

# Zip Shift Encoding of $m$ -to-1 Local Homeomorphisms

## Zip-Shift codage des $M$ -TO-1 homéomorphismes locaux

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(In memory of my student Carlos Freitas Dias)

**ABSTRACT.** We develop topological partitions for  $m$ -to-1 local homeomorphisms on compact metric spaces—maps that arise naturally in non-invertible dynamical systems, such as expanding and covering maps. These partitions enable a symbolic representation of the dynamics via the zip shift, an extended bilateral shift in the non-invertible setting. Inspired by Smale's horseshoe construction, this approach generalizes topological partitions to a broader class of systems and opens new directions for studying their topological and ergodic properties.

**2010 Mathematics Subject Classification.** Primary 37B10; Secondary 37D05, 37C29.

**KEYWORDS.** zip shift, topological partition, local homeomorphisms

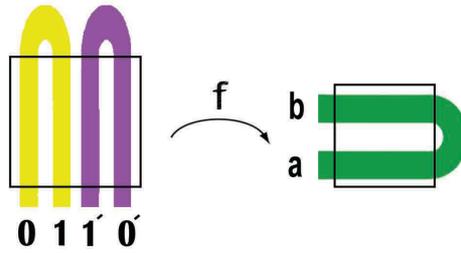
### 1. Introduction

One can study the dynamics of a discrete system by finding a symbolic encoding—a mapping of the phase space to sequences of symbols where the system's evolution becomes conjugate or semi-conjugate to a shift map. The natural way to associate a symbolic sequence with an orbit is to trace its orbit through a partition [1]. A symbolic code requires a partition with specific properties. For invertible dynamics, topological and Markov partitions have been developed since 1967 through the works of Smale [14], Anosov [2], Berg [4], Sinai [13], Fathi [6], Weiss [15], Adler [1], et al. These topological and Markov partitions, which are foundational to smooth hyperbolic theory, establish a key connection to the equilibrium theory of statistical mechanics, as developed by mathematicians such as Bowen [5], Ruelle [12], et al.

The absence of a continuation of these works over the past years, from both historical and structural perspectives, is noteworthy. In particular, the transition from the context of diffeomorphisms to that of local diffeomorphisms, using similar approaches, can be an important development. In this work we aim to develop a topological partition that codifies certain non-invertible dynamical systems through a new symbolic dynamics. As is well known, determining the behavior of dynamical systems with chaotic characteristics is not an easy task. The construction of a 1-to-1 horseshoe map by S. Smale [14] in 1967 paves the way for this endeavor and provides a framework for studying such dynamics using symbolic dynamics. In [9, 10], the authors present the construction and codifying of  $n$ -to-1 Smale horseshoe [9] (see Figure 1 for a 2-to-1 horseshoe), which introduced this extended symbolic dynamic known as zip shift. These  $n$ -to-1 horseshoe maps, by their construction, represent an intrinsic example of a Markov partition for non-invertible dynamics. We anticipate that this initial work represents a new beginning that will enable us, in the near future, to establish a comprehensive extended definition and construction of Markov partitions for encoding specific classes of hyperbolic dynamics.

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**Figure 1.** The 2-to-1 Smale type horseshoe

## 2. Terminology

Let  $X$  be a compact metric space, and let  $\{X_i\}_{i=1}^N$  be a disjoint collection of open connected subsets of  $X$ , such that  $\bigcup_{i=1}^N \overline{X_i} \subseteq X$ . A surjective continuous map  $\phi : X \rightarrow X$  is called an  $m$ -to-1 local homeomorphism if there exist disjoint sets  $X_i \subseteq X$  for  $i = 1, \dots, m$ , such that the map  $\phi|_{X_i} : X_i \rightarrow \phi(X_i)$  is a homeomorphism for each  $i = 1, \dots, m$ . Here, the sets  $X_i$  are assumed to be the same sets introduced earlier and the union of their closure covers or are included in  $X$ .

**Definition 2.1** (Principal Domain). We say that  $Y \subset X$  is a *principal domain* when  $\phi(Y) = X$  and  $\phi|_Y : Y \rightarrow X$  is a homeomorphism.

For  $m$ -covering maps defined on a connected compact metric space,  $\phi^{-1}(X)$  induces  $m$  principal domains (Theorem 22.1 of [3]).

**Definition 2.2.** Let  $(X, d)$  be a compact metric space. Then  $\mathcal{P} = \{P_0, \dots, P_{N-1}\}$  is a “*topological partition*” if:

- each  $P_i$  is open;
- $P_i \cap P_j = \emptyset$  for  $i \neq j$ ;
- $X = \bigcup_i \overline{P_i}$

**Definition 2.3.** Let  $R$  and  $S$  represent two finite coverings of  $X$  such that  $X = \bigcup_i \overline{R_i}$  and  $X = \bigcup_j \overline{S_j}$ , where  $R = \{R_0, R_1, \dots, R_{N-1}\}$  and  $S = \{S_0, S_1, \dots, S_{M-1}\}$ . We define their common topological refinement  $R \vee S$  as,

$$R \vee S = \{R_i \cap S_j : R_i \in R, S_j \in S\}.$$

Note that the common topological refinement of two topological partitions is a topological partition [1, Proposition 5.3].

**Definition 2.4.** Let  $\mathcal{R} = \{R_1, \dots, R_k\}$  and  $\mathcal{R}'$  be two topological partitions. We say that  $\mathcal{R}' = \{R'_1, \dots, R'_m\}$  is finer than  $\mathcal{R}$  if  $|\mathcal{R}'| \geq |\mathcal{R}|$  and  $\forall R'_i \in \mathcal{R}', \exists R_j \in \mathcal{R} : R'_i \subseteq R_j$ .

Let  $\mathcal{R}$  be a topological partition. Then define

$$\text{diam}(\mathcal{R}) := \max_{R_i \in \mathcal{R}} \sup_{x, y \in R_i} d(x, y).$$

**Definition 2.5.** For a dynamical system  $(X, \phi)$ , let

$$\bigvee_{k=-n}^n \phi^{-k}(\mathcal{R}) := \left\{ \bigcap_{k=-n}^n \phi^{-k}(R_k) : R_k \in \mathcal{R} \right\}.$$

Then  $\mathcal{R}$  is a “generator” if  $\lim_{n \rightarrow \infty} \text{diam}(\bigvee_{-n}^n \phi^{-k}(\mathcal{R})) = 0$ .

Note that If  $\mathcal{R}$  is a generator and  $\mathcal{A}$  represents a set of finite alphabets, then  $\lim_{n \rightarrow \infty} \text{diam}(\bigcap_{-n}^n \phi^{-k}(R_{s_k})) = 0$  for  $R_{s_k} \in \mathcal{R}, (s_k)_{k \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ .

In [11] we study the S-expansivity of m-to-1 local homeomorphisms and we show that for such maps, having a generator is equivalent to their S-expansivity.

**Proposition 2.6.** Let  $\mathcal{R}$  be a topological partition which is a generator. Then any topological partition  $\mathcal{R}'$  which is finer than  $\mathcal{R}$  is also a generator.

*Proof.* By Definitions 2.4 and 2.5, when  $\mathcal{R}'$  is finer than  $\mathcal{R}$ , for all  $n$ , the  $\text{diam}(\bigvee_{-n}^n \phi^{-k}(\mathcal{R}')) \leq \text{diam}(\bigvee_{-n}^n \phi^{-k}(\mathcal{R}))$ . Therefore,

$$\lim_{n \rightarrow \infty} \text{diam}(\bigvee_{-n}^n \phi^{-k}(\mathcal{R}')) = 0.$$

■

### 3. Zip shift space

Consider two finite sets of alphabets  $\mathcal{S} = \{a_1, \dots, a_M\}$  and  $\mathcal{S}' = \{0, \dots, N-1\}$  with  $M \leq N$ . Let  $\tau : \mathcal{S}' \rightarrow \mathcal{S}$  be an onto map (not necessarily invertible). Let  $Y = \{y = (y_i)_{i \in \mathbb{Z}}, y_i \in \mathcal{S}'\}$  be a two-sided shift space (see [7] for a definition) with its homeomorphism shift map  $\sigma$ . To any point  $y \in Y$ , correspond a point  $x = (x_i)_{i \in \mathbb{Z}} = (\dots, x_{-2}, x_{-1}; x_0, x_1, x_2, \dots)$  such that

$$x_i = \begin{cases} y_i \in \mathcal{S}' & \forall i \geq 0 \\ \tau(y_i) \in \mathcal{S} & \forall i < 0. \end{cases} \quad (3.1)$$

According to set  $Y$ , define the *zip space* as the set

$$\Sigma_{\mathcal{S}, \mathcal{S}'} := \{x = (x_i)_{i \in \mathbb{Z}} : x_i \text{ satisfies (3.1)}\}.$$

Note that in the construction of  $\Sigma_{\mathcal{S}, \mathcal{S}'}$ , the symbols for negative indices are mapped by  $\tau$ , whereas for nonnegative indices, they remain in  $\mathcal{S}'$ . This collapse phenomenon, applied by  $\tau$ , in fact provides the main construction for the branching of pre-images and makes possible the definition of a “shift-like” map, which is a local homeomorphism (see the definition of *zip shift map* in the proceeding lines).

Let  $M : \Sigma \rightarrow \mathbb{N} \cup \{0\}$  be given as,

$$M(x, y) = \begin{cases} \infty, & \text{if } x = y \\ \min\{|i|; x_i \neq y_i\}. & \text{if } x \neq y \end{cases}$$

We equip  $\Sigma_{\mathcal{S}, \mathcal{S}'}$  with a distance  $\bar{d}$ :

$$\bar{d}(x, y) := \frac{1}{2^{M(x, y)}}.$$

Note that according to the definition of  $M(x, y)$ , we have  $\bar{d}(x, y) = 0$  iff  $x = y$ . The topology induced by  $(\Sigma_{\mathcal{S}, \mathcal{S}'}, \bar{d})$  is equivalent to the product topology on  $\Sigma_{\mathcal{S}, \mathcal{S}'}$ .

**Definition 3.1.** We define the *zip shift map* on  $\Sigma_{\mathcal{S},\mathcal{S}'}$  as follows:

$$(\sigma_{\tau}(x))_i = \begin{cases} \tau(x_0) & \text{if } i = -1 \\ x_{i+1} & \text{otherwise.} \end{cases}$$

Any closed subspace  $\Sigma \subseteq \Sigma_{\mathcal{S},\mathcal{S}'}$  which is invariant under  $\sigma_{\tau}$  is called a *sub zip shift space* or simply a *zip shift space* of  $(\Sigma_{\mathcal{S},\mathcal{S}'}, \sigma_{\tau})$ . Whenever  $Y$  is a full shift space,  $\Sigma = \Sigma_{\mathcal{S},\mathcal{S}'}$  is called a *full zip shift space*. The readers can find more details and information on SFT-zip shift spaces, K-step Markov spaces, and sofic spaces, which are different types of zip shift spaces, in [10].

**Example 3.2.** Let  $\mathcal{S}' = \{0, 0', 1, 1'\}$  and  $\mathcal{S} = \{a, b\}$  be the alphabet sets. Suppose  $\tau(0) = \tau(0') = a$  and  $\tau(1) = \tau(1') = b$ . Let  $x \in \Sigma_{\mathcal{S},\mathcal{S}'}$ , then

$$\begin{aligned} x &= (\dots \quad x_{-3} \quad x_{-2} \quad x_{-1} \quad ; \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad \dots) \\ &= (\dots \quad b \quad a \quad b \quad ; \quad \mathbf{1}' \quad 0' \quad 1 \quad 1 \quad 1' \quad \dots), \\ \sigma_{\tau}(x) &= (\dots \quad b \quad a \quad \mathbf{b} \quad \mathbf{b} \quad ; \quad 0' \quad 1 \quad 1 \quad 1' \quad \dots). \end{aligned}$$

The 2-to-1 Smale-type horseshoe shown in Figure 1 is topologically conjugate to the zip shift map represented in this example (see [9] for more details).

It is noteworthy that the zip shift map is a local homeomorphism that extends the notion of a two-sided shift homeomorphism [10]. Note that if one takes  $\mathcal{S} = \mathcal{S}'$  and  $\tau = Id$ ; then the zip shift map represents a two-sided shift map. In [8], the authors give different examples of finite-to-1 covering maps which are topologically conjugated with some appropriate zip shift maps. The main idea behind such conjugacies is based on appropriate topological and Markov partitions for such maps. Figure 1 shows an appropriate topological (Markov) partition for a 2-to-1 horseshoe map.

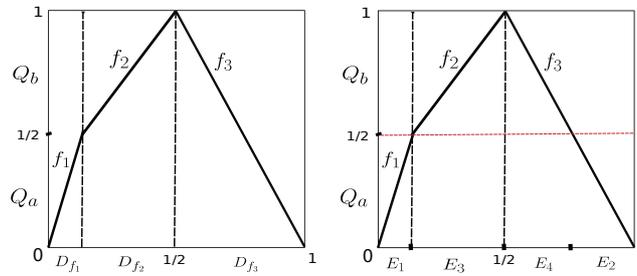
#### 4. Extended Topological Partition

In this section we assume all partitions being a finite partitions.

**Definition 4.1 (Domain topological partition).** Consider an  $m$ -to-1 local homeomorphism  $\phi$  defined on a compact connected metric space  $X$  with principal domains  $P_i, i = 1, \dots, m$ . Let  $\{P_{i_1}, \dots, P_{i_{k_i}}\}, k_i \in \mathbb{N}$  be a topological partition for  $P_i$  (see Definition 2.2), where each  $P_{i_j} \subseteq P_i$ . We define the *Domain Topological Partition (DTP)* as the collection of all elements in  $\mathcal{P} = \{P_{i_1}, \dots, P_{i_{k_i}}\}_{i=1}^m$  with  $k_j \in \mathbb{N}, 1 \leq j \leq m$ . (see figure 2)

**Definition 4.2 (Image topological partition).** Let  $\phi : X \rightarrow X$  be an  $m$ -to-1 local homeomorphism with a domain topological partition  $\mathcal{P} = \{P_{i_1}, \dots, P_{i_{k_i}}\}_{i=1}^m (k_j \in \mathbb{N}, 1 \leq j \leq m)$ . We say that  $\mathcal{Q} = \{Q_1, \dots, Q_k\}$  is an *Image topological partition (ITP)* associated with  $\mathcal{P}$ , if  $Q_j \in \bigvee \phi(\mathcal{P}) := \bigvee_{i_j} \phi(P_{i_j})$  with  $P_{i_j} \in \mathcal{P}$ , is a topological partition for  $X$ . Note that by definition  $\phi(P_i) = X$  where  $P_i$  is a principal domain. (see figure 2)

**Proposition 4.3.** Let  $\phi$  be an  $m$ -to-1 local homeomorphism on a connected compact metric space  $X$  and  $\mathcal{P} = \{P_{i_1}, \dots, P_{i_{k_i}}\}_{i=1}^m$  a DTP ( $k_j \in \mathbb{N}, 1 \leq j \leq m$ ). Then for all  $n, \phi^n(\mathcal{P})$  induces an image topological partition.



**Figure 2.** In left figure, the domain of the local functions  $\mathcal{R} = \{D_{f_1}, D_{f_2}, D_{f_3}\}$  is a DTP and  $\mathcal{Q} = \{Q_a, Q_b\}$  is its associated ITP. In right figure,  $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$  is the ETP obtained from  $\mathcal{Q}$ .

*Proof.* Let  $\mathcal{P} = \{P_{i_1}, \dots, P_{i_{k_i}}\}_{i=1}^m$  be the domain topological partition of  $X$ . By Definition 2.1 and Definition 4.1,  $\mathcal{P}$ , is a topological partition structurally based on principal domains  $P_i$ . This implies that  $\phi(P_i) = X$ . Since  $\cup_{j=1}^{k_i} \overline{P_{i_j}} = P_i$ , we have  $\phi(\cup_{j=1}^{k_i} \overline{P_{i_j}}) = X$  (overline means the closure). Indeed, for any  $i$ , the set  $\phi(P_i)$ , where  $P_i$  for  $i = 1, \dots, m$  is a principal domain, can be regarded as a topological partition for  $X$ . One can define  $\mathcal{Q}_1 = \bigvee_i \phi(P_i)$ . Let  $\mathcal{Q}_2 = \bigvee_i \phi^2(P_i)$ . Since  $\phi$  is an  $m$ -to-1 map,  $\phi^n$  becomes an  $m^n$ -to-1 map. Note that as  $\phi$  is a surjective map, if  $P_i$  is a principal domain for  $\phi$ , then it is also a principal domain for  $\phi^n$ . Therefore, by induction on  $n$ , one can define  $\mathcal{Q}_n = \bigvee_i \phi^n(\mathcal{P}) = \bigvee_i \phi(\mathcal{Q}_{n-1})$ , which is an image topological partition associated with  $\phi^n(\mathcal{P})$ . ■

*Remark 4.4.* Associated with any  $\mathcal{Q}$ , one can redefine a domain topological partition. To do so, it is enough to consider this new DTP, made up of the elements that are the pre-images of  $\mathcal{Q}$ . We call it the **Extended Topological Partition (ETP)** associated with  $\mathcal{Q}$ . The extended topological partitions are the main objects in the coding process (see figure 2).

**Proposition 4.5.** *Let  $\mathcal{R}$  be a domain topological partition and  $\phi : X \rightarrow X$  be an  $m$ -to-1 local homeomorphism. For every  $p \in X$ , there is a sequence  $O_p = (Q_{s_{k-1}}, R_{s_k})$  ( $k \geq 0$ , for  $R_{s_k}$  and  $k \leq 0$  for  $Q_{s_{k-1}}$ ) of sets in  $\mathcal{R}$  and  $\mathcal{Q}$ , such that*

$$p \in \left[ \bigcap_{-\infty}^0 \phi^{-k}(Q_{s_{k-1}}) \right] \cap \left[ \bigcap_0^{\infty} \phi^{-k}(R_{s_k}) \right].$$

Where  $\mathcal{Q}$  is the image topological partition associated to  $\mathcal{R}$ .

*Proof.* Assume that  $\phi : X \rightarrow X$  is an  $m$ -to-1 local homeomorphism and let  $\mathcal{R} = \{R_{i_1}, \dots, R_{i_{k_i}}\}_{i=1}^m$  with  $k_j \in \mathbb{N}, 1 \leq j \leq m$ , be a domain topological partition associated with the principal domains  $R_i, i = 1, 2, \dots, m$ . One obtains the image topological partition  $\mathcal{Q}$  associated with  $\mathcal{R}$  and defines an extended topological partition associated with  $\mathcal{Q}$ . Let  $\mathcal{Q} = \{Q_{a_1}, \dots, Q_{a_l}\}$ , and for the rest of the proof, let us assume that  $\mathcal{R}$  is the extended topological partition associated with  $\mathcal{Q}$ . Any element of  $\mathcal{Q}$  has  $m$  pre-images. Indeed,  $\mathcal{R}$  is a domain topological partition with  $lm$  elements, where  $l = \#(\mathcal{Q})$ . Let  $\mathcal{R} = \{R_1, R_2, \dots, R_{lm}\}$ . By Proposition 4.3, for all iterates  $k \geq 0$ ,  $\phi^k(\mathcal{R})$  induces an image topological partition  $\mathcal{Q}_k$ . For  $k \geq 0$ ,  $R_{s_k} \in \bigvee_{i=0}^k \phi^{-i}(\mathcal{R})$  (see Definition 4.2), which is a refinement of  $\mathcal{R}$ , and  $Q_{s_{-k-1}} \in \bigvee_{i=0}^k \phi^{i+1}(\mathcal{R}) = \bigvee_{i=0}^k \phi^i(\mathcal{Q})$ , which is a refinement of  $\mathcal{Q}$ .

Since  $\mathcal{Q} = \bigvee_i \phi(R_i)$ , one deduces some surjective map  $\tau_\phi : \mathcal{R} \rightarrow \mathcal{Q}$  between partition elements of  $\mathcal{R}$  and  $\mathcal{Q}$ . Using  $\tau_\phi$  one associates a partition representation to the orbit of  $p$ , denoted by  $O_p = (Q_{s_{-k-1}}, R_{s_k})$ , as follows.

- For all  $k \geq 0$ ;  $\exists R_{s_j} \in \mathcal{R}, 0 \leq j \leq k : p \in R_{s_0} \cap \phi^{-1}(R_{s_1}) \cap \dots \cap \phi^{-k}(R_{s_k})$ ; This is the same as saying that  $p \in R_{s_0}, \phi(p) \in R_{s_1}, \dots, \phi^k(p) \in R_{s_k}$ .
- for all  $k \geq 0$ ,

$$\exists Q_{s_{j-1}} \in \mathcal{Q}, -k \leq j \leq 0 : p \in Q_{s_{-1}} \cap \phi(Q_{s_{-1}}) \cap \dots \cap \phi^k(Q_{s_{-k}});$$

Since  $\mathcal{R}$  and  $\mathcal{Q}$  are topological partitions, their topological refinement is also a topological partition, therefore  $p \in [\bigvee_{k=-n}^0 \phi^{-k}(Q_{s_{k-1}})] \vee [\bigvee_0^n \phi^{-k}(R_{s_k})]$  for any refinement. In particular, one can conclude that  $p \in [\overline{\bigcap_{-\infty}^0 \phi^{-k}(Q_{s_{k-1}})}] \cap [\overline{\bigcap_0^\infty \phi^{-k}(R_{s_k})}]$ .

■

**Remark 4.6.** It is noteworthy that in general  $[\bigcap_{-\infty}^0 \phi^{-k}(Q_{s_{k-1}})] \cap [\bigcap_0^\infty \phi^{-k}(R_{s_k})] \subset [\bigcap_{-\infty}^0 \phi^{-k}(\overline{Q_{s_{k-1}}})] \cap [\bigcap_0^\infty \phi^{-k}(\overline{R_{s_k}})] \subseteq [\overline{\bigcap_{-\infty}^0 \phi^{-k}(Q_{s_{k-1}})}] \cap [\overline{\bigcap_0^\infty \phi^{-k}(R_{s_k})}]$  and equality happens in case we have  $\mathcal{R}$  as a generator [11].

**Theorem 4.7. [Principal]** Consider the dynamical system  $(X, \phi)$  where  $X$  is a compact connected metric space and  $\phi$  is an  $m$ -to-1 local homeomorphism with a domain topological partition that is a generator. Then there exists some invariant and closed set  $\Sigma$  (a sub-zip shift space) and a factor map  $\pi : \Sigma \rightarrow X$ , whose satisfies:

1.  $\pi \sigma_\tau = \phi \pi$
2.  $\pi$  is continuous
3.  $\pi$  is surjective.

*Proof.* The proof of the theorem proceeds in two steps:

In the first step, we find suitable image and consequently, extended topological partitions, alphabet sets  $\mathcal{S}'$ ,  $\mathcal{S}$ , and the set  $\Sigma$ . Additionally, we show that  $\Sigma$  is a closed invariant set of a zip shift space  $\Sigma_{\mathcal{S}, \mathcal{S}'}$ . In the second step, we define  $\pi : \Sigma \rightarrow X$  and show that it satisfies properties (1), (2), and (3).

• **Step I:**

As  $\phi$  is an  $m$ -to-1 local homeomorphism, we first consider the principal domains  $\{P_i : i = 1, \dots, m\}$  and define a domain topological partition (DTP, Definition 4.1). Associated with this DTP, one can find an image topological partition  $\mathcal{Q}$ , which is used to obtain the extended topological partition  $\mathcal{R}$ . Let  $\mathcal{Q} = \{Q_{a_1}, \dots, Q_{a_l}\}$ . Then the extended topological partition  $\mathcal{R}$  can be represented as  $\mathcal{R} = \{R_1, R_2, \dots, R_{ml}\}$ . Next, we introduce two alphabet sets derived from the partitions  $\mathcal{R}$  and  $\mathcal{Q}$ . Let  $\mathcal{S}' = \{b_1, b_2, \dots, b_{ml}\}$  be the alphabet set associated with  $\mathcal{R}$ , and let  $\mathcal{S} = \{a_1, a_2, \dots, a_l\}$  be the alphabet set associated with  $\mathcal{Q}$ . The process of constructing the image partition  $\mathcal{Q}$  and its relation to the extended topological partition  $\mathcal{R}$  induces a surjective map  $\tau_\phi : \mathcal{R} \rightarrow \mathcal{Q}$ .

Associated with  $\tau_\phi$  one can define some  $\tau : \mathcal{S}' \rightarrow \mathcal{S}$  and consider

$$\Sigma = \left\{ (\dots s_{-1}; s_0 s_1 \dots) : \left[ \bigcap_{-\infty}^0 \phi^{-k}(Q_{s_{k-1}}) \right] \cap \left[ \bigcap_0^\infty \phi^{-k}(R_{s_k}) \right] \neq \emptyset \right\}. \tag{4.1}$$

We show that  $\Sigma \subseteq \overline{\Sigma_{\mathcal{S}, \mathcal{S}'}}$  is a closed invariant set. It is closed as a consequence of  $X$  being compact and the fact that  $\overline{[\bigcap_{k=-n}^0 \phi^{-k}(Q_{s_{k-1}})]} \vee \overline{[\bigcap_{k=0}^n \phi^{-k}(R_{s_k})]}$  has the finite intersection property. So, if  $s \in \overline{\Sigma}$  then it belongs to  $\Sigma$ .

Before verifying the invariance of  $\Sigma$ , we need to address an important point. As is well known, in general, when  $\phi$  is not an invertible map, it holds that  $\phi(A \cap B) \subseteq \phi(A) \cap \phi(B)$ , and the map may not commute with the closure operator. However, in our case, we are using the fact that  $\phi$  is an  $m$ -to-1 local homeomorphism. Note that, by construction, the elements of the extended topological partition are subsets of (or equal to) the principal domains, and for any  $P_i \in \mathcal{R}$ , the restriction  $\phi|_{P_i} : P_i \rightarrow \phi(P_i)$  is a homeomorphism.

The  $\Sigma$  is (full) invariant under  $\sigma_\tau$ , because if  $s = (\cdots s_{-1}; s_0 s_1 \cdots) \in \Sigma$ , then for all  $n \geq 0$ ,

$$\begin{aligned} & \exists p \in \overline{[\bigcap_{k=-n}^0 \phi^{-k}(Q_{s_{k-1}})]} \cap \overline{[\bigcap_{k=0}^n \phi^{-k}(R_{s_k})]} \\ \Rightarrow & \overline{[\bigcap_{k=-n}^0 \phi^{-k}(Q_{s_{k-1}})]} \cap \overline{[\bigcap_{k=0}^n \phi^{-k}(R_{s_k})]} \neq \emptyset. \end{aligned}$$

Thereupon, for  $\sigma_\tau(s) = (\cdots s_{-1} \tau(s_0); s_1 \cdots)$ ,

$$\begin{aligned} & \overline{([\bigcap_{k=-n}^0 \phi^{-k+1}(Q_{s_{k-1}})] \cap [\bigcap_{k=0}^n \phi^{-k+1}(R_{s_k})])} \\ = & \overline{([\bigcap_{k=-n}^0 \phi^{-k+1}(Q_{s_{k-1}})] \cap \overline{Q_{\tau(s_0)}} \cap [\bigcap_{k=1}^n \phi^{-k+1}(R_{s_k})])} \neq \emptyset. \end{aligned}$$

Here  $Q_{\tau(s_0)}$  is obtained by  $\phi(R_{s_0})$ . Note we are using the fact that, if  $p \in \overline{[\bigcap_{k=-n}^0 \phi^{-k}(Q_{s_{k-1}})]} \cap \overline{[\bigcap_{k=0}^n \phi^{-k}(R_{s_k})]}$  then,

$$\begin{aligned} \phi(p) & \in \phi(\overline{([\bigcap_{k=-n}^0 \phi^{-k}(Q_{s_{k-1}})] \cap [\bigcap_{k=0}^n \phi^{-k}(R_{s_k})])}) \\ & = \overline{([\bigcap_{k=-n}^0 \phi^{-k+1}(Q_{s_{k-1}})] \cap \overline{Q_{\tau(s_0)}} \cap [\bigcap_{k=1}^n \phi^{-k+1}(R_{s_k})])}. \end{aligned} \tag{4.2}$$

From the other side,

$$\sigma_\tau^{-1}(s) = \{(\cdots s_{-2}; \tau^{-1}(s_{-1}) s_0 s_1 \cdots) : \tau^{-1}(s_{-1}) \in \mathcal{S}'\}.$$

One needs to guarantee that these elements belong to  $\Sigma$  as well. In order to show that, one needs to verify if

$$\overline{[\bigcap_{k=-n}^0 \phi^{-k-1}(Q_{s_{k-1}})]} \cap \overline{[\bigcap_{k=0}^n \phi^{-k-1}(R_{s_k})]} \neq \emptyset. \tag{4.3}$$

Note that again, by construction  $\phi|_{R_i}$  is a homeomorphism. Moreover,  $Q_{s_{-1}} \in \mathcal{Q} = \phi(\mathcal{R})$  and  $\phi^{-1}(Q_{s_{-1}}) \in \{R_{s'_0} : R_{s'_0} \in \tau_\phi^{-1}(Q_{s_{-1}})\} \subset \mathcal{R}$ . Therefore,

$$\overline{[\bigcap_{k=-n}^{-1} \phi^{-k-1}(Q_{s_{k-1}})]} \cap \overline{\phi^{-1}(Q_{s_{-1}})} \cap \overline{[\bigcap_{k=0}^n \phi^{-k-1}(R_{s_k})]} \neq \emptyset.$$

This is because for all  $n$ , if  $p \in \overline{[\bigcap_{k=-n}^0 \phi^{-k}(Q_{s_{k-1}})]} \cap \overline{[\bigcap_{k=0}^n \phi^{-k}(R_{s_k})]}$  then  $\phi^{-1}(p)$  belongs to  $\phi^{-1}(\overline{[\bigcap_{k=-n}^0 \phi^{-k}(Q_{s_{k-1}})]} \cap \overline{[\bigcap_{k=0}^n \phi^{-k}(R_{s_k})]})$ , which in case, is equal to the expression (4.3).

- **Step II:** In this part, first we claim that if  $(X, \phi)$  has a domain topological partition which is a generator, the extended topological partition is a generator as well. Assume that  $\mathcal{R} = \{R_{i_1}, \dots, R_{i_{k_i}}\}_{i=1}^m$  for some natural number  $k_i$  represent a domain topological partition associated with some adequate principal domain  $\mathcal{P} = \{P_1, \dots, P_m\}$ , and, that  $\mathcal{R}$  is a generator. The elements of the associated image topological partition  $\mathcal{Q}$  are denoted by  $Q_i \in \bigvee_i \phi(P_i)$ . The extended topological partition by construction is defined as the collection of disjoint elements of  $\phi^{-1}(Q_i)$  for  $Q_i \in \mathcal{Q}$ . Note that if  $Q_i = \phi(R_{i_{j_1}}) \cap \phi(R_{i_{j_2}})$  then  $Q_i \subset \phi(R_{i_{j_1}})$  and  $Q_i \subset \phi(R_{i_{j_2}})$  (for  $R_{i_{j_1}}, R_{i_{j_2}} \in \mathcal{R}$ ). Therefore the elements of the extended topological partition are of the form  $\phi^{-1}(Q_i) \cap R_{i_{j_1}} \subset R_{i_{j_1}}$  and  $\phi^{-1}(Q_i) \cap R_{i_{j_2}} \subset R_{i_{j_2}}$ . Thereupon, the extended topological partition is in fact finer than  $\mathcal{R}$  and by Proposition 2.6, the extended topological partition is a generator as well.

Consider the zip shift map  $\sigma_\tau : \Sigma \rightarrow \Sigma$ . When topological partition  $\mathcal{R}$  is a generator, by Definition 2.5, the non-empty intersection  $\overline{[\bigcap_{-\infty}^0 \phi^{-k}(Q_{s_{k-1}})]} \cap \overline{[\bigcap_0^\infty \phi^{-k}(R_{s_k})]}$  contains a single point and one can define a map  $\pi : \Sigma \rightarrow X$  as follows.

$$\pi(s) = \overline{\bigcap_{k=-\infty}^0 \phi^{-k}(Q_{s_{k-1}})} \cap \overline{\bigcap_{k=0}^\infty \phi^{-k}(R_{s_k})}. \quad (4.4)$$

The set  $\sigma$  and the map  $\pi$  are defined as (4.1) and (4.4). We show that  $\pi$  satisfies the properties (1), (2) and (3).

1. Let  $s = (s_n)_{n \in \mathbb{Z}} \in \Sigma$ , then,

$$\pi\sigma((s_n)_{n \in \mathbb{Z}}) = \pi((s_{n+1})_{n \in \mathbb{Z}}) = \overline{\bigcap_{k=-\infty}^{-1} \phi^{-k}(Q_{s_k})} \cap \overline{\bigcap_{k=-1}^\infty \phi^{-k}(R_{s_{k+1}})}.$$

On the other hand,

$$\begin{aligned} \phi(\pi(s)) &= \phi\left(\overline{\bigcap_{k=-\infty}^0 \phi^{-k}(Q_{s_{k-1}})} \cap \overline{\bigcap_{k=0}^\infty \phi^{-k}(R_{s_k})}\right) \\ &= \overline{\bigcap_{-\infty}^0 \phi^{-k+1}(Q_{s_{k-1}})} \cap \overline{\bigcap_0^\infty \phi^{-k+1}(R_{s_k})} \\ &= \overline{\bigcap_{k=-\infty}^{-1} \phi^{-k}(Q_{s_k})} \cap \overline{\bigcap_{k=-1}^\infty \phi^{-k}(R_{s_{k+1}})} \\ &= \pi\sigma((s_n)_{n \in \mathbb{Z}}). \end{aligned} \quad (4.5)$$

Again here in (4.5) we are using the fact that  $\phi$  is a local homeomorphism.

2. By hypothesis, the domain topological partition is a generator. By construction, the extended topological partition is finer than the domain topological partition  $\mathcal{R}$ . This is because the image topological partition's elements belong to  $\bigvee_{i_j} \phi(R_{i_j})$ , where  $R_{i_j} \in \mathcal{R}$  and the pre-image of these elements which are the elements of the extended topological partition, are

subsets or equal to the elements of  $\mathcal{P}$ . Thus by Proposition 2.6,

$$\lim_{n \rightarrow \infty} \text{diam}(\overline{\bigvee_{-n}^0 \phi^{-k}(Q_{s_{k-1}})} \vee \overline{\bigvee_0^n \phi^{-k}(R_{s_k})}) = 0.$$

Which implies that  $\lim_{n \rightarrow \infty} \text{diam}(\overline{\bigcap_{-n}^{-1} \phi^{-k} Q_{s_{k-1}}} \cap \overline{\bigcap_0^n \phi^{-k}(R_{s_k})}) = 0$ . Therefore, given  $\epsilon > 0$ , there is  $N \in \mathbb{Z}$  such that

$$\text{diam}(\overline{\bigcap_{k=-N}^0 \phi^{-k} Q_{s_{k-1}}} \cap \overline{\bigcap_{k=0}^N \phi^{-k} R_{s_k}}) < \epsilon.$$

Taking  $s, t \in \Sigma$  and  $0 < \delta \leq \frac{1}{2^N}$  if  $d(s, t) < \delta$ , then  $s_i = t_i$  for all  $i$  with  $|i| < N$ , so  $\pi(s), \pi(t) \in (\overline{\bigcap_{-N}^0 \phi^{-k} Q_{s_{k-1}}} \cap \overline{\bigcap_0^N \phi^{-k}(R_{s_k})})$  and so  $d(\pi(s), \pi(t)) < \epsilon$ , which means that  $\pi$  is continuous.

3. Consider  $p \in X$ . We associate the orbit of  $p$  with a sequence  $s = (x_n) \in \Sigma$  such that  $x_0 = s_0$  if  $p \in R_{s_0}$ . Using Proposition 4.5, there exists a sequence  $O_p = (Q_{s_{k-1}}, R_{s_k})$  associated to any  $p \in X$ , such that

- for all  $k \geq 0$ ;  $x_k = s_k$  if  $p \in R_{s_0} \cap \phi^{-1}(R_{s_1}) \cap \dots \cap \phi^{-k}(R_{s_k})$ ;
- for all  $k > 0$ ,  $x_{-k} = s_{-k}$  if  $p \in Q_{s_{-1}} \cap \phi(Q_{s_{-1}}) \cap \dots \cap \phi^k(Q_{s_{-k}})$ .

Therefore  $\pi$  is surjective. ■

**Example 4.8.** Expanding maps and Anosov endomorphisms  $\phi : T^2 \rightarrow T^2$  are  $m$ -to-1 covering maps. Thus, Theorem 4.7 provides a method for obtaining topological partitions and, consequently, a bilateral coding for such dynamics.

### Dataavailability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

### Conflicts of interest

The authors declare no conflict of interest.

### Founding

The first author gratefully acknowledges partial financial support from FAPEMIG, Brazil.

### Acknowledgments

We sincerely thank the anonymous referee for their thorough and sincere feedback on our paper. Their precise comments were instrumental in refining the contents of the paper.

## References

- [1] Adler, R.: Symbolic dynamics and Markov partitions, *Bull. Amer. Math. Soc.* 35(1), 1–56 (1998)
- [2] Anosov, D. V.: Geodesic flows on closed Riemannian manifolds of negative curvature, *Proc. Steklov Inst. Math.* 90, 1–137 (1967)
- [3] Aoki, N., Hiraide, K.: Topological theory of dynamical systems: recent advances, Vol. 52, Elsevier Science (1994)
- [4] Berg, M.: Markov partitions for systems with hyperbolic dynamics, *J. Math. Phys.* 11(3), 918–923 (1970)
- [5] Bowen, R.: Equilibrium states and the ergodic theory of Anosov diffeomorphisms, *Lecture Notes in Mathematics*, Vol. 470, Springer (1975)
- [6] Fathi, A.: Weak KAM theorem in Lagrangian dynamics, *Publ. Math. Univ. Paris* (1993)
- [7] Lind, D., Marcus, B.: An introduction to symbolic dynamics and coding, Cambridge Univ. Press, Cambridge (1995)
- [8] Lamei, S., Mehdipour, P., Ahmadi Dastjerdi, D.: Examples of finite-to-one Smale horseshoe maps, preprint
- [9] Lamei, S., Mehdipour, P.: An N-to-one Smale horseshoe map, [arXiv:2502.14441](https://arxiv.org/abs/2502.14441), submitted (2023)
- [10] Lamei, S., Mehdipour, P.: Zip shift space, [arXiv:2502.11272](https://arxiv.org/abs/2502.11272), submitted (2022)
- [11] Mehdipour, P., Lamei, S., Vargas, W.: S-expansiveness and zip shift maps in symbolic dynamics, [arXiv:2510.12980](https://arxiv.org/abs/2510.12980), submitted (2024)
- [12] Ruelle, D.: Statistical mechanics: rigorous results, World Scientific (1995)
- [13] Sinai, Ya. G.: Dynamical systems with elastic collisions, *Russ. Math. Surv.* 27(3), 65–94 (1972)
- [14] Smale, S.: Differentiable dynamical systems, *Bull. Amer. Math. Soc.* 73, 747–817 (1967)
- [15] Weiss, H.: Markov partitions and hyperbolic dynamics, *J. Dyn. Control Syst.* 1(4), 563–578 (1991)