^+(f,g)\right)=\partial^\alpha_{\bar{p}}\left(\frac{1}{p^0}Q^+(f,g)\right)=\sum_{k\leq\alpha}C^k_\alpha\,\partial^k_{\bar{p}}\left(\frac{1}{p^0}\right)\partial^{\alpha-k}_{\bar{p}}Q^+.$$

That is:

$$\partial_{\bar{p}}^{\beta+\beta'}\left(\frac{1}{p^0}Q^+(f,g)\right)=\frac{1}{p^0}\partial_{\bar{p}}^{\alpha}Q^+ + \sum_{k\leq\alpha,|k|\geq1} C_\alpha^k\,\partial_{\bar{p}}^k\left(\frac{1}{p^0}\right)\partial_{\bar{p}}^{\alpha-k}Q^+.$$

This implies:

$$\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial_{\bar{p}}^{\alpha}\left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{L^2(\mathbb{R}^3)}\leq\left\|(1+|\bar{p}|)^{d+|\alpha|}\frac{1}{p^0}\partial_{\bar{p}}^{\alpha}Q^+\right\|_{L^2(\mathbb{R}^3)} \tag{52}$$

$$+\left\|(1+|\bar{p}|)^{d+|\alpha|}\sum_{k\leq\alpha,|k|\geq1} C_\alpha^k\,\partial_{\bar{p}}^k\left(\frac{1}{p^0}\right)\partial_{\bar{p}}^{\alpha-k}Q^+\right\|_{L^2(\mathbb{R}^3)}. \tag{53}$$

Defining:

$$\begin{cases} A_1=\left\|(1+|\bar{p}|)^{d+|\alpha|}\frac{1}{p^0}\partial_{\bar{p}}^{\alpha}Q^+\right\|_{L^2(\mathbb{R}^3)},\\ A_2=\left\|(1+|\bar{p}|)^{d+|\alpha|}\sum_{k\leq\alpha,|k|\geq1} C_\alpha^k\,\partial_{\bar{p}}^k\left(\frac{1}{p^0}\right)\partial_{\bar{p}}^{\alpha-k}Q^+\right\|_{L^2(\mathbb{R}^3)}. \end{cases}$$

Using the factorization:

$$A_2=\left\|(1+|\bar{p}|)^{d+|\alpha|}\sum_{k\leq\alpha,|k|\geq1} C_\alpha^k(1+|\bar{p}|)^{|k|}\partial_{\bar{p}}^k\left(\frac{1}{p^0}\right)\partial_{\bar{p}}^{\alpha-k}Q^+\frac{1}{(1+|\bar{p}|)^{|k|}}\right\|_{L^2(\mathbb{R}^3)}.$$

Using Lemma (3.6), we obtain:

$$A_2=C_2\sum_{k\leq\alpha,|k|\geq1}\left\|(1+|\bar{p}|)^{d+|\alpha|-|k|}\frac{1}{p^0}\partial_{\bar{p}}^{\alpha-k}Q^+\right\|_{L^2(\mathbb{R}^3)}.$$

Since $|\alpha|-|k|\leq3$, applying the induction hypothesis yields:

$$A_2\leq C_2\|f\|_{V_d^3(\mathbb{R}^3)}\|g\|_{V_d^3(\mathbb{R}^3)}. \tag{54}$$

We now need to establish that:

$$A_1\leq C\|f\|_{V_d^3(\mathbb{R}^3)}\|g\|_{V_d^3(\mathbb{R}^3)}. \tag{55}$$

We have:

$$\partial_{\bar{p}}^{\alpha}Q^+(f,g)=\int_{\mathbb{R}^3}\frac{abce^{-2x}}{q^0}\,d\bar{q}\int_{S^2}\partial_{\bar{p}}^{\alpha}(fg\sigma)\,d\omega,$$

Applying Leibniz's formula twice in succession, we obtain:

$$\partial_{\bar{p}}^{\alpha}(fg\sigma)=\sum_{k\leq\alpha}C_\alpha^k\,\partial_{\bar{p}}^{\alpha-k}\sigma\sum_{\lambda\leq k}C_\lambda^k\,\partial_{\bar{p}}^{\lambda}f\,\partial_{\bar{p}}^{k-\lambda}g.$$

This leads to:

$$\left\|(1+|\bar{p}|)^{d+|\alpha|}\frac{1}{p^0}\partial_{\bar{p}}^{\alpha}Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)}^2$$

$$\leq \int\limits_{\mathbb{R}^3}\left[(1+|\bar{p}|)^{d+|\alpha|}\frac{1}{p^0}\int\limits_{\mathbb{R}^3}\frac{abce^{-2x}}{q^0}\,d\bar{q}\int\limits_{S^2}\sum_{k\leq\alpha}C_\alpha^k\,\partial_{\bar{p}}^{\alpha-k}\sigma\sum_{\lambda\leq k}C_\lambda^k\,\partial_{\bar{p}}^{\lambda}f\,\partial_{\bar{p}}^{k-\lambda}g\,d\omega\right]^2 d\bar{p}.$$

By invoking Remark (3.5), we obtain:

$$\left\|(1+|\bar{p}|)^{d+|\alpha|}\frac{1}{p^0}\partial_{\bar{p}}^{\alpha}Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)}^2 \leq C\sum_{k\leq\alpha}A_k,$$

where:

$$A_k=\int\limits_{\mathbb{R}^3}\left[(1+|\bar{p}|)^{d+|\alpha|}\frac{1}{p^0}\int\limits_{\mathbb{R}^3}\frac{abce^{-2x}}{q^0}d\bar{q}\int\limits_{S^2}\partial_{\bar{p}}^{\alpha-k}\sigma\sum_{\lambda\leq k}C_k^{\lambda}\partial_{\bar{p}}^{\lambda}f\,\partial_{\bar{p}}^{k-\lambda}g\,d\omega\right]^2 d\bar{p}.$$

We need to show that for all $k\in\mathbb{N}^3$ such that $0\leq|k|\leq|\alpha|$, we have:

$$A_k\leq C\|f\|_{V_d^3(\mathbb{R}^3)}^2\|g\|_{V_d^3(\mathbb{R}^3)}^2. \tag{56}$$

Let $s_1\in\mathbb{N}^3$ be arbitrarily chosen such that $s_1\leq\alpha$. For $k=s_1$, by Remark (3.5), we obtain:

$$A_{s_1}\leq C\sum_{\lambda\leq s_1}I_\lambda,$$

where:

$$I_\lambda=\int\limits_{\mathbb{R}^3}\left[(1+|\bar{p}|)^{d+|\alpha|}\frac{1}{p^0}\int\limits_{\mathbb{R}^3}\frac{abce^{-2x}}{q^0}\,d\bar{q}\int\limits_{S^2}\partial_{\bar{p}}^{\alpha-s_1}\sigma\partial_{\bar{p}}^{\lambda}f\,\partial_{\bar{p}}^{s_1-\lambda}g\,d\omega\right]^2 d\bar{p}.$$

Let $s_2\in\mathbb{N}^3$ be arbitrarily chosen such that $s_2\leq s_1$.

Estimation of I_λ for $\lambda=s_2$.

We have:

$$I_{s_2}=\int\limits_{\mathbb{R}^3}\left[\frac{(1+|\bar{p}|)^{d+|\alpha|}}{p^0}\int\limits_{\Omega}\frac{abce^{-2x}}{q^0}\,d\bar{q}\int\limits_{S^2}\partial_{\bar{p}}^{\alpha-s_1}\sigma\partial^{s_2}\bar{p}f\partial_{\bar{p}}^{s_1-s_2}g\,d\omega\right]^2 d\bar{p}.$$

Applying Cauchy-Schwarz to the integral over $\mathbb{R}^3\times S^2$ and using the assumptions on the collision kernel σ, we obtain:

$$I_{s_2}\leq C\int\limits_{\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d+2|\alpha|}}{(p_0)^2}\,d\bar{p}\int\limits_{\mathbb{R}^3\times S^2}\frac{|\partial_{\bar{p}}^{s_2}f|^2|\partial_{\bar{p}}^{s_1-s_2}g|^2}{q^0}\,d\bar{q}d\omega.$$

Applying $|\alpha| \leq |\beta| + 1$, and taking into account:

$$\begin{cases} (1+|\bar{p}|)^2 \leq C_1 p^0 q^0 p'^0 q'^0, \\ \dfrac{d\bar{p}d\bar{q}}{p^0 q^0} = -\dfrac{d\bar{p}'d\bar{q}'}{p'^0 q'^0}, \\ (1+|\bar{p}|) \leq C_1(1+|\bar{p}'|)(1+|\bar{q}'|), \end{cases} \tag{57}$$

we obtain the following estimate:

$$\begin{aligned} I_{s_2} &\leq C \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(1+|\bar{p}|)^{2d+2|\alpha|}|\partial^{s_2} f|^2 |\partial^{s_1-s_2} g|^2}{(q_0 p_0)^2}\, d\bar{q}\, d\bar{p} \\ &\leq C \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} (1+|\bar{p}'|)^{2d+2|\beta|}|\partial^{s_2} f|^2 (1+|\bar{q}'|)^{2d+2|\beta|}|\partial^{s_1-s_2} g|^2 \frac{(1+|\bar{p}|)^2}{(q^0 p^0)^2}\, d\bar{p}\, d\bar{q} \\ &\leq C \int\limits_{\mathbb{R}^3} (1+|\bar{p}'|)^{2d+2|\beta|}|\partial^{s_2} f|^2 d\bar{p}' \int\limits_{\mathbb{R}^3} (1+|\bar{q}'|)^{2d+2|\beta|}|\partial^{s_1-s_2} g|^2 d\bar{q}'. \end{aligned}$$

Thus:

$$I_{s_2} \leq C\|f\|^2_{V_d^3(\mathbb{R}^3)} \|g\|^2_{V_d^3(\mathbb{R}^3)}. \tag{58}$$

Since s_2 is arbitrary, summing over $\lambda \leq s_1$ yields:

$$A_{s_1} \leq C^2\|f\|^2_{V_d^3(\mathbb{R}^3)} \|g\|^2_{V_d^3(\mathbb{R}^3)}. \tag{59}$$

Since s_1 is arbitrary, this ensures (56) holds, leading to (55).

Equations (55) and (54) complete the proof of Proposition (3.8). ☐

Proposition 3.9. *Let $T > 0$ and $d \in \left]\frac{5}{2}, +\infty\right[$ be real numbers. Suppose that $f, g \in V_d^3(\mathbb{R}^3)$. Then we have:*

1. $\frac{1}{p^0}Q^+(f,g), \frac{1}{p^0}Q^-(f,g) \in V_d^3(\mathbb{R}^3)$.
2. *There exists a constant $C_2 = C_2(T) > 0$ such that:*

$$\begin{aligned} \left\|\frac{1}{p^0}Q^+(f,g)\right\|_{V_d^3(\mathbb{R}^3)} &\leq C_2\|f\|_{V_d^3(\mathbb{R}^3)}\|g\|_{V_d^3(\mathbb{R}^3)}, \\ \left\|\frac{1}{p^0}Q^-(f,g)\right\|_{V_d^3(\mathbb{R}^3)} &\leq C_2\|f\|_{V_d^3(\mathbb{R}^3)}\|g\|_{V_d^3(\mathbb{R}^3)}. \end{aligned}$$

Proof. For the proof of the first result of the proposition, we show that for all $\beta \in \mathbb{N}^3$ such that $0 \leq |\beta| \leq 3$, we have:

$$\left\|(1+|p|)^{d+|\beta|}\partial^\beta \left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{L^2(\mathbb{R}^3)} \leq C_2\|f\|_{V_d^3(\mathbb{R}^3)}\|g\|_{V_d^3(\mathbb{R}^3)}.$$

To prove this, we proceed by induction on the order of the multi-index β. The conclusion follows from Proposition (3.8).

For the second result of the proposition, we use an argument identical to the one used in the first case. ☐

Proposition 3.10. *Let $d \in \left]\frac{5}{2}, +\infty\right[$ be a real number, and let $f, g \in H^3_d(\mathbb{R}^3_{\bar{p}})$. Then,*

1. $\frac{1}{p^0} Q(f,g) \in H^3_d(\mathbb{R}^3_{\bar{p}})$.
2. *There exists a constant $C_2 = C_2(T) > 0$ such that:*

   (i) $\left\| \frac{1}{p^0} Q(f,g) \right\|_{H^3_d(\mathbb{R})} \leq C_2 \|f\|_{H^3_d(\mathbb{R}^3)} \|g\|_{H^3_d(\mathbb{R}^3)}$,

   (2i) $\left\| \frac{1}{p^0} Q(f,f) - \frac{1}{p^0} Q(g,g) \right\|_{H^3_d(\mathbb{R}^3)} \leq 2C_2 \left(\|f\|_{H^3_d(\mathbb{R}^3)} + \|g\|_{H^3_d(\mathbb{R}^3)} \right) \|f-g\|_{H^3_d(\mathbb{R}^3)}$.

Before proving this proposition, we first state the following useful lemma.

Lemma 3.11. *Let $d \in \left]\frac{5}{2}, +\infty\right[$ be a real number, and let $f, g \in H^3_d(\mathbb{R}^3)$. There exists a constant $C_2 = C_2(T) > 0$ such that:*

$$\left\| \frac{1}{p^0} Q^+(f,f) - \frac{1}{p^0} Q^+(g,g) \right\|_{H^3_d(\mathbb{R}^3)} \leq C_2 \left(\|f\|_{H^3_d(\mathbb{R}^3)} + \|g\|_{H^3_d(\mathbb{R}^3)} \right) \|f-g\|_{H^3_d(\mathbb{R}^3)},$$

$$\left\| \frac{1}{p^0} Q^-(f,f) - \frac{1}{p^0} Q^-(g,g) \right\|_{H^3_d(\mathbb{R}^3)} \leq C_2 \left(\|f\|_{H^3_d(\mathbb{R}^3)} + \|g\|_{H^3_d(\mathbb{R}^3)} \right) \|f-g\|_{H^3_d(\mathbb{R}^3)}.$$

Proof. Using the bilinearity of the collision operator, we obtain:

$$\begin{aligned}\left\| \frac{1}{p_0} Q^+(f,f) - \frac{1}{p_0} Q^+(g,g) \right\|_{H^3_d(\mathbb{R}^3)} &= \left\| \frac{1}{p_0} Q^+(f,f-g) + \frac{1}{p_0} Q^+(f-g,g) \right\|_{H^3_d(\mathbb{R}^3)} \\ &\leq C_2 \left(\|f\|_{H^3_d(\mathbb{R}^3)} + \|g\|_{H^3_d(\mathbb{R}^3)} \right) \|f-g\|_{H^3_d(\mathbb{R}^3)}.\end{aligned}$$

This yields the first estimate. Similarly, the second estimate is established. □

Proof of Proposition (3.10). By density of $V^3_d(\mathbb{R}^3)$ in $H^3_d(\mathbb{R}^3_{\bar{p}})$, we prove the result for $f, g \in V^3_d(\mathbb{R}^3)$, and the conclusion follows by taking the limit.

Let $f, g \in V^3_d(\mathbb{R}^3)$ and $\beta \in \mathbb{N}^3$ with $0 \leq |\beta| \leq 3$, we have:

$$\frac{1}{p_0} Q(f,g) = \frac{1}{p_0} Q^+(f,g) - \frac{1}{p_0} Q^-(f,g).$$

By Proposition (3.9), we establish that: $\frac{1}{p_0} Q(f,g) \in V^3_d(\mathbb{R}^3)$. Moreover, there exists $C_2 = C_2(T) > 0$ such that $\left\| \frac{1}{p_0} Q(f,g) \right\|_{V^3_d(\mathbb{R}^3)} \leq C_2 \|f\|_{V^3_d(\mathbb{R}^3)} \|g\|_{V^3_d(\mathbb{R}^3)}$.

For (2i), we leverage the bilinear structure of the collision operator along with the results established in Lemma 3.11. □

4. Existence, uniqueness, and well-posedness

We establish the local existence of solutions for the Boltzmann equation in the cosmological framework of Bianchi type III space-time under the influence of a Maxwell field given by (44),

where the modified force $\tilde{P}^i$ are given by (45), the electric charge density e is given by (20), the electromagnetic field by (23) and (29).

4.1. *Local existence*

We prove that for any initial data $f_0 \in H^3_{d,r}(\mathbb{R}^3)$, there exists a $T > 0$ such that a unique solution f exists in $C([0,T]; H^3_{d,r}(\mathbb{R}^3))$.

Lemma 4.1. *Let $f \in H^3_{d,r}(0,T,\mathbb{R}^3_{\bar{p}})$. For all $i,j,k \in \{1,2,3\}$, the following expressions are bounded:*

$$\partial_{p^j}(\tilde{P}^i), \quad \partial^2_{p^i p^j}(\tilde{P}^i)(1+|\bar{p}|), \quad \partial^3_{p^i p^j p^k}(\tilde{P}^i)(1+|\bar{p}|)^2.$$

Proof. We decompose $\tilde{P}^i$ in (45) as $\tilde{P}^i = T^i_1 + T^i_2$, where:

$$T^1_1 = -2\frac{\dot{a}}{a}p^1 - \frac{b^2 e^{-2x}}{a^2}\frac{(p^2)^2}{p^0}, \qquad T^1_2 = \left(F^{01} + \frac{g^{11}p^j\rho_{1j}}{p^0}\right)e(t,x), \tag{60}$$

$$T^2_1 = -2\frac{\dot{b}}{b}p^2 + 2\frac{p^1p^2}{p^0}, \qquad T^2_2 = \left(F^{02} + \frac{g^{22}p^j\rho_{2j}}{p^0}\right)e(t,x), \tag{61}$$

$$T^3_1 = -2\frac{\dot{c}}{c}p^3, \qquad T^3_2 = \left(F^{03} + \frac{g^{33}p^j\rho_{3j}}{p^0}\right)e(t,x). \tag{62}$$

The boundedness of $\partial_{p^j}\tilde{P}^i$ follows from showing that $\partial_{p^j}T^i_1$ and $\partial_{p^j}T^i_2$ are bounded.

For T^i_2, differentiating with respect to p^j yields:

$$\partial_{p^j}(T^i_2) = \left(\frac{g^{ii}\delta^{kj}\rho_{ji}}{p^0} - \frac{g^{ii}p^k\rho_{ki}g^{jj}p^j}{(p^0)^3}\right)\left(\int_{\mathbb{R}^3} f(t,\bar{p})abce^{-2x}\,d\bar{p} + \frac{F^{10}}{4\pi}\right).$$

Since g^{ii} are regular functions from the metric and assumed bounded, and ρ_{ji} are constants, the first factor is bounded. Furthermore, the integral term is also bounded since $f(t) \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, ensuring that $\partial_{p^j}(T^i_2)$ is bounded.

Similarly, for T^1_1, differentiating and simplifying yields:

$$\partial_{p^j}(T^1_1) = -2\frac{\dot{a}}{a}\delta^{j1} - \frac{b^2e^{-2x}}{a^2}\left[\frac{2p^2\delta^{j2}}{p^0} - \frac{(p^2)^2g^{jj}p^j}{(p^0)^3}\right].$$

Since $\dot{a}/a$ is bounded, and all terms involving p^j are controlled, $\partial_{p^j}(T^1_1)$ is bounded.

By similar arguments, we establish the boundedness of $\partial^2_{p^ip^j}(\tilde{P}^i)(1+|\bar{p}|)$ and $\partial^3_{p^ip^jp^k}(\tilde{P}^i)(1+|\bar{p}|)^2$, completing the proof. □

4.2. *Recursive construction of approximate solutions*

Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions constructed recursively to approximate the solution of the Boltzmann equation. At each step n, we solve the linearized equation for f_{n+1}, fixing f_n in the non-linear term of the collision operator $Q(f,f)$. The linearized equation takes the form:

$$\frac{\partial f_{n+1}}{\partial t} + \tilde{P}^i_n\frac{\partial f_{n+1}}{\partial p^i} = \frac{1}{p^0}Q(f_n,f_n), \tag{63}$$

with initial conditions $f_{n+1}(0,\bar{p}) = f_0(\bar{p})$.

We define the approximate solution f_{n+1} as a linear combination of a Hilbert basis $(v_k)_{k\in\mathbb{N}^*}$ of $H^3_d(\mathbb{R}^3_{\bar{p}})$, following the approach in [1]:

$$f^M_{n+1}(t,\bar{p}) = \sum_{k=1}^{M} \lambda^M_k(t) v_k(\bar{p}), \quad M \in \mathbb{N}^*, \tag{64}$$

where the coefficients $\lambda^M_k(t)$ are determined by solving the system of ordinary differential equations:

$$\left(\frac{\partial f^M_{n+1}}{\partial t}/v_j\right) + \left(\tilde{P}^i_n \frac{\partial f^M_{n+1}}{\partial p^i}/v_j\right) = \left(\frac{1}{p^0} Q(f_n,f_n)/v_j\right). \tag{65}$$

Since M is arbitrary, we conclude that:

$$\frac{\partial f^M_{n+1}}{\partial t} + \tilde{P}^i_n \frac{\partial f^M_{n+1}}{\partial p^i} = \frac{1}{p^0} Q(f_n,f_n), \tag{66}$$

showing that f^M_{n+1} is a solution of the linearized Boltzmann equation (63).

We now establish key energy estimates to ensure the boundedness and convergence of the approximations.

Proposition 4.2. *Let $d \in \left]\frac{5}{2},+\infty\right[$, and suppose $f^M_{n+1} \in H^3_d(\mathbb{R}^3)$. For multi-indices $\alpha,\beta \in \mathbb{N}^3$ with $|\alpha| \le |\beta| \le 3$, we have the estimate:*

$$\begin{aligned}
&\left|\left\langle (1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}\left[\tilde{P}^i_n \partial_{p^i} f^M_{n+1}\right], (1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1}\right\rangle_{L^2(\mathbb{R}^3)}\right| \\
&\qquad \le C\left(\sum_{|\alpha|\le|\beta|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial^\alpha_{\bar{p}} f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)}\right)\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Proof. The proof follows from Lemmas 4.3 and 4.4 below: □

Lemma 4.3. *Under the assumptions of Proposition 4.2, we have:*

$$\left|\left\langle (1+|\bar{p}|)^d \tilde{P}^i_n \partial_{p^i} f^M_{n+1}, (1+|\bar{p}|)^d f^M_{n+1}\right\rangle_{L^2(\mathbb{R}^3)}\right| \le C\left\|(1+|\bar{p}|)^d f^M_{n+1}\right\|^2_{L^2(\mathbb{R}^3)}.$$

Proof. Using the product differentiation rule, we obtain:

$$\partial_{p^i}\left((1+|\bar{p}|)^d f^M_{n+1}\right) = \partial_{p^i}(1+|\bar{p}|)^d f^M_{n+1} + (1+|\bar{p}|)^d \partial_{p^i} f^M_{n+1}.$$

Thus, the inner product decomposes as:

$$\left\langle \tilde{P}^i_n (1+|\bar{p}|)^d \partial_{p^i} f^M_{n+1}, (1+|\bar{p}|)^d f^M_{n+1}\right\rangle = B_1 + B_2,$$

where B_1 and B_2 are given by:

$$\begin{aligned}
B_1 &= \left\langle \tilde{P}^i_n \partial_{p^i}\left((1+|\bar{p}|)^d f^M_{n+1}\right), (1+|\bar{p}|)^d f^M_{n+1}\right\rangle, \\
B_2 &= -\left\langle \tilde{P}^i_n \partial_{p^i}(1+|\bar{p}|)^d f^M_{n+1}, (1+|\bar{p}|)^d f^M_{n+1}\right\rangle.
\end{aligned}$$

For B_1, integrating by parts and using symmetry, we obtain:

$$B_1 = -\frac{1}{2}\left\langle \partial_{p^i}\tilde{P}^i_n(1+|\bar{p}|)^d f^M_{n+1}, (1+|\bar{p}|)^d f^M_{n+1}\right\rangle.$$

Thus, applying supremum estimates, we deduce:

$$|B_1| \le C\|(1+|\bar{p}|)^d f^M_{n+1}\|^2_{L^2}.$$

For B_2, since

$$\partial_{p^i}(1+|\bar{p}|)^d = \frac{dp^i(1+|\bar{p}|)^{d-1}}{|\bar{p}|},$$

we similarly obtain:

$$|B_2| \le C\|(1+|\bar{p}|)^d f^M_{n+1}\|^2_{L^2}.$$

Thus, we conclude:

$$\left|\left\langle (1+|\bar{p}|)^d \tilde{P}^i_n \partial_{p^i} f^M_{n+1}, (1+|\bar{p}|)^d f^M_{n+1}\right\rangle\right| \le C\|(1+|\bar{p}|)^d f^M_{n+1}\|^2_{L^2}.$$

□

Lemma 4.4. *Let $d \in [\frac{5}{2}, +\infty)$ be a real number, and let $f^M_{n+1} \in H^3_d(\mathbb{R}^3_{\bar{p}})$ with $\beta \in \mathbb{N}^3$, $|\beta| \le 3$. If*

$$\begin{aligned}&\left|\left\langle (1+|\bar{p}|)^{d+|\beta|}\partial_{p^j}\left[\tilde{P}^i_n\partial_{p^i} f^M_{n+1}\right], (1+|\bar{p}|)^{d+|\beta|}\partial_{p^j} f^M_{n+1}\right\rangle_{L^2(\mathbb{R}^3)}\right| \\ &\quad \le C\left(\sum_{|\alpha|\le|\beta|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial^\alpha_{\bar{p}} f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)}\right)\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_{p^j} f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)},\end{aligned}$$

then for all $\beta' \in \mathbb{N}^3$ with $|\beta'| = 1$, we have:

$$\begin{aligned}&\left\langle (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{\bar{p}}\left[\tilde{P}^i_n\frac{\partial f^M_{n+1}}{\partial p^i}\right], (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{\bar{p}} f^M_{n+1}\right\rangle_{L^2(\mathbb{R}^3)} \\ &\quad \le C\left(\sum_{|\alpha|\le|\beta+\beta'|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial^\alpha_{\bar{p}} f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)}\right)\left\|(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{\bar{p}} f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Proof. The proof follows by induction. The base case is established in Lemma 4.3.

We define $\lambda = \beta + \beta'$. Using Leibniz's rule for multi-index derivatives, we obtain:

$$\partial^\lambda_{\bar{p}}\left[\tilde{P}^i_n\frac{\partial f^M_{n+1}}{\partial p^i}\right] = \sum_{k\le\lambda} C^k_\lambda \partial^k_{\bar{p}}\tilde{P}^i_n\partial^{\lambda-k}_{\bar{p}}\left(\frac{\partial f^M_{n+1}}{\partial p^i}\right).$$

Taking the weighted L^2 inner product yields:

$$\left|\left\langle (1+|\bar{p}|)^{d+|\beta|+1}\partial^\lambda_{\bar{p}}\left[\tilde{P}^i_n\frac{\partial f^M_{n+1}}{\partial p^i}\right], (1+|\bar{p}|)^{d+|\beta|+1}\partial^\lambda_{\bar{p}} f^M_{n+1}\right\rangle\right| \le K_1 + K_2, \tag{67}$$

where:

$$K_1 = \left\langle (1+|\bar{p}|)^{d+|\beta|+1} \tilde{P}^i_n \partial^\lambda_{\bar{p}} \left[\frac{\partial f^M_{n+1}}{\partial p^i} \right], (1+|\bar{p}|)^{d+|\beta|+1} \partial^\lambda_{\bar{p}} f^M_{n+1} \right\rangle,$$

$$K_2 = \left\langle (1+|\bar{p}|)^{d+|\beta|+1} \sum_{k \leq \lambda, |k| \geq 1} C^k_\lambda \partial^k_{\bar{p}} \tilde{P}^i_n \partial^{\lambda-k}_{\bar{p}} \left(\frac{\partial f^M_{n+1}}{\partial p^i} \right), (1+|\bar{p}|)^{d+|\beta|+1} \partial^\lambda_{\bar{p}} f^M_{n+1} \right\rangle.$$

For K_1, differentiating the weight function yields:

$$|K_1| \leq |K_1'| + |K_1''|,$$

where:

$$K_1' = \left\langle \tilde{P^i}_n \partial_{p^i} \left[(1+|\bar{p}|)^{d+|\beta|+1} \partial^\lambda_{\bar{p}} f^M_{n+1} \right], (1+|\bar{p}|)^{d+|\beta|+1} \partial^\lambda_{\bar{p}} f^M_{n+1} \right\rangle,$$

$$K_1'' = \left\langle \tilde{P^i}_n \partial_{p^i} \left[(1+|\bar{p}|)^{d+|\beta|+1} \right] \partial^\lambda_{\bar{p}} f^M_{n+1}, (1+|\bar{p}|)^{d+|\beta|+1} \partial^\lambda_{\bar{p}} f^M_{n+1} \right\rangle.$$

Using boundedness of $\tilde{P}^i$ and the derivative estimate:

$$|K_1| \leq C \left\| (1+|\bar{p}|)^{d+|\beta|+1} \partial^\lambda_{\bar{p}} f^M_{n+1} \right\|^2_{L^2(\mathbb{R}^3)}. \tag{68}$$

This concludes the estimate for K_1.

Estimation of K_2

From Lemma (4.1), the function $(1+|\bar{p}|)^{|k|-1} \partial^k_{\bar{p}} \tilde{P}^i$ is bounded. Consequently, we estimate:

$$|K_2| \leq C \sum_{k \leq \lambda, |k| \geq 1} C^k_\lambda \left| \left\langle (1+|\bar{p}|)^{d+|\beta|-|k|+2} \partial^{\lambda-k}_{\bar{p}} \left(\frac{\partial f^M_{n+1}}{\partial p^i} \right), (1+|\bar{p}|)^{d+|\beta|+1} \partial^{\beta+\beta'}_{\bar{p}} f^M_{n+1} \right\rangle_{L^2(\mathbb{R}^3)} \right|.$$

Introducing $\gamma \in \mathbb{N}^3$ with $|\gamma| = 1$, we rewrite:

$$|K_2| \leq C \sum_{k \leq \lambda, |k| \geq 1} C^k_\lambda \left\| (1+|\bar{p}|)^{d+|\beta|-|k|+2} \partial^{\lambda+\gamma-k}_{\bar{p}} f^M_{n+1} \right\|_{L^2(\mathbb{R}^3)} \left\| (1+|\bar{p}|)^{d+|\beta|+1} \partial^{\beta+\beta'} f^M_{n+1} \right\|_{L^2(\mathbb{R}^3)}. \tag{69}$$

Applying inequalities (67), (68), and (69), the proof of Lemma (4.4) follows.

Finally, considering cases $|\beta| = 0, 1, 2, 3$, yields to the proof of Proposition (4.2). □

We establish a uniform bound on the weighted Sobolev norm of the iterates in the approximation scheme for the Boltzmann equation.

Proposition 4.5. *Let $d \in [\frac{5}{2}, +\infty)$ and $f^M_{n+1} \in H^3_d(\mathbb{R}^3)$. For any multi-index $\beta \in \mathbb{N}^3$ such that $|\beta| \leq 3$, we have the uniform estimate:*

$$\|(1+|\bar{p}|)^{d+|\beta|} \partial^\beta_{\bar{p}} f^M_{n+1}\|_{L^2(\mathbb{R}^3)} \leq C, \tag{70}$$

where C is a constant independent of n and M.

Proof. From Proposition (3.10), we have:

$$\frac{1}{p^0}Q(f_n, f_n) \in H^3_d((0,T)\times\mathbb{R}^3). \tag{71}$$

Since $f_n \in H^3_{d,r}(\mathbb{R}^3)$, it follows from the definition of $H^3_d(\mathbb{R}^3)$ that:

$$(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}\left(\frac{1}{p^0}Q(f_n, f_n)\right) \in L^2(\mathbb{R}^3), \quad |\beta| \le 3. \tag{72}$$

Additionally, as $f^M_{n+1} \in H^3_d(\mathbb{R}^3)$, we also have:

$$(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1} \in L^2(\mathbb{R}^3). \tag{73}$$

Taking the inner product in $L^2(\mathbb{R}^3)$ and using equation (66), we obtain:

$$\begin{aligned}&\left\langle (1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}\left[\partial_t f^M_{n+1} + \tilde{P}^i_n \partial_{p^i} f^M_{n+1}\right], (1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1}\right\rangle_{L^2(\mathbb{R}^3)} \\ &= \left\langle (1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}\left(\frac{1}{p^0}Q(f_n, f_n)\right), (1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1}\right\rangle_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{74}$$

Introducing Propositions (4.2) and (4.5) into (74), we obtain after simplification:

$$\begin{aligned}\frac{d}{dt}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta(\bar{p},\tilde{q}) f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)} &\le C\left(\sum_{|\alpha|\le|\beta|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial^\alpha(\bar{p},\tilde{q}) f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)}\right) \\ &\quad + \left\|\frac{1}{p^0}L(f_n, f_n)\right\|_{H^3_d(\mathbb{R}^3)}.\end{aligned} \tag{75}$$

Summing (75) over $|\beta| = 0, 1, 2, 3$, we obtain:

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{|\beta|\le 3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)}\right) \\ &\le C\left(\sum_{|\beta|\le 3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1}\right\|_{L^2(\mathbb{R}^3)} + \left\|\frac{1}{p^0}Q(f_n, f_n)\right\|_{H^3_d(\mathbb{R}^3)}\right).\end{aligned} \tag{76}$$

Applying integration by parts and the Cauchy-Schwarz inequality, we obtain a differential inequality for the weighted norm of f^M_{n+1}. Using Proposition (4.2), this leads to:

$$\frac{d}{dt}\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1}\|^2_{L^2(\mathbb{R}^3)} \le C\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}} f^M_{n+1}\|_{L^2(\mathbb{R}^3)}. \tag{77}$$

By Gronwall's lemma, we conclude that the weighted Sobolev norm remains uniformly bounded, completing the proof. □

Theorem 4.6. *Let $f_0 \in H^3_{d,r}(\mathbb{R}^3)$, and let $f_n \in H^3_{d,r}(0,T,\mathbb{R}^3)$ be given. For fixed parameters $(\tilde{a}, \tilde{b}, \tilde{c})$, the linearized Boltzmann equation* (63) *admits a unique local weak solution in $H^3_d(0,T,\mathbb{R}^3)$.*

Proof. The proof consists of two steps:

Existence: Using Proposition (4.5), the sequence $\{f_{n+1}^M\}_M$ is bounded in the reflexive Hilbert space $H_d^3(0,T,\mathbb{R}^3)$. Hence, by the Banach-Alaoglu theorem, there exists a subsequence $\{f_{n+1}^{M_l}\}_{M_l}$ that converges weakly to f_{n+1} in $H_d^3(0,T,\mathbb{R}^3)$. This ensures that f_{n+1} satisfies the linearized Boltzmann equation (22) with initial condition $f_{n+1}(0,\bar{p}) = f_0$. **Uniqueness:** Suppose f_1 and f_2 are two solutions of (63) with the same initial data f_0. Define $f = f_1 - f_2$, which satisfies the homogeneous problem:

$$\frac{\partial f}{\partial t} + \tilde{P}^i \frac{\partial f}{\partial p^i} = 0, \quad f(0) = 0. \tag{78}$$

Using an energy estimate similar to Proposition 4.5, we obtain

$$\|f\|_{H_d^3} \leq 0 \Rightarrow f \equiv 0. \tag{79}$$

Thus, uniqueness is established. ☐

Remark 4.7. *Since $H_d^3(\mathbb{R}^3) \hookrightarrow C^1(\mathbb{R}^3)$, we observe that any weak solution belonging to $H_d^3(\mathbb{R}^3)$ is in $C^1(\mathbb{R}^3)$ and thus qualifies as a strong solution.*

Theorem 4.8. *Let $f_0 \in H_{d,r}^3(\mathbb{R}^3)$, with fixed $\tilde{g} = (\tilde{a}, \tilde{b}, \tilde{c})$. The Boltzmann equation* (44)

admits a unique local solution $f \in H_{d,r}^3(\mathbb{R}^3)$ such that $f(0,\bar{p}) = f_0(\bar{p})$.

Proof. From the linearized Boltzmann equation (63), we define the following Cauchy problems:

$$(P_n)_{n\in\mathbb{N}^*} : \begin{cases} \frac{\partial f_{n+1}}{\partial t} + \tilde{P}^i{}_n \frac{\partial f_{n+1}}{\partial p_i} = \frac{1}{p^0} Q(f_n, f_n), \\ f_{n+1}(0,\bar{p}) = f_0. \end{cases} \tag{80}$$

We use Theorem (4.6) to construct a sequence $(f_n)_n$ of solutions for the Cauchy problems $(P_n)_{n\in\mathbb{N}^*}$.

For f_0, there exists a unique and bounded solution f_1 for the Cauchy problem (P_0) in $H_d^3(\mathbb{R}^3)$. Similarly, for f_1, there exists a unique and bounded solution f_2 for (P_1) in $H_d^3(\mathbb{R}^3)$. Thus, recursively, we construct the sequence $(f_n)_{n\in\mathbb{N}} \subset H_d^3(\mathbb{R}^3)$ of solutions for the Cauchy problems $(P_n)_{n\in\mathbb{N}^*}$.

Existence: We need to prove that the sequence (f_n) is bounded in $H_d^3(\mathbb{R}^3)$.

Suppose $\|f_n\|_{H_d^3} \leq r$. From (77) and (76), we get:

$$\frac{d}{dt}\left(\sum_{|\beta|\leq 3} \left\|(1+|\bar{p}|)^{d+|\beta|} \partial_{\bar{p}}^{\beta} f_{n+1}^M\right\|_{L^2(\mathbb{R}^3)}\right) \leq C\left(\|f_0\|_{H_d^3(\mathbb{R}^3)} + T\|f_n\|^2_{H_d^3(\mathbb{R}^3)} + \left\|\frac{1}{p^0} Q(f_n,f_n)\right\|_{H_d^3(\mathbb{R}^3)}\right). \tag{81}$$

Integrating this inequality over $[0,t]$ for $0 \leq t < T$ and using $\left\|\frac{1}{p^0}Q(f_n,f_n)\right\|_{H_d^3(\mathbb{R}^3)} \leq r^2$, we obtain:

$$\|f_{n+1}^M\|_{H_d^3(\mathbb{R}^3)} \leq \|f_0\|_{H_d^3(\mathbb{R}^3)} + C\left(\|f_0\|_{H_d^3(\mathbb{R}^3)} T + r^2 T^2\right). \tag{82}$$

If we take $f_0 \in H_{d,r}^3(\mathbb{R}^3)$ and $T > 0$ such that:

$$\|f_0\|_{H_d^3(\mathbb{R}^3)} \leq \frac{r}{2}, \quad \text{and} \quad C\left(\|f_0\|_{H_d^3(\mathbb{R}^3)} T + r^2 T^2\right) \leq \frac{r}{2}, \tag{83}$$

the bound $\|f^M_{n+1}\|_{H^3_d(\mathbb{R}^3)} \leq r$ follows. By Theorem (4.6), we have $f^M_{n+1} + f_{n+1}$ in $H^3_d(\mathbb{R}^3)$, implying that $\|f_{n+1}\|_{H^3_d(\mathbb{R}^3)} \leq r$. Hence, the sequence $(f_n)_n$ is bounded.

Since $H^3_d(\mathbb{R}^3)$ is a reflexive Hilbert space, we can extract a weakly convergent subsequence (f_{n_k}) that converges weakly to a solution f of the Boltzmann equation (44) in $H^3_d(\mathbb{R}^3)$ with $f(0,\bar{p}) = f_0$.

Uniqueness: Let f and g be two solutions of the Boltzmann equation (44) in $H^3_{d,r}(\mathbb{R}^3)$ with the same initial data f_0. Setting $H = f - g$, we obtain:

$$\begin{cases} \frac{\partial H}{\partial t} + \tilde{P}^i \frac{\partial H}{\partial p_i} = \frac{1}{p^0} Q(f,H) + \frac{1}{p^0} Q(H,g), \\ H(0) = 0. \end{cases} \tag{84}$$

Using (76), we get:

$$\begin{aligned} &\frac{d}{dt} \sum_{|\beta| \leq 3} \left\| (1+|\bar{p}|)^{d+|\beta|} D^\beta_{\bar{p}} H \right\|_{L^2(\mathbb{R}^3)} \\ &\leq C \sum_{|\beta| \leq 3} \left\| (1+|\bar{p}|)^{d+|\beta|} D^\beta_{\bar{p}} H \right\|_{L^2(\mathbb{R}^3)} + \left\| (1+|\bar{p}|)^{d+|\beta|} D^\beta_{\bar{p}} \left(\frac{1}{p^0} Q(f,H) + \frac{1}{p^0} Q(H,g) \right) \right\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{85}$$

Integrating over $[0,t]$ and applying Gronwall's lemma, we obtain:

$$\|H\|_{H^3_d(\mathbb{R}^3)} \leq CT \|H\|_{H^3_d(\mathbb{R}^3)}. \tag{86}$$

If $T > 0$ is chosen such that $T \times C < 1$, then $\|H\|_{H^3_d(\mathbb{R}^3)} = 0$, leading to $f = g$. □

Theorem 4.9. *Let $f_0 \in H^3_{d,r}(\mathbb{R}^3)$ be given. The solution (f, F) of the Maxwell-Boltzmann system in the Bianchi type III spacetime obtained in Theorem* (4.8)*, satisfies the following estimates:*

$$\|f\|_{H^3_d(\mathbb{R}^3)} \leq C \|f_0\|_{H^3_d(\mathbb{R}^3)}, \quad \|F\|_{C([0,T],\mathbb{R}^{16})} \leq C \|F_0\|_{C([0,T],\mathbb{R}^{16})}. \tag{87}$$

Proof. By applying inequality (82) from Theorem (4.8) and taking the limit as $M \to \infty$, we obtain:

$$\|f_{n+1}\|_{H^3_d(\mathbb{R}^3)} \leq \|f_0\|_{H^3_d(\mathbb{R}^3)} + C \left(\|f_0\|_{H^3_d(\mathbb{R}^3)} T + r^2 T^2 \right). \tag{88}$$

Since the sequence (f_n) converges to f as guaranteed by Theorem (4.8), the previous inequality implies:

$$\|f\|_{H^3_d(\mathbb{R}^3)} \leq \|f_0\|_{H^3_d(\mathbb{R}^3)} + C \left(\|f_0\|_{H^3_d(\mathbb{R}^3)} T + r^2 T^2 \right). \tag{89}$$

The function $C = C(a_0, b_0, c_0, r, T, e)$ depends continuously on its parameters, which are themselves continuous functions of time on the compact interval $[0,T]$. Consequently, there exists a constant C_0 that represents the maximum value of C over this interval. This allows us to rewrite (89) as:

$$\|f\|_{H^3_d(\mathbb{R}^3)} \leq \|f_0\|_{H^3_d(\mathbb{R}^3)} + C_0 T \left(\|f_0\|_{H^3_d(\mathbb{R}^3)} + r^2 T \right). \tag{90}$$

By selecting T such that $Tr^2 \leq \|f_0\|_{H^3_d(\mathbb{R}^3)}$, we derive the desired bound:

$$\|f\|_{H^3_d(\mathbb{R}^3)} \leq C \|f_0\|_{H^3_d(\mathbb{R}^3)}. \tag{91}$$

Moreover, from (29), we have:

$$|F^{0i}(t)| = \left|\frac{a_0 b_0 c_0}{abc} F^{0i}(0)\right| \le \left|\frac{a_0 b_0 c_0}{abc}\right| |F^{0i}(0)|. \tag{92}$$

Define the constant $C = \sup_{t\in[0,T]} \left|\frac{a_0 b_0 c_0}{abc}\right|$, then we immediately have:

$$\sup_{t\in[0,T]} |F^{0i}(t)| \le C \sup_{t\in[0,T]} |F^{0i}(0)|. \tag{93}$$

Also, (23) yields:

$$\sup_{t\in[0,T]} |F_{ij}(t)| = \sup_{t\in[0,T]} |F_{ij}(0)|. \tag{94}$$

From (93) and (94), we obtain:

$$\|F(t)\|_{\mathbb{R}^{16}} \le C\|F(0)\|_{\mathbb{R}^{16}}. \tag{95}$$

Finally, taking the sup over $t \in [0, T]$ yields:

$$\sup_{t\in[0,T]} \|F(t)\|_{\mathbb{R}^{16}} \le C \sup_{t\in[0,T]} \|F(0)\|_{\mathbb{R}^{16}}, \tag{96}$$

which completes the proof of Theorem (4.9). □

5. Conclusion

We have demonstrated the well-posedness of the magnetized Boltzmann equation in a Bianchi type III space-time.By employing a rigorous functional analysis framework, we established the existence and uniqueness of solutions in the weighted Sobolev space $H^3_d(0, T, \mathbb{R}^3)$. Through careful estimates and compactness arguments, we ensured the stability of our solution.

Furthermore, we showed that our solution has a good regularity by leveraging the embedding $H^3_d(\mathbb{R}^3) \hookrightarrow C^1(\mathbb{R}^3)$, as highlighted in Remark 4.7. These findings provide a solid foundation for further investigations into the magnetized Boltzmann equation in anisotropic relativistic settings. Future work will investigate the coupling with the Einstein system and analyze its asymptotic behavior.

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