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From a defective Segre-Veronese embedding to a non-defective one adding a factor

D'un plongement Segre-Veronese défectif à un autre non-défectif en ajoutant un facteur

Edoardo Ballico

Dept. of Mathematics, University of Trento, 38123 Povo (TN), Italy, https://orcid.org/0000-0002-1432-7413
edoardo.ballico@unitn.it

ABSTRACT. Fix $x \in \mathbb{N}$, a multiprojective space Y and a very ample line bundle L on Y. We say that (Y, L) satisfies $\pm x\star$-non-defectivity if the s-secant variety of (Y, L) has the expected dimension if either $(\dim Y + 1)(s + x) \le h^0(L)$ or $(\dim Y + 1)(s - x) \ge h^0(L)$. Natural examples arise when L is the Segre line bundle and all factors have the same dimension (Abo - Ottaviani - Peterson). We take integers $r > 0$, $t \ge 2$ and set $X := Y \times \mathbb{P}^r$. Let $L[t]$ be the line bundle on X coming from L and $\mathcal{O}_{\mathbb{P}^r}(t)$. Under certain assumptions on x, $\dim Y$, $h^0(L)$, r and t we prove that $L[t]$ is not secant defective. Two of the main results are for $r \le 2$. In particular we extend a recent result by Ballico, Bernardi and Mańduz on the non-defectivity of Segre-Veronese embeddings of multidegree $(t_1, \dots, t_k)$ of $(\mathbb{P}^2)^k$, $k \ge 3$, to the case in which $t_i = 1$ for $y > 0$ integers i: we require $y \ge 9$.
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1. Introduction

For very good reasons there is a huge literature on the dimension of the secant varieties of the Segre-Veronese varieties ([1, 2, 3, 4, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 27, 25, 26]). These dimensions give the dimensions of the set of all partially symmetric tensors with a prescribed partially symmetric rank. Low rank approximation of (partially symmetric) tensors is a powerful toolkit of applied linear algebra.

A Segre-Veronese variety $X \subset \mathbb{P}^r$ is the embbeding in $\mathbb{P}^r$ of the multiprojective space $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ by the complete linear system $|\mathcal{O}_Y(t_1, \dots, t_k)|$, where $t_1, \dots, t_k$ are positive integers. The embedding $X \subset \mathbb{P}^r$ is said to be *non-defective* if for all positive integer s the s-secant variety $\sigma_s(X)$ has the "expected dimension", i.e., $\dim \sigma_s(X) = \min\{r, s(\dim X + 1) - 1\}$ for all s. By [28, Th. 1.5] knowing that a secant variety $\sigma_s(X) \subsetneq \mathbb{P}^r$ has the expected dimension gives an identifiability result (a general partially symmetric tensor of rank $s - 1$ has a unique rank 1 decomposition).

We start with a multiprojective space Y, a very ample line bundle $\mathcal{L}$ on Y and a positive integer r. Set $X := Y \times \mathbb{P}^r$. Let $p_1 : X \to Y$ and $p_2 : X \to \mathbb{P}^r$ denote the projections. For all integers t set $\mathcal{L}[t] := p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^r}(t))$. The line bundle $\mathcal{L}[t]$ is very ample if and only if $t > 0$. Under certain assumptions we are able to prove that $\mathcal{L}[t]$, $t \ge 2$, is not defective even without assuming that $\mathcal{L}$ is not defective.

We give the following general definition which we think it is quite useful.

The author is a member of GNSAGA of INdAM (Italy).

Definition 1. *Fix $x \in \mathbb{N}$. Take a very ample line bundle $\mathcal{R}$ on an integral projective variety W. Set $n := \dim W$ and $\alpha := h^0(\mathcal{R})$. We say that $\mathcal{R}$ is $\pm x\star$-nondefective if a general union $Z \subset W$ of t double points satisfies $h^0(\mathcal{I}_Z \otimes \mathcal{R}) = \alpha - t(n+1)$ if $(t+x)(n+1) \leq \alpha$ and $h^0(\mathcal{I}_Z \otimes \mathcal{R}) = 0$ if $(t-x)(n+1) \geq \alpha$. Let $M \subset \mathbb{P}^r$ be an integral and non-degenerate projective variety. We say that the embedding $M \subset \mathbb{P}^r$ is $\pm x\star$-nondefective if $\dim \sigma_s(M) = s(\dim M + 1) - 1$ if $(s-x)(\dim M + 1) \leq r+1$ and $\sigma_s(M) = \mathbb{P}^r$ if $(s-x)(\dim M + 1) \geq r+1$.*

Definition 1 has the following motivation. Take for instance the case $M \subset \mathbb{P}^r$. The case $x = 0$ is the classical definition of non-defective. Fix an integer $x > 0$. If we know that M is $\pm x\star$-non-defective, but it is $\pm(x-1)\star$-defective, then we have a precise measure of its defectivity. If we only know that M is $\pm x\star$-nondefective often this information is sufficient from the practical point of view. For instance, it is a good approximation of the unknown dimension of the set of partially symmetric tensors with prescribed format and rank. In this paper we prove that sometimes it is sufficient to have M $\pm x\star$-nondefective to get the non-defectivity of a related embedded variety (Proposition 1, Theorems 1 and 3 and Corollary 1).

At the end of the paper (Remark 6) we discuss one-sided definitions of bounded non-defectivity.

In the results stated in the introduction the reader will see among the assumptions lower bounds for an integer $\alpha := h^0(L)$. The assumptions appearing in this paper are the best we are able to do. Some assumptions are needed (the ones in [14] are sharp). Weaker assumptions on α are a key step in many proofs, e.g., [16, Th. 3] and [15]. Moreover (as in [14, 16]) these statements are used to get results in other papers and allow the reader to decrease the number of cases done by computer. It is far quicker to avoid tricky initial cases doing a numerical computation. Often the number of initial cases is huge and there is no alternative to a numerical check. However, for the positive characteristic case we loose any information for an unknown number of primes.

We adapt the proofs in [14] to get the following result weaker than the case $x = 0$ (non-defectivity), i.e., [14, Th. 2]) and the case of $\pm 1\star$-nondefectivity, i.e., [14, Th. 3].

Proposition 1. *Let Y be a multiprojective space and $\mathcal{L}$ a very ample line bundle on Y which is $\pm x\star$-nondefective. Set $\alpha := h^0(\mathcal{L})$, $X := Y \times \mathbb{P}^1$, $m := \dim Y$ and assume $(m-1)\alpha \geq (m+1)^2 + (m+2)(m-1)((m+1)x + m)$. Then all $\mathcal{L}[t]$, $t \geq 2$, are non-defective.*

For $x = 0$ the next result does not cover [16, Th. 1]. Note that [14, 16] are (for $x = 0$) more general that our set-up, because they do not require that Y is a multiprojective space.

Theorem 1. *Let Y be a multiprojective space and $\mathcal{L}$ a very ample line bundle on Y which is $\pm x\star$-nondefective. Set $\alpha := h^0(\mathcal{L})$, $X := Y \times \mathbb{P}^2$, $m := \dim Y$ and assume*

$$(m-1)\alpha \geq (m+1)^2 + (m+2)(m-1)((m+1)x + m), \tag{1}$$

$$\alpha \geq (m+2)^2(3m+7)/6, \tag{2}$$

Then all $\mathcal{L}[t]$, $t \geq 2$, are non-defective.

Using Theorem 1 and [6, Th. 5.2] we may extend [16, Th. 3], which is the case $y = 0$.

Theorem 2. *Take $X := (\mathbb{P}^2)^k$, $k \geq 3$, and let $\mathcal{R} := \mathcal{O}_X(t_1, \dots, t_k)$ be a very ample line bundle on X. Let y be number of all $i \in \{1, \dots, k\}$ such that $t_i = 1$. Assume $y < k$ and that either $y = 0$ or $y \geq 9$. Then $\mathcal{R}$ is not defective.*

For a specific integer $t \geq 3$ we prove the following results in which we use the following notation. Set $X = Y \times \mathbb{P}^r$ and let $p_2 : X \to \mathbb{P}^r$ denote the projection. Set $\mathcal{O}_X(0,1) := p_2^*(\mathcal{O}_{\mathbb{P}^r}(1))$. Note that $\dim |\mathcal{O}_X(0,1)| = r$ and that each $H \in |\mathcal{O}_X(0,1)|$ is isomorphic to $Y \times \mathbb{P}^{r-1}$.

Theorem 3. *Fix $x \in \mathbb{N}$. Let Y be a multiprojective space and $\mathcal{L}$ a very ample line bundle on Y. Set $\alpha := h^0(\mathcal{L})$. Fix an integer $r \geq 1$ and set $X := Y \times \mathbb{P}^r$ and $n := \dim X$. Take $H \in |\mathcal{O}_X(0,1)|$. Fix an integer $t \geq 3$ and assume that $(H, \mathcal{L}[t]_{|H})$ is $\pm x\star$-nondefective and $\binom{r+t-2}{r-1}(n - r/t)\alpha \geq n^2(nx + n - 1) + n^2(n+3)/2$. Then $\mathcal{L}[t]$ is not defective.*

Corollary 1. *Let Y be a multiprojective space and $\mathcal{L}$ a very ample line bundle on Y. Set $\alpha := h^0(\mathcal{L})$. Fix an integer $r \geq 1$ and set $X := Y \times \mathbb{P}^r$ and $n := \dim X$. Take $H \in |\mathcal{O}_X(0,1)|$. Assume that $(H, \mathcal{L}[3]_{|H})$ is $\pm x\star$-nondefective and that $\binom{r+1}{2}(n - r/3)\alpha \geq n^2(n+3)/2 + n^2(nx + n - 1)$. Then all $\mathcal{L}[t]$, $t \geq 3$, are non-defective.*

Our proofs use the Differential Horace Lemma ([9, 10, 11, 12] and a result of B. Ådlandsvik ([7, Prop. 2.1(ii)]) rediscovered by A. T. Blomenhofer and A. Casarotti who used it to prove a very general result: the non-defectivity of many secant varieties of homogeneous varieties ([18]). This very general result opened the path for further improvements ([5, 15]).

The interested reader may adapt our proofs in the following way. Start with a $\pm x$-nondefective line bundle $\mathcal{L}$ and get a $\pm y$-nondefective line bundle $\mathcal{L}[t]$ under weaker assumptions on $\dim Y$, $h^0(\mathcal{L})$, r and t. This is easy for Theorem 2 and hence Corollary 1 and for the case $t = 2$ of Proposition 1 and Theorem 1. One could try this approach for some $0 < y < x$ to improve the case $t \geq 3$ of Proposition 1 and Theorem 1.

We work over an algebraically closed field with characteristic zero.

1.1. *Roadmap of the paper*

Section 2 contains the preliminaries and notation used in the paper.

Section 3 only contains the proof of Proposition 1 and may be omitted (in the next section we see the corollaries of the statement of Proposition 1, never of its proof). This part is single out, because to avoid several pages of repetitive work it uses the notation and proofs of [14].

Section 4 starts with a long explanation of the Differential Horace Strategy used to prove Theorems 3 and 1. Then we give a corollary of the statement of Proposition 1 and a lemma. Then we give the proof of Theorem 3. We are careful to distinguish each step in smaller claims, because the explicit values of the integers involved are essential. Hence again we introduce the integers w_t, w_t', z_t and e_t. We may use induction for $\mathcal{L}[t-1]$ for $t \geq 4$ (or for Theorem 1 for $t \geq 3$) and induction for $\mathcal{L}[t-2]$. The proof of Theorem 1 is the one more similar to our Differential Horace Strategy. For Theorem 3 the assumption $t \geq 3$ comes from the use of [7] for the very ample line bundle $\mathcal{L}[1]$. The other results of the paper require short proofs, because they are corollaries of one of these theorems together with older references.

The last short section contains some remarks (one-sided bounded defectivity, the positive characteristic case and not algebraically closed fields).

2. Preliminaries

Let W be a projective variety and $D \subset W$ an effective Cartier divisor. For any smooth point p of W let $(2p, W)$ denote the closed subscheme of W with $(\mathcal{I}_p)^2$ as its ideal sheaf. The scheme $(2p, W)$ is zero-dimensional, $\{p\} = (2p, W)_{\mathrm{red}}$ and $\deg((2p, W)) = \dim W + 1$. For any finite set $S \subset W_{\mathrm{reg}}$ set $(2S, W) := \cup_{p\in S}(2p, W)$. If W is the multiprojective space X we write $2p$ and $2S$ instead of $(2p, X)$ and $(2S, X)$, respectively. Now assume that $W \subset \mathbb{P}^r$ is embedded in a projective space. The Terracini Lemma gives that the dimension of the s-secant variety $\sigma_s(W) \subseteq \mathbb{P}^r$ of W is the dimension of the linear span $\langle(2S, W)\rangle$ of $(2S, W)$, where S is a general union of s points of W ([7, Lemma 1.1], [27]).

For any zero-dimensional scheme $Z \subset X$ the residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of W with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. If $Z = A \cup B$ with $A \cap B = \emptyset$, then $\mathrm{Res}_D(Z) = \mathrm{Res}_D(A) \cup \mathrm{Res}_D(B)$. If $Z \cap D = \emptyset$, then $\mathrm{Res}_D(Z) = Z$. If p is a smooth point of both W and D, then $\mathrm{Res}_D((2p, W)) = \{p\}$ and $(2p, W) \cap D = (2p, D)$. For any line bundle $\mathcal{R}$ on W and any Z the following sequence of coherent sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{R}(-D) \to \mathcal{I}_{Z\cap D,D} \otimes \mathcal{R}_{|D} \to 0$$

is exact and it is often called the residual exact sequence of Z with respect to D.

We use the Differential Horace Lemma for double points ([9, 10, 11, 12]) in the following form.

Remark 1. *Let W be a projective variety, $\mathcal{R}$ a line bundle on W and D an effective Cartier divisor of W. Fix a zero-dimensional scheme $Z \subset W$. Take general finite sets $S \subset W$ and $S_1 \subset D$ such that $\#S = \#S_1$. Hence all points of S are smooth points of W, all points S_1 are smooth points of D, $S \cap D = \emptyset$ and $S \cap Z = S_1 \cap Z = \emptyset$. For $i = 0, 1$ we have*

$$h^i(W, \mathcal{I}_{Z\cup 2S} \otimes \mathcal{R}) \leq h^i(W, \mathcal{I}_{\mathrm{Res}_D(Z)\cup S_1} \otimes \mathcal{R}(-D)) + h^i(D, \mathcal{I}_{(Z\cap D)\cup(2S_1,D)} \otimes \mathcal{R}_{|D}).$$

Remark 2. *Let $Z \subset W$ be zero-dimensional subschemes of the projective variety M and $\mathcal{R}$ a line bundle on M. Obviously, $h^0(M, \mathcal{I}_Z \otimes \mathcal{R}) \geq h^0(M, \mathcal{I}_W \otimes \mathcal{R})$. Now assume $h^0(\mathcal{I}_W \otimes \mathcal{R}) = h^0(\mathcal{R}) - \deg(W)$ (resp. $h^1(M, \mathcal{I}_W \otimes \mathcal{R}) = 0$). Since $\dim W = 0$, we have $h^1(W, \mathcal{I}_{Z,W} \otimes \mathcal{R}_{|W}) = 0$. Thus the long cohomology exact sequence of the exact sequence*

$$0 \to \mathcal{I}_{W,M} \otimes \mathcal{R} \to \mathcal{I}_{Z,M} \otimes \mathcal{R} \to \mathcal{I}_{Z,W} \otimes \mathcal{R}_{|W} \to 0$$

gives $h^0(M, \mathcal{I}_Z \otimes \mathcal{R}) = h^0(\mathcal{R}) - \deg(Z)$ (resp. $h^1(M, \mathcal{I}_Z \otimes \mathcal{R}) = 0$).

Remark 3. *Fix a projective variety X, a line bundle $\mathcal{R}$ on X, an effective divisor $D \subset X$ and a zero-dimensional scheme Z. Fix an integer $s > 0$ and a general $S \subset D$ such that $\#S = s$. Since S is general in D, by induction on s it is easy to prove that $h^0(\mathcal{I}_{Z\cup S} \otimes \mathcal{R}) = h^0(\mathcal{I}_Z \otimes \mathcal{R}) - s$ if $h^0(\mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{R}(-D)) \leq h^0(\mathcal{I}_Z \otimes \mathcal{R}) - s$.*

We state a general result due to B. Ådlandsvik ([7, Prop. 2.1(ii)]) in a stronger form ([15, Th. 2.1]) which was inspired by [18].

Theorem 4. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety such that no proper secant variety of X is a cone. Let $W \subset \mathbb{P}^r$ be an integral variety (we allow the case $W = \emptyset$ and the case W degenerate). Set $n := \dim X$ and $w := \dim W$, with the convention $w = -1$ if $W = \emptyset$. For any set $E \subset \mathbb{P}^r$ let $\langle E \rangle$ denote its linear span. Let B be a general tangent space of W and A the union of x general tangent spaces of X. Then*

(a) If $w + (n+1)x \le r - (n+1)n/2$, then $\dim\langle A \cup B\rangle = w + (n+1)x$.

(b) If $w + (n+1)x \ge r + (n+1)n/2$, then $\langle A \cup B\rangle = \mathbb{P}^r$.

We use Theorem 4 with X a variety homogeneous for the action of a connected algebraic group G and with the embedding $X \subset \mathbb{P}^r$ given by the complete linear system of a rank 1 G-bundle on X giving an irreducible representation.

3. Proof of Proposition 1

Proof of Proposition 1: We use a notation as near to the one of [14] as we are able to do, adding a tilde to symbols in [14] if they conflict with the notation seen elsewhere in this paper. Set $n := m+1$. For any integer $t > 0$ set $w_t := \lfloor (t+1)\alpha/(n+1) \rfloor$, $w'_t := \lceil (t+1)\alpha/(n+1) \rceil$, $e := \lfloor \alpha/n \rfloor - x$ and $f := \alpha - ne$. Note that $nx \le f \le nx + n - 1$. Since $\alpha \ge n(nx+n-1)$, $e \ge nx + n - 1 \ge f$.

Since $\mathcal{L}$ is $\pm x\star$-nondefective, a general union $Z \subset Y$ of c double points satisfies $h^1(Y, \mathcal{I}_Z \otimes \mathcal{L}) = 0$ if $c \le e$ and $h^0(\mathcal{I}_Z \otimes \mathcal{L}) = 0$ if $nc \ge \alpha + nx$. Fix $o \in \mathbb{P}^1$ and set $H := Y \times \{o\} \in |\mathcal{O}_X(0,1)|$.

Observation 1: We fix t and take $\tilde{z} \in \{w_t, w'_t\}$. By Remark 2 to prove the proposition for the integer t it is sufficient to prove $h^0(\mathcal{I}_W \otimes \mathcal{L}[t]) = \max\{0, (t+1)\alpha - (n+1)\tilde{z}\}$ for a general union $W \subset X$ of $\tilde{z}$ double points of X.

Observation 2: Since $ne + f = \alpha$, $f \le nx + n - 1$ and $\alpha \ge (n+1)(nx+n-1)$, we have $e \ge f$.

Set $\tilde{w} := \tilde{z} - e - f$.

Observation 3: The inequality $\tilde{w} \ge e$, i.e., $n(n+1)\tilde{z} \ge (n+1)n(2e+f)$, holds for the following reasons. Since $t \ge 2$, we have $(n+1)\tilde{z} \ge 3\alpha - n$. We have $ne + f = \alpha$ and hence $(n+1)n(2e+f) = (n+1)(2\alpha + (n-2)f)$ with $nx \le f \le nx+n-1$. Thus it is sufficient to use that $(n-2)\alpha \ge n^2 + (n+1)(n-2)(nx+n-1)$, i.e., $(m-1)\alpha \ge (m+1)^2 + (m+2)(m-1)((m+1)x+m)$.

Let $S \subset H$ (resp. $S' \subset H$, resp. $S'' \subset H$) be a general subset of H with e (resp. f, resp. $e-f$) points.

Observation 4: Fix integers c and c_1 such that $c \ge c_1$, $c - c_1 \le e$ and $n(c-c_1) + c_1 \le \alpha$. Let $E \subset X$ be a general union of c double points of X. By [14, Prop. 1] we have $h^1(\mathcal{I}_E \otimes \mathcal{L}[1]) = 0$.

Set $\Delta := (t+1)\alpha + n - (n+1)\tilde{z}$. We have $0 \le \Delta \le n$.

Claim 1: We have $\Delta + \tilde{z} \ge tf + e + n$.

Proof of Claim 1: To get verbatim the proof of the same name claim in the proof of [14, Th. 2] it is sufficient to prove (last line) that $ne \ge (n+1)f$. Hence it is sufficient to have $\alpha - f \ge (n+1)f$. Thus it is sufficient to assume $\alpha \ge (n+2)(nx+n-1)$.

Claim 2: We have $nf + n(\tilde{w} - e) + e \leq \alpha$.

Proof of Claim 2: By Observation 3 we have $\tilde{z} \geq te + f$. Assuming this inequality the definition of Δ gives that Claims 1 and 2 are equivalent.

(a) Assume $t = 2$. Set $\varepsilon := 0$ if $\tilde{z} = w'_t$ and $\varepsilon := 3\alpha - (n+1)w_2$ if $\tilde{z} = w_t$. We specialize a general union of $\tilde{z}$ double points of X to $E \cup Z'$ with $E \cap Z' = \emptyset$ with E a general union of $\tilde{w} - e - f$ double points of X and $Z'_{\mathrm{red}} = S \cup S'$. We apply the Differential Horace Lemma (Remark 1) to each point of S'. We get that to prove the case $t = 2$ it is sufficient to prove that $h^0(\mathcal{I}_{E \cup S \cup (2S',H)} \otimes \mathcal{L}[1]) = \varepsilon$. Since S is general in H, Remark 3 shows that it is sufficient to prove that $h^1(\mathcal{I}_{E \cup (2S'H)} \otimes \mathcal{L}[1]) = 0$ and that $h^0(\mathcal{I}_E \otimes \mathcal{L}[0]) \leq \max\{0, h^1(\mathcal{I}_{E \cup (2S'H)} \otimes \mathcal{L}[1]) - e\}$.

Claim 3: Either $h^1(\mathcal{I}_{E \cup (2S'H)} \otimes \mathcal{L}[1]) = 0$ or $h^0(\mathcal{I}_{E \cup (2S'H)} \otimes \mathcal{L}[1]) = 0$.

Proof of Claim 3: In the proof in [14] of the same named claim we only use Observation 3.

Obviously, if $h^0(\mathcal{I}_{E \cup (2S'H)} \otimes \mathcal{L}[1]) = 0$, then $\tilde{z} = w'_2$ and $h^0(\mathcal{I}_G \otimes \mathcal{L}[2]) = 0$ for a general union G of $\tilde{z}$ double points of X. Hence by Claim 3 to conclude the proof of the case $t = 2$ it is sufficient to prove the following claim.

Claim 4: We have $h^0(\mathcal{I}_E \otimes \mathcal{L}[0]) \leq \max\{0, h^0(\mathcal{I}_{E \cup (2S',H)} \otimes \mathcal{L}[1]) - e\}$ with $h^0(\mathcal{I}_{E \cup (2S'H)} \otimes \mathcal{L}[1]) = 2\alpha - \deg(E) - nf$ and.

Proof of Claim 4: We have $h^0(\mathcal{I}_E \otimes \mathcal{L}[0]) = h^0(Y, \mathcal{I}_U \otimes \mathcal{L})$ with $U \subset Y$ a general union of $\gamma := \tilde{w} - e - f = \tilde{z} - 2e - 2f$ double points of Y. We have $\deg(E) = (n+1)\gamma$ and $h^0(\mathcal{I}_{E \cup (2S'H)} \otimes \mathcal{L}[1]) = 2\alpha - (n+1)\gamma - nf = \alpha - \gamma - (n-1)f$.

If $\gamma \leq e$, then $h^1(Y, \mathcal{I}_U \otimes \mathcal{L}) = 0$ and hence $h^0(Y, \mathcal{I}_U \otimes \mathcal{L}) = \alpha - n\gamma$. We only need $\alpha \geq \gamma + nf = \tilde{z} - 2e + (n-2)f$. We use that $m \geq 3$. By [23] a general tangent vector of Y gives 2 independent conditions to each linear system on Y of dimension at least 2. Hence $h^0(\mathcal{I}_U \otimes \mathcal{L}) = 0$ if $\gamma \geq e + f/2$. Now assume $e < \gamma < e + f/2$. We get $h^0(\mathcal{I}_U \otimes \mathcal{L}) \leq f - 2(\gamma - e)$, which is better than in [14], which required $\alpha \geq e + (n+1)f - 2$, i.e., $n\alpha + 2n \geq ne + n(n+1)f = \alpha + (n^2 - n + 1)f$, i.e.,

$(n-1)\alpha + 2n \geq (n^2 - n + 1)f$ with $f \leq nx + n - 1$. Thus it is sufficient to have $m\alpha \geq (m^2 + m)((m+1)x + m)$, which is true because $\alpha \geq (m+3)((m+1)x + m)$.

(b) In this step we assume $t > 2$ and prove that $\mathcal{L}[t]$ is not defective. By step (a) and induction on t we may assume that $\mathcal{L}[y]$ is not defective for all $2 \leq y < t$. We only do the case $t = 3$, since the case $t \geq 4$ follows from the inductive assumption as in [14]. Take $\tilde{z} \in \{w_3, w'_3\}$. We have $w' := \tilde{z} - e - f \geq 0$. Let $W \subset X$ be a general union of w' double points of X. As in step (a) using Remark 3 it is sufficient to prove that $h^0(\mathcal{I}_W \otimes \mathcal{L}[1]) \leq \max\{0, 3\alpha - \deg(W)\}$. If $w' \geq 2e + f$ using twice the differential Horace lemma with respect to H we get $h^0(\mathcal{I}_W \otimes \mathcal{L}[1]) = 0$. If $w' \leq e$ using once the Horace Lemma we get $h^1(\mathcal{I}_W \otimes \mathcal{L}[1]) = 1$. If $e \leq w' \leq 2e$ using twice the Horace Lemma we get $h^1(\mathcal{I}_W \otimes \mathcal{L}[1]) \leq 2e - w'$, which is enough. If $w' > 2e$ we degenerate W as in part (b) of the proof of [14, Th. 2], which only requires the inequality $e \geq f$. $\square$

4. Proofs of the other results

In this section we do not need the proofs of [14]. In the proofs of Theorems 1 and 3 we use the following set-up, which we call the Differential Horace Strategy. The other results of this papers follow from the literature and previously proved results in this paper, not needing the full force of the Differential Horace Strategy.

You have a pair $(X, \mathcal{R})$ with $\mathcal{R}$ very ample line bundle on X with $h^1(\mathcal{R}) = 0$. Suppose you want to prove that this pair is not defective. Set $\beta := h^0(\mathcal{R})$, $n := \dim X$, $w := \lfloor \beta/(n+1) \rfloor$ and $w' := \lceil \beta/(n+1) \rceil$. By Remark 2 it is sufficient to prove that $h^1(\mathcal{I}_{W_1} \otimes \mathcal{R}) = 0$ for a general union W_1 of w double points of X and $h^0(\mathcal{I}_{W_2} \otimes \mathcal{R}) = 0$ for a general union W_2 of w' double points of X. Of course, these cases are the hardest ones, but we have specific integers w and w' and we may use inequalities for these integers. We have an effective divisor $H \subset X$ such that $h^1(\mathcal{R}(-H)) = 0$ and hence $h^0(H, \mathcal{R}_{|H}) = h^0(\mathcal{R}) - h^0(\mathcal{R}(-H))$. By some inductive assumption we may assume that $(H, \mathcal{R}_{|H})$ is not defective. But we cannot assume that $(X, \mathcal{R}(-H))$ is not defective. We have $\dim H = n-1$. Set $z := \lfloor h^0(H, \mathcal{R}_{|H})/n \rfloor$ and $e := h^0(H, \mathcal{R}_{|H}) - nz$. We have $0 \le e \le n$. We outline the strategy to prove that $h^1(\mathcal{I}_{W_1} \otimes \mathcal{R}) = 0$ using the Differential Horace Lemma and Remark 3. To get $h^1(\mathcal{I}_{W_1} \otimes \mathcal{R}) = 0$ we degenerate W_1 to the union of $w - z - e$ general double points of X, z double points of X with as their reduction general points of H and apply the Differential Horace Lemma (Remark 1) with respect to e general points of H. Let $Z' \subset X$ be a general union of $w - z - e$ double points of X. Since $(H, \mathcal{R}_{|H})$ is not defective, we get $h^1(\mathcal{I}_{W_1} \otimes \mathcal{R}) \le h^1(\mathcal{I}_{Z' \cup S' \cup (2S'', H) \cup S''} \otimes \mathcal{R}(-H))$, where S' is a general subset of H with cardinality z and S'' is a general subset of H with cardinality e. The integer e is very small, $e \le n$. We first prove that $h^1(\mathcal{I}_{Z' \cup (2S'', H)} \otimes \mathcal{R}(-H)) = 0$. Although we do not know if $\mathcal{R}(-H)$ is not defective, the integer z is so large (but for each theorem we need different numerical checks) that we can prove that $h^1(\mathcal{I}_{Z' \cup (2S'', H)} \otimes \mathcal{R}(-H)) = 0$. The set S' is general in H, not in X. We have $\mathrm{Res}_H(Z' \cup (2S'', H)) = Z'$. We use Remark 3 to get $h^1(\mathcal{I}_{Z' \cup S' \cup (2S'', H) \cup S''} \otimes \mathcal{R}(-H))$ using Z' for $\mathcal{R}(-2H)$.

In Theorem 1 we have $X = Y \times \mathbb{P}^2$ and $\mathcal{R} = \mathcal{L}[t]$ for some $t \ge 0$ and $H \in |\mathcal{O}_X(0,1)|$. Thus $\mathcal{R}(-H) = \mathcal{L}[t-1]$ and $\mathcal{R}(-2H) = \mathcal{L}[t-2]$. We write w_t, w'_t, z_t and e_t instead of w, w', z and e if $\mathcal{R} = \mathcal{L}[t]$. The hardest case is $t = 2$, because $\mathcal{L}[0]$ is not even ample. Since $|\mathcal{L}[0]|$ induces the projection $Y \times \mathbb{P}^2 \to Y$ onto the first factor, a general union of $w_2 - z_2 - e_2$ double points of X gives at most $(w_2 - z_2 - e_2)(n-1)$ conditions to $|\mathcal{L}[0]|$. We use that $\mathcal{L}$ is ± 1-nondefective and $(w_2 - z_2 - e_2) \ge 1 + \alpha/(n-1)$ (a numerical check made at the end of step (a) of the proof of Theorem 1).

Each step of the proofs of Theorems 1 and 3 is divided in several small steps (called Claims and Observations) with each Claim having a separate proof, while the statements in the Observations have one-line proofs, sometimes quoting previous claims or observations.

Corollary 2. *Let Y be a multiprojective space and $\mathcal{L}$ a very ample line bundle on Y. Set $m := \dim Y$ and assume $\alpha \ge (m+3)((m+1)x + m)$. Fix an integer $r \ge 1$ and integers $t_i \ge 2$, $1 \le i \le r$. Set $W := (\mathbb{P}^1)^r$, $X := Y \times W$ and let $p_1 : X \to Y$ and $p_2 : X \to W$ denote the projections. Set $E := \mathcal{O}_Y(t_1, \dots, t_r)$ and $\mathcal{R} := p_1^*(\mathcal{L}) \otimes p_2^*(E)$. Then $\mathcal{R}$ is not defective.*

Proof. The case $r = 1$ is true by Proposition 1. Hence we may assume $r \ge 2$ and use induction on the integer r. To apply [14] it is sufficient to prove that $\prod_{i=1}^{r-1}(t_i + 1)\alpha > (m+r)^2$. It is sufficient to use that $2^{r-1}(m+3)m > (m+r)^2$. □

Remark 4. *Take $Y = (\mathbb{P}^a)^y$ and as $\mathcal{L}$ the Segre line bundle $\mathcal{O}_Y(1,\dots,1)$ of Y. By [6, Th. 5.2 and Cor. 5.4] or [8] the pair $(Y,\mathcal{L})$ is $\pm a\star$-nondefective. These papers give more precise results, depending on the the integer $\delta_y \in \{0,\dots,a\}$ such that $\delta_y \equiv \lfloor (a+1)^y/(ay+1)\rfloor \mod a$.*

Lemma 1. *Take $Y := (\mathbb{P}^a)^y$ with $y \ge 3$ and let $\mathcal{L}$ be the Segre line bundle of Y. Assume that $a+1$ is a prime and that $\mathcal{L}$ is numerically perfect, i.e., assume $(a+1)^y \equiv 0 \pmod{ay+1}$. Then $\mathcal{L}$ is not secant defective.*

Proof. By assumption $s_y := (a+1)^y/(ay+1)$ is an integer. Since $a+1$ is a prime, $s_y = (a+1)^b$ for some positive integer b. Thus $s_y \equiv 0 \pmod{a+1}$. With the terminology of [6, Th. 5.2] we have $\delta_y = 0$. Hence [6, Th. 5.2] gives $\dim \sigma_{s_y}(Y) = (a+1)^y - 1$. Hence $\mathcal{L}$ is not secant defective. □

Proof of Theorem 3: Set $w := \lfloor \alpha\binom{r+t}{r}/(n+1)\rfloor$, $w' := \lceil \alpha\binom{r+t}{r}/(n+1)\rceil$, $z := \lfloor \alpha\binom{r+t-1}{r-1}/n\rfloor - x$ and $e := \alpha\binom{r+t-1}{r-1} - nz$. Note that $nx \le e \le nx+n-1$. Let $W_1 \subset X$ (resp. $W_2 \subset X$) be a general union of w (resp. w') double points of X. To show that $\mathcal{L}[t]$ is not secant defective it is sufficient to prove that $h^0(\mathcal{I}_{W_2}\otimes\mathcal{L}[t]) = 0$ and $h^1(\mathcal{I}_{W_1}\otimes\mathcal{L}[t]) = 0$. Take a general $S'\cup S'' \subset H$ such that $S'\cap S'' = \emptyset$, $\#S' = z$ and $\#S'' = e$.

Observation 1: Since $\binom{r+t-1}{r-1} \ge \binom{r+t-2}{r-1}$, we have $\binom{r+t-2}{r-1}(n-t/n) \le \binom{r+t-1}{r-1}n$. Thus our assumption implies $\binom{r+t-1}{r-1}\alpha \ge n(nx+n-1) + n^2(n+3)/2$.

Claim 1: We have $h^1(H,\mathcal{I}_{(2S'',H)}\otimes\mathcal{L}[t-1]_{|H}) = 0$.

Proof of Claim 1: Since H is homogeneous and $e \le nx+n-1$, our plan is to use Theorem 4 with H instead of X. Since $\dim H = n-1$, it is sufficient to prove that $h^0(\mathcal{L}[t-1]) = \binom{r+t-2}{r-1}\alpha \ge n(nx+n-1)+n(n-1)/2$. The latter inequality is true because we assumed $\binom{r+t-2}{r-1}(n-r/t)\alpha \ge n^2(nx+n-1)+n^2(n+1)/2$.

Fix general $S,\hat{S} \subset X$ such that $\#S = w'-z-e$ and $\#\hat{S} = w-z-e$. Let $Z \subset X$ (resp. $\hat{Z} \subset X$) be the union of the double points of X with S (resp. $\hat{S}$) as its reduction.

Observation 2: We have $h^i(H,\mathcal{I}_{(2S',H)\cup S''}\otimes\mathcal{L}[t]_{|H}) = 0$, $i = 0,1$, because $\mathcal{L}[t]_{|H}$ is $\pm x\star$-nondefective.

(a) In this step we prove that $h^0(\mathcal{I}_{W_2}\otimes\mathcal{L}[t]) = 0$.

By Observation 2 and the Differential Horace Lemma (Remark 1) to prove that $h^0(\mathcal{I}_{W_2}\otimes\mathcal{L}[t]) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_{Z\cup S'\cup(2S'',H)}\otimes\mathcal{L}[t-1]) = 0$.

Claim 2: We have $h^1(\mathcal{I}_{Z\cup(2S'',H)}\otimes\mathcal{L}[t-1]) = 0$.

Proof of Claim 2: By Claim 1 the e-secant variety W of the pair $(H,\mathcal{L}[t-1]_{|H})$ has dimension $en-1$. Thus to apply Theorem 4 to $\mathcal{L}[t-1]$ with this variety W it is sufficient to prove that $\deg(Z)+ne \le h^0(\mathcal{L}[t-1]) - n(n+1)/2$. We have $\deg(Z)+ne = (n+1)w' - (n+1)z - (n+1)e + ne$ with $(n+1)w' \le \binom{t+r}{r}\alpha + n$, $nz+e = \binom{r+t-1}{r-1}\alpha$ and $h^0(\mathcal{L}[t-1]) = \binom{r+t-1}{r}\alpha$. Thus it is sufficient to have $z \ge n + n(n+1)/2$, i.e., $nz \ge n^2(n+3)/2$. Since $\binom{r+t-1}{r-1}\alpha - nz \le nx+n-1$, it is sufficient to use that $\binom{r+t-1}{r-1}\alpha \ge n(nx+n-1)+n^2(n+3)/2$ (Observation 1).

By Claim 2 and the generality of S' in H, to conclude the proof that $h^0(\mathcal{I}_{W_2}\otimes\mathcal{L}[t]) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_Z\otimes\mathcal{L}[t-2]) \le \max\{0, h^0(\mathcal{L}[t-1]) - \deg(Z) - z - ne\}$. We have $h^0(\mathcal{L}[t-$

$1]) - h^0(\mathcal{L}[t-2]) = \binom{r+t-2}{r-1}\alpha$ and $z + x \le \binom{r+t-1}{r-1}\alpha/n$. Note that $n\binom{r+t-2}{r-1} - \binom{r+t-1}{r-1} = \binom{r+t-2}{r-1} = \binom{r+t-2}{r-1}(n - r/t)$. Since $ne \le n(nx+n-1)$, Theorem 4 shows that it is sufficient to assume the inequality $\binom{r+t-2}{r-1}(n-r/t)\alpha \ge n^2(nx+n-1) + n^2(n+1)/2$.

(b) In this step we prove that $h^1(\mathcal{I}_{W_1} \otimes \mathcal{L}[t]) = 0$. By Observation 1 and the Differential Horace Lemma it is sufficient to prove that $h^1(\mathcal{I}_{\hat{Z}\cup S'\cup(2S'',H)} \otimes \mathcal{L}[t-1]) = 0$. Since $\hat{Z} \subseteq Z$, Claim 2 gives $h^1(\mathcal{I}_{\hat{Z}\cup(2S'',H)} \otimes \mathcal{L}[t-1]) = 0$. Note that $\deg(\hat{Z}) + z + ne = h^0(\mathcal{L}[t]) - (n+1)w \ge 0$. Hence the generality of S' in H, show that to conclude the proof for W_1 it is sufficient to prove that $h^0(\mathcal{I}_{\hat{Z}} \otimes \mathcal{L}[t-2]) \le h^0(\mathcal{L}[t-1]) - \deg(\hat{Z}) - z - ne$. We have $h^0(\mathcal{L}[t-1]) - h^0(\mathcal{L}[t-2]) = \binom{r+t-2}{r-1}\alpha$. Hence as in step (a) it is sufficient to assume that $\binom{r+t-2}{r-1}(n-r/t)\alpha \ge n^2(nx+n-1) + n^2(n+3)/2$. □

Proof of Corollary 1: Note that $\binom{r+1}{2}(n - r/3) \le \binom{r+t-2}{r-1}(n - r/t)$ for all $t \ge 3$. Hence we may apply Theorem 3. □

Proof of Theorem 1: Set $n := m+2$. Fix $H \in |\mathcal{O}_X(0,1)|$. Set $w_t := \lfloor \binom{t+2}{2}\alpha/(n+1)\rfloor$, $w'_t := \lceil \binom{t+2}{2}\alpha/(n+1)\rceil$, $f_t := \binom{t+2}{2}\alpha - (n+1)w_t$, $z_t := \lfloor (t+1)\alpha/n\rfloor$ and $e_t := (t+1)\alpha - nz_t$. By Remark 2 to prove the theorem for the line bundle $\mathcal{L}[t]$ it is sufficient to prove $h^1(\mathcal{I}_{W_1} \otimes \mathcal{L}[t]) = 0$ and $h^0(\mathcal{I}_{W_2} \otimes \mathcal{L}[t]) = 0$, where W_1 is a general union of w_t different double points of X and W_2 is a general union of w'_t double points of X.

Claim 1: We have $w_t \ge z_t + e_t$ for all $t \ge 2$.

Proof of Claim 1: Assume $w_t \le z_t + e_t - 1$. We get $\binom{t+2}{2}\alpha - n \le (n+1)z_t + (n+1)e_t - n - 1 = \binom{t+2}{2}\alpha + z_t + ne_t - n - 1$, i.e., $\binom{t+1}{2}\alpha \le z_t + ne_t - 1$. Since $nz_t + e_t = (t+1)\alpha$, $\binom{t+1}{2} \ge t+1$ and $e_t \le nx + n - 1$, we get $z_t \le nx + n - 1$. Hence $(t+1)\alpha \le n(n-1) + n - 1 + 2nx$, a contradiction.

Fix a general $S_t \subset X$ (resp. $\hat{S}_t \subset X$) such that $\#S_t = w'_t - z_t - e_t$ (resp. $\#\hat{S}_t = w_t - z_t - e_t$) and a general $S'_t \cup S''_t \subset H$ such that $S'_t \cap S''_t = \emptyset$, $\#S'_t = z_t$ and $\#S''_t = e_t$.

Observation 1: Fix $t \ge 2$. Since $(m-1)\alpha \ge (m+1)^2 + (m+2)(m-1)((m+1)x+m)$, Proposition 1 gives that $\mathcal{L}[t]_{|H}$ is not defective. Hence we have the vanishing $h^i(H, \mathcal{I}_{(2S'_t,H)\cup S''_t} \otimes \mathcal{L}[t]_{|H}) = 0$, $i = 0,1$. Applying the Differential Horace Lemma (Remark 1) to each point of S''_t we see that to prove that $h^0(\mathcal{I}_{W_2} \otimes \mathcal{L}[t]) = 0$ (resp. $h^1(\mathcal{I}_{W_1} \otimes \mathcal{L}[t]) = 0$) it is sufficient to prove that $h^0(\mathcal{I}_{Z_t\cup S'_t\cup(2S''_t,H)} \otimes \mathcal{L}[t-1]) = 0$ (resp. $h^1(\mathcal{I}_{\hat{Z}_t\cup S'_t\cup(2S''_t,H)} \otimes \mathcal{L}[t-1]) = 0$).

Observation 2: Since $(n+1)w_t \le (t+1)\alpha \le (n+1)w'_t$, we have

$$\deg(\hat{Z} \cup S' \cup (2S'', H)) \le h^0(\mathcal{L}[t-1]) \le \deg(Z \cup S' \cup (2S'', H)).$$

(a) In this step we take $t = 2$ and prove that $\mathcal{L}[2]$ is not defective.

Claim 3: We have $h^1(\mathcal{I}_{Z_2\cup(S''_2,H)} \otimes \mathcal{L}[1]) = 0$.

Proof of Claim 3: By Observation 1, Theorem 4 applied to H and the inequalities $ne_2 \le n(n-1)$ and $n(n-1) + n(n-1)/2 \le 2\alpha$, we have $h^1(H, \mathcal{I}_{(2S''_2,H),H} \otimes \mathcal{L}[1]_{|H}) = 0$. Hence to apply Theorem 4 to $\mathcal{L}[1]$ with as W the e_2-secant variety of the pair $(H, \mathcal{L}[1]_{|H})$ it is sufficient to have $(n+1)(w'_2 - z_2 - e_2) + ne_2 \le 3\alpha - n(n+1)/2$. Since $nz_2 + e_2 = 3\alpha$, $(n+1)w'_2 \le 6\alpha + n$ and $e_2 \le n-1$, it is sufficient to have $z_2 \ge n^2 + n(n+1)/2$, i.e., $nz_2 \ge n^2(3n+1)/2$. Since $n(n+1)/2$ is an integer, it is sufficient to assume $3\alpha \ge n^2(3n+1)/2$, i.e., $\alpha \ge (m+2)^2(3m+7)/6$.

By Claim 3 and Observation 2 to prove the theorem for $t = 2$ and both W_1 and W_2 it is sufficient to prove that $h^0(\mathcal{I}_{\hat{Z}_2} \times \mathcal{L}[0]) = 0$. Since $\mathcal{L}$ is $\pm x\star$-nondefective, it is sufficient to prove that $(n-1)(w_2-z_2-e_2-x) \geq \alpha$, i.e., $(n+1)n(n-1)(w_2-z_2-e_2-x) \geq (n+1)n\alpha$. Since $nz_2+e_2 = 3\alpha$, $(n+1)w_2 \geq 6\alpha-n$ and $e_2 \leq n-1$, it is sufficient to have $(2n^2-7n+3)\alpha \geq +x(n+1)n(n-1)+n^2(n-1)+(n+1)(n-1)^3$. Since $n^2(n-1)+(n+1)(n-1)^3 = (n-1)(n^3-4n^2+3n-1)$, it is sufficient to prove the inequality

$$(2m^2+m-3)\alpha \geq (m+1)(m^3+2m^2-m-7+x(m+2)(m+3)). \tag{3}$$

Now we check that (1) implies (3). Indeed, since $\alpha \geq (m+3)m((m+1)x+m)$, it is sufficient to check that $(2m^2+m-3)(m+3)m \geq (m+1)(m^3+2m^2-m-7)$ and $(2m^2+m-3)(m+3)(m+1)x \geq (m+1)(m+2)(m+3)x$.

(b) Now we assume $t \geq 3$. By induction on the integer t and part (a) we may assume that $\mathcal{L}[y]$ is not defective for all $2 \leq y < t$. Since $t \geq 3$ and $e_t \leq n-1$, $z_{t-1} \geq e_t$. Let $S_1 \subset H \setminus S_t$ be a general subset with cardinality $z_{t-1} - e_t$.

(b1) In this step we prove that $h^1(\mathcal{I}_{Z_t \cup (S_t'', H)} \otimes \mathcal{L}[t-1]) = 0$. If $\deg(Z_t)/(n+1) < z_{t-1} - e_t$, then degenerating Z_t to a general union of double points of X with reduction contained in H we get $h^1(\mathcal{I}_{Z_t \cup (S_t'', H)} \otimes \mathcal{L}[t-1]) = 0$. Thus we may assume $w_t' - z_t - z_{t-1} \geq 0$. Let $E_t \subset X$ be a general union of $w_t' - z_t - z_{t-1}$ double points of X. We degenerate Z_t to $E_t \cup 2S_1$. Since $\mathcal{L}[t-1]_{|H}$ is not defective, $h^1(H, \mathcal{I}_{(2(S_1 \cup S_t'', H)), H} \otimes \mathcal{L}[t-1]_{|H}) = 0$. Hence to conclude the proof of step (b1) it is sufficient to prove $h^1(\mathcal{I}_{E_t} \cup S_1 \otimes \mathcal{L}[t-2]) = 0$. Since $(n+1)w_t' - \binom{t+2}{2}\alpha + n$ and $e_{t-1} \leq n-1$, we have $\deg(E_t) + \#S_1 \leq \binom{t}{2}\alpha + 2n - 1 - z_t \leq \binom{t}{2}\alpha$.

Claim 4: We have $h^1(\mathcal{I}_{E_t} \otimes \mathcal{L}[t-2]) = 0$

Proof of Claim 4: If $t \geq 4$ it is sufficient to use the inductive assumption. Now assume $t = 3$. By Theorem 4 it is sufficient to prove that $\deg(E_3) \leq 3\alpha - n(n+1)/2$. Since $(n+1)w_3' - 6\alpha \leq n$ and $e_2 \geq 0$, we have $\deg(E_3) \leq 3\alpha + n - z_2$. Hence it is sufficient to have $z_2 \geq n(n+3)/2$, i.e. $nz_2 \geq n^2(n+3)/2$. Since $n^2(n+3)/2$ is an integer, to prove Claim 4 it is sufficient to use the inequality $3\alpha \geq n^2(n+3)/2 = (m+2)^2(m+5)/2$.

(b2) We have $h^0(\mathcal{L}[t-2]) - h^0(\mathcal{L}[t-3]) = (t-1)\alpha$. Since $h^1(\mathcal{I}_{Z_t \cup (S_t'', H)} \otimes \mathcal{L}[t-1]) = 0$ (step (b1)), $h^1(\mathcal{I}_{\hat{Z}_t \cup (S_t'', H)} \otimes \mathcal{L}[t-1]) = 0$. By Remark 3 to prove that $h^0(\mathcal{I}_{W_2} \otimes \mathcal{L}[t]) = 0$ it is sufficient to prove that

$$h^0(\mathcal{I}_{Z_t} \otimes \mathcal{L}[t-2]) \leq \max\{0, h^0(\mathcal{L}[t-2]) - \deg(Z_t) - z_t - ne_t\}.$$

First assume $t \geq 4$. The inductive assumption gives that either $h^0(\mathcal{I}_{Z_t} \otimes \mathcal{L}[t-2]) = 0$ or $h^1(\mathcal{I}_{Z_t} \otimes \mathcal{L}[t-2]) = 0$, i.e. $h^0(\mathcal{I}_{Z_t} \otimes \mathcal{L}[t-2]) = h^0(\mathcal{L}[t-2]) - \deg(Z_t)$. In the latter case it is sufficient to prove that $(t+1)\alpha \geq z_t + net_t$, i.e., $n(t+1)\alpha \geq nz_t + n^2e_t$. The same proof gives $h^1(\mathcal{I}_{W_1} \otimes \mathcal{L}[t]) = 0$.

Now assume $t = 3$. To get $h^0(\mathcal{I}_{Z_3} \otimes \mathcal{L}[1]) = 0$, it is sufficient to have $\deg(Z_3) \geq 3\alpha + n(n+1)/2$. To prove that $h^1(\mathcal{I}_{W_1} \otimes \mathcal{L}[t]) = 0$ we will check that $h^0(\mathcal{I}_{\hat{Z}_t} \otimes \mathcal{L}[1])) = 0$. Hence to conclude the proof of the theorem it is sufficient prove that $\deg(\hat{Z}_3) \geq 3\alpha + n(n+1)/2$. We have $\deg(\hat{Z}_3) \geq 10\alpha - n - (n+1)z_3 - ne_3 \geq 10\alpha - n^2 - (n+1)z_3$ with $e_3 \leq n-1$. Hence it is sufficient to have $10n\alpha \geq n^3 + (n+1)4\alpha$, which is obviously true. $\square$

Proof of Theorem 2: The case $y = 0$ is true by [16, Th. 2]. Now assume $9 \leq y < k$. Let $\mathcal{L}$ be the Segre line bundle of $(\mathbb{P}^2)^y$. By Remark 4 $\mathcal{L}$ is $\pm 2\star$-nondefective. We have $\alpha := 3^y$. and $m :=$

$\dim Y = 2y$. Set $W := Y \times \mathbb{P}^2$ and $E := \mathcal{O}_W(1,\dots,1,t_{y+1})$. Since $t_{y+1} \ge 2$, it is sufficient to check the assumptions of Theorem 1 with $x = 2$. First, we check the inequality (2), i.e., the inequality $3^y \ge (2y+2)^2(6y+7)/6$, i.e., $3^{y+1} \ge 2(y+1)^2(6y+7)$, which is true for $y \ge 9$. The inequality (1), i.e., $(2y-1)3^y \ge (2y+1)^2 + (2y+2)(2y-1)(6y+2)$ is true for all $y \ge 9$. □

5. Concluding Remarks

Remark 5. *As in [16] from Theorem 2 and [14] we may add another factor $W := (\mathbb{P}^1)^r$ with $\mathcal{O}_W(x_1,\dots,x_r)$ and $x_i \ge 2$ for all i. Using [16, Th. 1] and the known cases in which the Segre variety of $(\mathbb{P}^2)^a$ is non-defective we may do other cases. Remark 4, Lemma 1 and [16, Th. 1] give the case $y = 4$.*

As in [6] and many of its descendants we work over an algebraically closed field with characteristic zero. Of course, secant non-defectivity is translated in the good "Hilbert function" of general unions of double points. Since a multiprojective space is defined over $\mathbb{Q}$ and its $\mathbb{Q}$-points are Zariski dense, we may find general unions of double point with good Hilbert function and with as reduction points defined over $\mathbb{Q}$, i.e. points with integers as multihomogeneous coordinates.

Remark 6. *In most of the proofs in [14, 16] we only use that $h^1(Y,\mathcal{I}_{2S}\otimes\mathcal{L}) = 0$ if $\#S + x \le h^0(\mathcal{L})/(\dim Y + 1)$, which should be called $-x\star$-nondefectivity. In this paper we often use that $h^0(Y,\mathcal{I}_{2S}\otimes\mathcal{L}) = 0$ if $\#S + x \le h^0(\mathcal{L})/(\dim Y + 1)$, which should be called $+\star$-nondefectivity, when we quote Theorem 4. The one-sided notions of bounded defectivity are useful to apply [6, Th. 5.2 and Cor. 5.4].*

The Differential Horace Lemma does not need the characteristic 0 assumption. The assumption that the base field has characteristic 0 is essential to use [7]. Moreover, several papers required characteristic 0, because some cases were done by computer, e.g., [24], and hence, they are certainly true over an algebraically closed field of characteristic 0, but no hint for all positive characteristics. Then these papers are used in other papers. Hence it would be very difficult to check what is proved for some fixed positive characteristic

6. Declarations

This research produced no database and hence data availability is not an issue.

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