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# Classifying symplectic metric Jordan superalgebras

## Classification des super-algèbres de Jordan métriques symplectiques

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**ABSTRACT.** This paper complements the description of finite-dimensional Jordan symplectic metric superalgebras on algebraically closed fields of characteristic zero. We discuss the graduation of the metric and the symplectic structures and use a new type of generalized double extension by two-dimensional Jordan superalgebras.
**KEYWORDS.** Jordan superalgebras, classification of algebras, symplectic algebras, double extension.

## 1 Introduction

A symplectic metric Jordan superalgebra is a Jordan superalgebra endowed with both a homogeneous symplectic form and a homogeneous associative non-degenerate supersymmetric bilinear form. The inductive classification problem of finite dimensional algebraic structures has been extensively studied. However, the results are relatively scarce in the context of superalgebras.

Most of the known results focus on the theory of Lie superalgebras. In 1977, V. G. Kac developed Kantor's method and applied it to obtain a classification of finite-dimensional simple Jordan superalgebras over an algebraically closed field of characteristic $0$ (cf. [10]). There was one missing case in this classification that was later described by I. L. Kantor in [11]. More than two decades later, M. L. Racine and E. I. Zelmanov took advantage of Jordan theoretic methods, like Peirce decomposition and representation theory (that are less sensitive to the characteristic of the base field) to obtain a classification of simple Jordan superalgebras over fields of characteristic different from 2 whose even part is semisimple (cf. [18]). The remaining case, namely simple Jordan superalgebras with nonsemisimple even part, was tackled in [14] by C. Martinez and E. I. Zelmanov. Other recent study of simple Jordan supralgebras can be found for example in [16, 12] However, non-simple Jordan superalgebras have less studied.

The purpose of this paper is to provide an inductive classification of the non-simple symplectic metric Jordan superalgebras. By inductive classification, we mean a method of constructing all symplectic metric Jordan superalgebras using only uni-dimensional and bi-dimensional Jordan superalgebras. We itemize the classification on a case-by-case basis. Towards this goal, we provide two generalizations of the process of symplectic double extension (cf. [7, 8, 13, 15]) using one dimensional and two dimensional Jordan superalgebras.

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Note that the metric Jordan superalgebras have descibed in [5] through the so-called double extension method. Describing symplectic Jordan algebras through double extensions techniques seems very complicated. This being so, the inductive description of symplectic metric Jordan algebras by double extensions methods was obtained in [4, 3].

The outline of the paper is as follow: The next section focuses on the introduction of notations and the development of certain necessary tools. The third section is devoted to a detailed classification of all symplectic metric Jordan superalgebras through case by case basis. When the symplectic and metric structures are odd, one constructs directly into Theorem 3.1 symplectic metric Jordan superalgebras using the process of double extensions. In addition, for the even case, we consider a generalized double extension by a uni-dimensional Jordan algebra and prove in Theorem 3.1 that each symplectic metric Jordan algebra is obtained through a sequence of symplectic double extensions using one dimensional Jordan superalgebra and/or generalized double extension. In the context of symplectic and metric structures of different graduation, we first define a new type of generalized double extension inducing a two dimensional Jordan superalgebra. Consequently, we then prove in Theorem 3.2 that any symplectic metric Jordan superalgebra is obtained as a finite sequence of such two dimentional generalized double extensions.

## 2 Backgrounds

In this section we give some basic notions and concepts used through the paper.

### 2.1 Superspaces and superalgebras

Throughout the paper, linear spaces are assumed to be finite-dimensional over a field $\mathbb{K}$ of characteristic zero. Recall that a $\mathbb{Z}_2$-graded vector space $\mathcal{V} = \mathcal{V}_{\bar{0}} \oplus \mathcal{V}_{\bar{1}}$ is called a $\mathbb{K}$-superspace or merely a superspace for brevity, where $\mathbb{Z}$isthe ring of integers. Elements of the subspace $\mathcal{V}_{\bar{0}}$ are said to be even and elements of $\mathcal{V}_{\bar{1}}$ to be odd. For a homogeneous element $v \in (\mathcal{V}_{\bar{i}})_{i\in\{0,1\}}$, we denote by $|v|$ its parity (i.e. $|v| = \bar{i}$). A $\mathbb{K}$-superalgebra (or merely a superalgebra) is an algebra whose underlying vector space is the superspace $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ such that $\mathcal{A}_{\bar{i}}\mathcal{A}_{\bar{j}} \subset \mathcal{A}_{\overline{i+j}}, \quad i, j \in \{0, 1\}$. In this case, $\mathcal{A}^2 = \mathrm{span}\{xy; x, y \in \mathcal{A}\}$ has a structure of superspace where:

$$(\mathcal{A}^2)_{\bar{0}} := \mathcal{A}_{\bar{0}}\mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{1}}\mathcal{A}_{\bar{1}} = \mathrm{span}\{xy + zt, \quad x, y \in \mathcal{A}_{\bar{0}}, \quad z, t \in \mathcal{A}_{\bar{1}}\}$$

and

$$(\mathcal{A}^2)_{\bar{1}} := \mathcal{A}_{\bar{0}}\mathcal{A}_{\bar{1}} + \mathcal{A}_{\bar{1}}\mathcal{A}_{\bar{0}} = \mathrm{span}\{xy + zt, \quad x, t \in \mathcal{A}_{\bar{0}}, \quad y, z \in \mathcal{A}_{\bar{1}}\}.$$

Let $\mathcal{V} = \mathcal{V}_{\bar{0}} \oplus \mathcal{V}_{\bar{1}}$ and $\mathcal{W} = \mathcal{W}_{\bar{0}} \oplus \mathcal{W}_{\bar{1}}$ be two superspaces. Then, $\tilde{\mathcal{V}} = \mathcal{V} \oplus \mathcal{W}$ is also a superspace with $\tilde{\mathcal{V}}_{\bar{0}} = \mathcal{V}_{\bar{0}} \oplus \mathcal{W}_{\bar{0}}$ and $\tilde{\mathcal{V}}_{\bar{1}} = \mathcal{V}_{\bar{1}} \oplus \mathcal{W}_{\bar{1}}$.

A linear map $f : \mathcal{V} \to \mathcal{W}$ is called homogeneous of degree $\alpha \in \{0, 1\}$ if $f(\mathcal{V}_{\bar{i}}) \subseteq \mathcal{W}_{\overline{\alpha+i}}, i \in \{0, 1\}$. We denote by $\mathrm{Hom}(\mathcal{V}, \mathcal{W})_{\bar{\alpha}}$ the space of homogeneous linear map from $\mathcal{V}$ to $\mathcal{W}$ of degree $\alpha$. Let $\mathrm{End}(\mathcal{V}) := \mathrm{Hom}(\mathcal{V}, \mathcal{V})$ and $\mathrm{End}_{\bar{\alpha}}(\mathcal{V}) := \mathrm{Hom}(\mathcal{V}, \mathcal{V})_{\bar{\alpha}}$. Let also $\mathcal{V}^* := \mathrm{Hom}(\mathcal{V}, \mathbb{K})$ and $\mathcal{V}^*_{\bar{\alpha}} := \mathrm{Hom}(\mathcal{V}, \mathbb{K})_{\bar{\alpha}}$, where $\mathbb{K}$ is considered as a superspace with $\mathbb{K}_{\bar{0}} := \mathbb{K}$ and $\mathbb{K}_{\bar{1}} := \{0\}$. The superspace $\mathcal{V}^* = \mathcal{V}^*_{\bar{0}} \oplus \mathcal{V}^*_{\bar{1}}$ is called the dual superspace of $\mathcal{V} = \mathcal{V}_{\bar{0}} \oplus \mathcal{V}_{\bar{1}}$.

## 2.2 Jordan superalgebras

A Jordan superalgebra $\mathfrak{J}$ is a $\mathbb{K}$-superalgebra such that for every homogeneous elements $a, b, c, d \in \mathfrak{J}$, one has:

- $ab = (-1)^{|a||b|}ba$ (super-commutativity).
- $(-1)^{|a||c|}(ab)(cd) + (-1)^{|a||b|}(bc)(ad) + (-1)^{|b||c|}(ca)(bd)$
$= (-1)^{|a||c|}a((bc)d) + (-1)^{|a||b|}b((ca)d) + (-1)^{|b||c|}c((ab)d)$ (super-Jordan identity).

For $a \in \mathfrak{J}$, the linear map $L_a : \mathfrak{J} \longrightarrow \mathfrak{J}$ defined by $L_a(x) = ax, \quad \forall x \in \mathfrak{J}$ is called the left multiplication by $a$.

A superderivation on $\mathfrak{J}$ is a homogeneous linear map $D : \mathfrak{J} \longrightarrow \mathfrak{J}$ that satisfies:

$$D(xy) = D(x)y + (-1)^{|x||D|}xD(y),\ x, y \in \mathfrak{J}.$$

A linear subspace $\mathfrak{I}$ of a Jordan superalgebra $\mathfrak{J}$ is called an ideal of $\mathfrak{J}$, if $xy \in \mathfrak{I}$ for any $x \in \mathfrak{I}$ and $y \in \mathfrak{J}$. An ideal $\mathfrak{I}$ of a Jordan superalgebra $\mathfrak{J}$ is called a graded ideal if whenever $x = x_{\bar{0}} + x_{\bar{1}} \in \mathfrak{I}$ (where $x_{\bar{i}} \in \mathfrak{J}_{\bar{i}}$ for $i \in \{0, 1\}$ ), both $x_{\bar{0}}$ and $x_{\bar{1}}$ belong to $\mathfrak{I}$. The annihilator of a Jordan superalgebra $\mathfrak{J}$ is defined by:

$$\mathrm{Ann}(\mathfrak{J}) := \{x \in \mathfrak{J};\ xy = yx = 0, \quad y \in \mathfrak{J}\}.$$

is a graded ideal.

Let $\mathfrak{J} = \mathfrak{J}_{\bar{0}} \oplus \mathfrak{J}_{\bar{1}}$ be a Jordan superalgebra.

A Jordan superalgebra is called solvable (resp. nilpotent) if there exists a natural number $n$ such that $\mathfrak{J}^{(n)} = \{0\}$ (resp. $\mathfrak{J}^n = \{0\}$) where the sequence $(\mathfrak{J}^{(k)})_{k\in\mathbb{N}}$ (resp. $(\mathfrak{J}^k)_{k\in\mathbb{N}}$) is defined inductively by the rules:

$$\mathfrak{J}^{(0)} = \mathfrak{J} \text{ and } \mathfrak{J}^{(k+1)} = \mathfrak{J}^{(k)}\mathfrak{J}^{(k)}.$$

$$(\text{resp. } \mathfrak{J}^0 = \mathfrak{J} \text{ and } \mathfrak{J}^{k+1} = \mathfrak{J}\mathfrak{J}^k).$$

For more details about these kind of algebras the reader can consult [1, 2, 17].

## 2.3 Metric and odd-metric Jordan superalgebras

A bilinear form $B$ of a Jordan superalgebra $\mathfrak{J}$ is said to be:

- Supersymmetric, if $B(a, b) = (-1)^{|a||b|}B(b, a),\ \ a \in \mathfrak{J}_{|a|},\ \ b \in \mathfrak{J}_{|b|}$.
- Non-degenerate, if $Rad(\mathfrak{J}) := \{x \in \mathfrak{J}\ ;\ B(x, y) = 0, \quad y \in \mathfrak{J}\} = \{0\}$.
- Associative, if $B(ab, c) = B(a, bc),\ \ a, b, c \in \mathfrak{J}$.
- Even, if $B(\mathfrak{J}_{\bar{0}}, \mathfrak{J}_{\bar{1}}) = B(\mathfrak{J}_{\bar{1}}, \mathfrak{J}_{\bar{0}}) = \{0\}$.
- Odd, if $B(\mathfrak{J}_{\bar{0}}, \mathfrak{J}_{\bar{0}}) = B(\mathfrak{J}_{\bar{1}}, \mathfrak{J}_{\bar{1}}) = \{0\}$.

An even (resp. odd) bilinear form $B$ on $\mathfrak{J}$ is called an associative (resp. odd associative) scalar product of $\mathfrak{J}$ if $B$ is supersymmetric, non-degenerate and associative. A Jordan superalgebra $\mathfrak{J}$ admitting an associative (resp. odd associative) scalar product is said to be metric (resp. odd-metric) Jordan superalgebra.

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A graded ideal $\mathcal{I}$ of an odd metric Jordan superalgebra $\mathfrak{J}$ is called non-degenerate, if the restriction of $B$ to $\mathcal{I} \times \mathcal{I}$ is non-degenerate. Such a graded ideal is called $B$-irreducible if it contains no non-degenerate graded ideals of $\mathfrak{J}$ other than $\{0\}$ and $\mathcal{I}$. A graded ideal that is not non-degenerate is called degenerate.

**Definition 2.1.** *An endomorphism $\phi$ of $\mathfrak{J}$ of degree $\alpha \in \{0, 1\}$ is said to be $B$-supersymmetric (resp. $B$-skewsupersymmetric) if for all homogeneous elements $x, y \in \mathfrak{J}$,*

$$B(\phi(x), y) = (-1)^{\alpha|x|}B(x, \phi(y)) \ \big(\textit{resp. } B(\phi(x), y) = -(-1)^{\alpha|x|}B(x, \phi(y))\big).$$

*A $B$-supersymmetric endomorphism $\phi$ of $\mathfrak{J}$ is said to be a symmetric Jordan operator of $\mathfrak{J}$ if it satisfies:*

$$\begin{aligned}&(-1)^{(|a|+|b|)|c|}(\phi(a)c)b + (-1)^{|a||c|}\phi((ab)c) + (-1)^{|a||b|}(\phi(b)c)a\\ &= \sum_{cycl}(-1)^{|b||c|}\phi(c)(ab) = \sum_{cycl}(-1)^{|a||c|}\phi(ab)c, \ \ (a, b, c) \in \mathfrak{J}_{|a|} \times \mathfrak{J}_{|b|} \times \mathfrak{J}_{|c|}.\end{aligned}$$

*The sub-superspace of End$(\mathfrak{J})$ consisting of the symmetric Jordan operators is denoted by $Op_s(\mathfrak{J})$.*

### 2.4 Symplectic metric Jordan superalgebras

A bilinear form $\omega$ on $\mathfrak{J}$ is said to be:

- Skew-Supersymmetric, if $\omega(x, y) = -(-1)^{|x||y|}\omega(y, x))$,
- Supercyclic, if $(-1)^{|x||z|}\omega(x, yz) + (-1)^{|x||y|}\omega(y, zx) + (-1)^{|y||z|}\omega(z, xy) = 0$,

for all homogeneous elements $x, y, z \in \mathfrak{J}$. In this case $\omega$ is also called a scalar 2-cocyle of $\mathfrak{J}$. An even (resp. odd) bilinear form $\omega$ on $\mathfrak{J}$ is called symplectic (odd symplectic) if $\omega$ is skew-supersymmetric, non-degenerate and supercyclic. A Jordan superalgebra $\mathfrak{J}$ admitting a symplectic (resp. odd symplectic) form is said to be symplectic (resp. odd-symplectic) Jordan superalgebra. □

## 3 Inductive classification symplectic metric Jordan superalgebras

In this section, we consider the setting of finite dimensional Jordan superalgebras over an algebraically closed field of characteristic zero. We first record two types of symplectic metric Jordan algebras according to the fact that the symplectic and metric structures have or fail to get the same graduation. Let us prove the following results:

**Proposition 3.1.** *Let $(\mathfrak{J}, B)$ be a metric (or odd-metric) Jordan superalgebra. Then, $\mathfrak{J}$ has a symplectic (or odd symplectic) form $\omega$, if and only if, there exists an invertible $B$-skew-supersymmetric superderivation $D$ of $\mathfrak{J}$ such that* $\deg D = \deg B + \deg \omega$, *and*

$$\omega(x, y) = B(D(x), y), \quad x, y \in \mathfrak{J}.$$

*Proof.* Let $\omega$ be a symplectic structure on $\mathfrak{J}$. Consider the linear map $D : \mathfrak{J} \to \mathfrak{J}$ defined for $x \in \mathfrak{J}$ by $D(x) = y$, where $y$ is the unique homogeneous element of $\mathfrak{J}$ such that $B(y, .) = \omega(x, .)$. Sinse $B$ and $\omega$ are non-degenerate, $D$ is invertible. Moreover, as $\omega$ is skew-supersymmetric and $B$ is supersymmetric, then $D$ is skew-supersymmetric with respect to $B$. The fact that $\omega$ is a scalar 2-cocyle of $\mathfrak{J}$, implies that

$$B\Big(D(x)y + (-1)^{|x|degD}xD(y) - D(xy), z\Big) = 0, \quad x, y, z \in \mathfrak{J}.$$

Thus, $D$ is a superderivation of $\mathfrak{J}$. The converse is easy to check. □

**Proposition 3.2.** *Let* $(\mathfrak{J}, B, \omega)$ *be a symplectic metric Jordan superalgebra. Then,* $\mathrm{Ann}(\mathfrak{J}) \neq \{0\}$.

*Proof.* Let $D$ be the $B$-skew-supersymmetric invertible superderivation of $\mathfrak{J}$ such that

$$\omega(x, y) = B(D(x), y), \quad x, y \in \mathfrak{J} \quad \text{and} \quad \deg D = \deg B + \deg w.$$

The Jordan algebra $\mathfrak{J}_{\bar{0}}$ has an invertible derivation equals to $D$ if $D$ is even and to $D^2$ otherwise. By (Lemma 7.2 of [4]), $\mathfrak{J}_{\bar{0}}$ is a nilpotent Jordan algebra. Thus, $\mathrm{Ann}(\mathfrak{J}) \neq \{0\}$. □

**Corollary 3.1.** *Let* $(\mathfrak{J}, B, \omega)$ *be an odd symplectic metric Jordan superalgebra. Then,* $\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{0}} \neq \{0\}$.

*Proof.* It is clear that $\mathrm{Ann}(\mathfrak{J}_{\bar{0}}) \neq \{0\}$. As $B$ is odd, associative and non-degenerate, then $\mathrm{Ann}(\mathfrak{J}_{\bar{0}}) \subset (\mathrm{Ann}(\mathfrak{J}))_{\bar{0}}$. Thus, $(\mathrm{Ann}(\mathfrak{J}))_{\bar{0}} \neq \{0\}$ and finally $\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{0}} \neq \{0\}$. □

### 3.1 The case where the metric and the symplectic structures have the same graduation

Let $\mathfrak{J}$ be a Jordan superalgebras which admits a metric structure $B$ and a symplectic structure $\omega$ of the same graduation and let $\delta$ be the derivation of $\mathfrak{J}$ is defined by $\omega(x, y) = B(\delta(x), y)$ for all $x, y \in \mathfrak{J}$. Let $\phi$ be a symmetric Jordan operator such that $\phi^2$ is an even superderivation and let $(D, A_0) \in Op_s(\mathfrak{J}))_{\bar{0}} \times \mathfrak{J}_{\bar{0}}$ be an admissible pair (c.f. [5], Definition 4.1). That is the pair satisfies :

$$\begin{aligned}
&\mathcal{C}_1: \quad 3D(aA_0) = aD(A_0) + 2D^3 a \\
&\mathcal{C}_2: \quad D(A_0 a) = A_0 D(a) \\
&\mathcal{C}_3: \quad D^2 A_0 = A_2^0 \\
&\mathcal{C}_4: \quad abA_0 = D^2(ab) + aD^2 b + (-1)^{|a||b|} bD^2(a) - 2D(a)D(b) \\
&\mathcal{C}_5: \quad (ab)A_0 - a(bA_0) = 2D(D(a)b) - 2D(a)D(b)
\end{aligned}$$

.

Let also $A \in \mathfrak{J}_{\bar{1}}, x_0 \in \mathfrak{J}_{\bar{0}}$ and $\alpha \in \mathbb{K}$.

Consider $(\mathfrak{J}_3, B_3)$ (resp. $(\mathfrak{J}_4, B_4)$) the double extension (resp. the generalized double extension) of $(\mathfrak{J}, B)$ by $(\mathbb{K}e)_{\bar{1}}$ (resp. by $(\mathbb{K}e)_{\bar{0}}$) by means of $\phi$ (resp. of $(D, A_0)$).

For $i \in \{3, 4\}$, define on $\mathfrak{J}_i$ the bilinear form $\omega_i$ by

$$\omega_i(x, y) = B_i(\delta_i(x), y), \quad x, y \in \mathfrak{J}_i,$$

where $\delta_i$ is the endomorphism of $\mathfrak{J}_i$ defined by:

$$\text{For } i = 3: \quad \delta_3(e^*) = \alpha e^*, \qquad \delta_3(e) = -\alpha e + A, \qquad \delta_3(x) = \delta(x) - B(x, A)e^*, \quad x \in \mathfrak{J}.$$

$$\text{For } i = 4: \quad \delta_4(e^*) = \alpha e^*, \qquad \delta_4(e) = -\alpha e + x_0, \qquad \delta_4(x) = \delta(x) - B(x, x_0)e^*, \quad x \in \mathfrak{J}.$$

**Proposition 3.3.** *(i) If* $[\delta, \phi] + \alpha\phi = L_A$*, then* $(\mathfrak{J}_3, B_3, \omega_3)$ *is a symplectic metric Jordan superalgebra.*

*(ii) If* $B$ *is even and* $D(x_0) = \alpha A_0 + (1/2)\delta(A_0)$ *and* $[\delta, D] + \lambda D = L_{x_0}$*, then* $\omega_4$ *is an even symplectic form on* $\mathfrak{J}_4$ *and* $(\mathfrak{J}_4, B_4, \omega_4)$ *is a symplectic metric Jordan superalgebra.*

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*Proof.* (i) Since $\deg B = \deg \omega$, $D$ is an even superderivation of $\mathfrak{J}$. The fact that $B$ is non-degenerate and $[\delta, \phi] + \alpha\phi = L_A$ give that

$$\delta_3\big((x+e) \star y\big) = \delta_3(x+e) \star y + (x+e) \star \delta_3(y), \quad x, y \in \mathfrak{J}.$$

Furthermore, $\delta_3(e \star e) = 0 = \delta_3(e) \star e + e \star \delta_3(e)$. Thus, $\delta_3$ is an even superderivation of $(\mathfrak{J}_3, B_3)$. Moreover, since $\delta$ is $B$-skew-supersymmetric, then $\delta_3$ is a $B_3$-skew-supersymmetric superderivation of $(\mathfrak{J}_3, B_3)$.

(ii) Since $\delta$ is invertible, then so is $\delta_4$. The fact that $\delta$ is an even superderivation of $\mathfrak{J}$, $B$ is non-degenerate and $\delta D - D\delta + \alpha D = L_{x_0}$ entails,

$$\delta_4(x \star y) = \delta_4(x) \star y + x \star \delta_4(y), \quad x, y \in \mathfrak{J}.$$

Moreover, as $\delta D - D\delta + \lambda D = L_{x_0}$ and $D(x_0) = \lambda A_0 + (1/2) \quad \delta(A_0)$ we get

$$\delta_4(x \star e) = \delta_4(x) \star e + x \star \delta_4(e), \quad x \in \mathfrak{J}.$$

By an easy computation, we obtain that $\delta_4(e \star e) = 2\delta_4(e) \star e$. Therefore, $\delta_4$ is an even superderivation of $\mathfrak{J}_4$. Furthermore, it is clear that $B_4(\delta_4(x), y) = -B_4(x, \delta_4(y))$, for all $x, y \in \mathfrak{J}$. $\square$

**Remark 3.1.** The symplectic forms $\omega_3$ and $\omega_4$ are of the same graduation of $\omega$.

## The main result

**Definition 3.1.** 1. The Jordan superalgebra $(\mathfrak{J}_3, B_3, \omega_3)$ obtained as in Proposition 3.3, (i) is said to be the symplectic metric double extension (abbreviated as s.m.d.e.1) of $(\mathfrak{J}, B, \omega)$ by $(\mathbb{K}e)_{\bar{1}}$ by means of $\phi, A$ and $\alpha$.

2. The Jordan superalgebra $(\mathfrak{J}_4, B_4, \omega_4)$ obtained obtained as in Proposition 3.3, (ii) is said to be a symplectic metric generalized double extension (abbreviated as g.s.m.d.e.1) of $(\mathfrak{J}, B, \omega)$ by $(\mathbb{K}e)_{\bar{0}}$ by means of $(D, A_0, x_0, \alpha)$.

Now we are prepared to state the main theorem of this section:

**Theorem 3.1.** *Let $(\mathfrak{J}, B, \omega)$ be a symplectic metric Jordan superalgebra.*

(i) *If $B$ and $\omega$ are even, then $\mathfrak{J}$ is obtained by a sequence of orthogonal direct sums and/or (s.m.d.e.1) and/or (g.s.m.d.e.1) of even symplectic metric Jordan superalgebras by one dimensional Jordan algebra.*

(ii) *If $B$ and $\omega$ are odd, then $\mathfrak{J}$ is obtained by a sequence of orthogonal direct sums and/or (s.m.d.e.1) of odd symplectic metric Jordan superalgebra by one-dimensional Jordan superalgebra with zero even part.*

Proof across multiple stages. Let us first prove the following lemma:

**Lemma 3.1.** *Let $(\mathfrak{J}, B, \omega)$ be an irreducible symplectic metric Jordan superalgebra such that* $\dim \mathfrak{J} > 1$ *and* $\deg B = \deg \omega$. *If* $\big(\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\overline{\deg\omega+1}}\big) \neq \{0\}$, *then $(\mathfrak{J}, B, \omega)$ is obtained as an (s.m.d.e.1).*

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*Proof.* Let $\delta$ be the even invertible $B$-skew-supersymmetric superderivation of $\mathfrak{J}$ such that

$$\omega(x,y) = B(\delta(x),y) \text{ for all } x,y \in \mathfrak{J}.$$

Let $e^* \in (\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\overline{\deg\omega+1}})\backslash\{0\}$ such that $\delta(e^*) = \alpha e^*, \alpha \in \mathbb{K}\backslash\{0\}$. Denote by $\mathcal{I}$ the orthogonal of $\mathbb{K}e^*$ with respect to $B$. Since $B$ is non-degenerate, there exists $e \in \mathfrak{J}_{\bar{1}}$ such that $B(e^*,e) = 1$. Suppose that $B$ and $\omega$ are odd. By (cf. [5], Theorem 5.1), we have , the metric Jordan superalgebra $(\mathfrak{J},B)$ turns out to be the double extension of $(\mathfrak{J}_1 = \mathcal{I}/\mathcal{I}^{\perp}, B_1 := B_{|\mathfrak{J}_1\times\mathfrak{J}_1})$ by $(\mathbb{K}e)_{\bar{1}}$ by means of $\phi$. As earlier, the product on $\mathfrak{J} = \mathbb{K}e \oplus \mathfrak{J}_1 \oplus \mathbb{K}e^*$ reads

$$\begin{cases} e \star x = \phi(x), \\ x \star y = x.y + (-1)^{|x|+|y|}B(\phi(x),y)e^*, \quad x,y \in \mathfrak{J}_1, \\ e^* \star x = 0, \end{cases}$$

where $x.y$ denotes the product of $x$ and $y$ in $\mathfrak{J}_1$. Let $x \in \mathfrak{J}_1$, we have:

$$B(\delta(x),e^*) = -B(x,\delta(e^*)) = -\alpha B(x,e^*) = 0.$$

Thus, $\delta(\mathfrak{J}_1) \subset \mathcal{I} = \mathbb{K}e^* \oplus \mathfrak{J}_1$ and consequently, the linear map $\delta : \mathfrak{J} \to \mathfrak{J}$ is determined by the identities:

$$\delta(e^*) = \alpha e^*, \quad \delta(e) = \nu e + A + \mu e^*, \quad \delta(x) = \delta_1(x) + \beta(x)e^*, \quad x \in \mathfrak{J}_1,$$

where $\alpha \in \mathbb{K}\backslash\{0\}$, $\nu$, $\mu \in \mathbb{K}$, $A \in (\mathfrak{J}_1)_{\bar{1}}$, $\beta(x) \in \mathbb{K}$ and $\delta_1 \in End(\mathfrak{J}_1)$.

The fact that $B\big(\delta(e+x+e^*),e\big) = -B\big(e+x+e^*,\delta(e)\big)$, for all $x \in \mathfrak{J}_1$, gives that $\mu = 0,\ \nu = -\alpha$ and $\beta(x) = -B(x,A)$. Thus,

$$\delta(e^*) = \alpha e^*, \quad \delta(e) = -\alpha e + A \quad \text{and} \quad \delta(x) = \delta_1(x) - B(x,A)e^*, \quad x \in \mathfrak{J}_1.$$

Moreover, from the fact that $\delta$ is an even invertible $B$-skew-supersymmetric superderivation of $\mathfrak{J}$, one easily deduces that $\delta_1$ is an even invertible $B_1$-skew-supersymmetric superderivation of $\mathfrak{J}_1$. Furthermore, since $\delta(e \star x) = \delta(e) \star x + e \star \delta(x)$, for every $x \in \mathfrak{J}_1$, then $\delta_1\big(\phi(x)\big) - \phi\big(\delta_1(x)\big) + \alpha\phi(x) = Ax$, for any $x \in \mathfrak{J}_1$. Consequently, $[\delta_1,\phi] + \alpha\phi = L_A$. Thus, $(\mathfrak{J},B,\omega)$ is a (s.m.d.e.1) of $(\mathfrak{J}_1,B_1,\omega_1)$ by $(\mathbb{K}e)_{\bar{1}}$. Likewise, using Corollary 5.2 of [5], the result follows in the case where $\omega$ is even. □

Back now to the proof of Theorem 3.1. (i) Suppose that $\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{0}} \neq \{0\}$. Let $e^* \in (\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{0}})\backslash\{0\}$ such that $\delta(e^*) = \lambda e^*, \lambda \in \mathbb{K}\backslash\{0\}$. Consider the graded ideal $\mathcal{I} = \mathbb{K}e^*$. Since $B(e^*,e^*) = 0$, then $\mathcal{I} \subset \mathcal{I}^{\perp}$. Let $e \in \mathfrak{J}_{\bar{0}}$ such that $B(e,e) = 0$ and $B(e^*,e) = 1$. By Theorem 5.3 of [5], $(\mathfrak{J},B)$ is a generalized double extension of $(\mathfrak{J}_1 = (\mathbb{K}e \oplus \mathbb{K}e^*)^{\perp}, B_1 := B_{|\mathfrak{J}_1\times\mathfrak{J}_1})$ by some one dimensional Jordan algebra by means of $(D,A_0) \in (Op_s(\mathfrak{J}_1))_{\bar{0}} \times (\mathfrak{J}_1)_{\bar{0}}$. As earlier, the product on $\mathfrak{J} = \mathbb{K}e \oplus \mathfrak{J}_1 \oplus \mathbb{K}e^*$ reads

$$\begin{cases} e \star e = A_0 + ke^*, \\ e \star x = D(x) + B(A_0,x)e^*, \quad x \in \mathfrak{J}_1, \\ x \star y = xy + B(D(x),y)e^*, \quad x,y \in \mathfrak{J}_1, \\ e^* \star x = 0, \quad x \in \mathfrak{J}. \end{cases}$$

It is easy to check that $\delta(\mathfrak{J}_1) \subset \mathcal{I}^{\perp} = \mathbb{K}e^* \oplus \mathfrak{J}_1$. Therefore,

$$\delta(e^*) = \lambda e^*, \qquad \delta(e) = \alpha e + x_0 + \beta e^*, \qquad \delta(x) = \delta(x) + \psi(x)e^*, \quad x \in \mathfrak{J}_1,$$

where $\lambda \in \mathbb{K}\backslash\{0\},\ \alpha,\ \beta \in \mathbb{K},\ x_0 \in (\mathfrak{J}_1)_{\bar{0}},\ \delta(x) \in \mathfrak{J}_1$ and $\psi(x) \in \mathbb{K}$.

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Since $\delta$ is a $B$-skew-supersymmetric superderivation of $\mathfrak{J}$, then $\beta = 0$, $\alpha = -\lambda$ and $\psi(x) = -B(x, x_0)$. Thus, $\delta : \mathfrak{J} \to \mathfrak{J}$ is determined by:

$$\delta(e^*) = \lambda e^*, \qquad \delta(e) = -\lambda e + x_0, \qquad \delta(x) = \Delta(x) - B(x_0, x)e^*, \quad \Delta \in End(\mathfrak{J}_1), \quad x \in \mathfrak{J}_1.$$

Moreover, the fact that $\delta$ is an even invertible $B$-skew-supersymmetric superderivation of $\mathfrak{J}$ implies that $\Delta$ is also an even invertible $B_1$-skew-supersymmetric superderivation of $\mathfrak{J}_1$. Furthermore, $\delta(e \star x) = \delta(e) \star x + e \star \delta(x)$, $x \in \mathfrak{J}_1$. Thus,

$$D(x_0) = \lambda A_0 + (1/2) \quad \delta(A_0) \text{ and } [\Delta, D] + \lambda D = L_{x_0}.$$

Thus, $(\mathfrak{J}, B, \omega)$ is a (g.s.m.d.e.1) of $(\mathfrak{J}_1, B_1, \omega_1)$. Now, if $\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{0}} = \{0\}$, then by Proposition 3.2, we have $\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{1}} \neq \{0\}$. In this case, by Lemma 3.1, $(\mathfrak{J}, B, \omega)$ is a (s.m.d.e.1) of $(\mathfrak{J}_1, B_1, \omega_1)$. We repeat the process of this proof inductively to obtain the result.

(ii) Since $B$ and $\omega$ are odd, then by Corollary 3.1, we have $\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{0}} \neq \{0\}$. We use inductively Lemma 3.1 to get the result.

### 3.2 The setting where the metric structure and the symplectic structure have different graduations

In this section, we generate an inductive description of Jordan superalgebra $\mathfrak{J}$, endowed with an associative scalar product $B$ and a symplectic form $\omega$ in the case where the degrees of $B$ and $\omega$ are different. To obtain that, we need to define the concept of double extensions by two dimensional Jordan superalgebra with zero product.

Consider a metric Jordan superalgebra $(\mathfrak{J}, B)$ and a two dimensional Jordan superalgebra $(\mathbb{K}e_0 \oplus \mathbb{K}e_1)$ with zero product such that $|e_i| = |e_i^*| = \bar{i}$, $i \in \{0, 1\}$. Let $(D, x_0) \in Op_s(\mathfrak{J})_{\bar{0}} \times \mathfrak{J}_{\bar{0}}$ be an admissible pair of $\mathfrak{J}$ and $x_1 \in \mathfrak{J}_{\bar{1}}$ such that $B(D(x_0), x_1) = 0$. Let also $\overline{D}$ be an odd Jordan operator of $\mathfrak{J}$ which satisfies the following properties:

- $\overline{D}^2$ is an even supersymmetric superderivation of $\mathfrak{J}$.
- $[D, \overline{D}]$ is an odd superderivation of $\mathfrak{J}$.
- $B(\overline{D}(x_0), x_0 + x_1) = 0$.

On the $\mathbb{Z}_2$-graded vector space $\widetilde{\mathfrak{J}} = \mathbb{K}e_0 \oplus \mathbb{K}e_1 \oplus \mathfrak{J} \oplus \mathbb{K}e_0^* \oplus \mathbb{K}e_1^*$, we define the following product:

$$\begin{cases} x \star y = x.y + B(D(x), \quad y)e^*_{degB} - (-1)^{degB} B(\overline{D}(x), \quad y)e^*_{(1-degB)}, \\ e_0 \star x = D(x) + B(x_0, \quad x)e^*_{degB} - (-1)^{degB} B(x_1, \quad x)e^*_{(1-degB)}, \\ e_1 \star x = \overline{D}(x) + B(x_1, \quad x)e^*_{degB}, \\ e_0 \star e_0 = ke_0^* + x_0, \quad k \in \mathbb{K}, \\ e_0 \star e_1 = (degB)\alpha e_1^* + x_1, \quad \alpha \in \mathbb{K}, \end{cases}$$

for all $x, y \in \mathfrak{J}$. We define on $\widetilde{\mathfrak{J}}$ the bilinear form $\widetilde{B}$ by

$$\widetilde{B}_{|\mathfrak{J}\times\mathfrak{J}} = B, \quad \widetilde{B}(e_0^*, \quad e_{degB})\widetilde{B}(e_1^*, \quad e_{(1-degB)}) = 1 \quad \text{and} \quad \widetilde{B}(e_0, \quad e_{degB}) = 0.$$

The following is immediate:

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**Lemma 3.2.** $(\widetilde{\mathfrak{J}}, \widetilde{B})$ *is a metric Jordan superalgebra if and only if:*

$$\overline{D}(x_0x) - \overline{D}(x_0)x = 2D(\overline{D}(D(x))) - 2D(x_1x), x \in \mathfrak{J},$$
$$\overline{D}^2(x_0) = 2D(\overline{D}(x_1)), \quad D^2(x_1) = x_0x_1 + (1/2)[D, \overline{D}](x_0),$$
$$D(\overline{D}(x_0)) = x_0x_1 \quad \textit{and} \quad [D, \overline{D}^2] = L_{\overline{D}(x_1)}.$$

Suppose now that we are under the assumptions of Lemma 3.2 and consider a symplectic structure $\omega$ on $\mathfrak{J}$ such that $\deg B \neq \deg \omega$. Let $\delta$ be the odd invertible skew-supersymmetric superderivation of $(\mathfrak{J}, B)$ such that $\omega(x, y) = B(\delta(x), y)$, for all $x, y \in \mathfrak{J}$. Let also $c_0 \in \mathfrak{J}_{\bar{0}}$, $c_1 \in \mathfrak{J}_{\bar{1}}$ and $\lambda \in \mathbb{K}\backslash\{0\}$ meeting the following conditions:

- $L_{c_1} = [\delta, D] - (-1)^{degB}\lambda\overline{D}$,
- $L_{c_0} = [\delta, \overline{D}] + (-1)^{degB}\lambda D$,
- $\delta(x_1) = D(c_0) - \overline{D}(c_1) - (-1)^{degB}\lambda x_0$,
- $\delta(x_0) = 2D(c_1) + (-1)^{degB}2\lambda x_1$,
- $\lambda k = B(c_0 + c_1, x_0) + 2B(c_1, x_1)$.

On $(\widetilde{\mathfrak{J}}, \widetilde{B})$, define the automorphism $\Delta$:

$$\begin{cases} \Delta(x) = -B(c_1, \quad x)e^*_{degB} + \delta(x) - (-1)^{degB}B(c_0, \quad x)e^*_{(1-degB)}, \text{ for all } x \in \mathfrak{J}, \\ \Delta(e_1) = -(-1)^{degB}\alpha e^*_0 - (-1)^{degB}\lambda e_0 + c_0, \quad \alpha \in \mathbb{K}, \\ \Delta(e_0) = (1 - degB)\alpha e^*_1 + (-1)^{degB}\lambda e_1 + c_1, \quad \alpha \in \mathbb{K}, \\ \Delta(e^*_0) = \lambda e^*_1 \quad \text{and} \quad \Delta(e^*_1) = \lambda e^*_0. \end{cases} \tag{3.1}$$

The following is then immediate:

**Proposition 3.4.** *Let* $\widetilde{\omega}$ *be the bilinear form defined on* $\widetilde{\mathfrak{J}}$ *by*

$$\widetilde{\omega}(x, y) := \widetilde{B}(\Delta(x), y), \quad x, y \in \widetilde{\mathfrak{J}}, \textit{ where } \Delta \textit{ is as in } (3.1).$$

*Then* $(\widetilde{\mathfrak{J}}, \widetilde{B}, \widetilde{\omega})$ *defines a symplectic metric Jordan superalgebra and* $\deg \widetilde{\omega} \neq \deg \widetilde{B}$.

**Definition 3.2.** The symplectic Jordan superalgebra $(\widetilde{\mathfrak{J}}, \widetilde{B}, \widetilde{\omega})$ is said to be the symplectic metric double extension (s.m.d.e.2) of $(\mathfrak{J}, B, \omega)$.

We now prove the following two results:

**Lemma 3.3.** *A Jordan superalgebra* $\mathfrak{J}$ *has simultaneously an even metric structure and an odd metric structure if and only if* $\mathfrak{J}^2 = \{0\}$ *and* $\dim \mathfrak{J}_{\bar{0}} = \dim \mathfrak{J}_{\bar{1}}$.

*Proof.* Let $B$ (resp. $\tilde{B}$) be an even scalar product (resp. an odd scalar product) on $\mathfrak{J}$. The fact that $(\mathfrak{J}, B)$ is metric implies that there exists an isomorphism of $\mathfrak{J}_{\bar{0}}$-module $\varphi : \mathfrak{J}_{\bar{0}} \to \mathfrak{J}^*_{\bar{0}}$ defined by $\varphi(x) := B(x, .)$, for all $x \in \mathfrak{J}_{\bar{0}}$ (c.f. [5]). Likewise, we can prove that the map $\psi : \mathfrak{J}_{\bar{1}} \to \mathfrak{J}^*_{\bar{0}}$ defined by $\psi(x) = \tilde{B}(x, .)$, for all $x \in \mathfrak{J}_{\bar{0}}$ is an isomorphism of $\mathfrak{J}_{\bar{0}}$-module. Consequently, the map $\phi : \psi^{-1} \circ \varphi$ is an isomorphism of $\mathfrak{J}_{\bar{0}}$-module between $\mathfrak{J}_{\bar{0}}$ and $\mathfrak{J}_{\bar{1}}$. Consider the bilinear form $f : \mathfrak{J}_{\bar{0}} \times \mathfrak{J}_{\bar{0}} \to \mathbb{K}$ defined by

$$f(x, y) = B'(\phi(x), \phi(y)), \quad \text{where} \quad B' = B_{|\mathfrak{J}_{\bar{1}} \times \mathfrak{J}_{\bar{1}}}.$$

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We can easily check that $f$ is a non-degenerate associative skewsymmetric bilinear form. Thus, $\mathfrak{J}_{\bar{0}}^2 = \{0\}$. Furthermore, the fact that $\tilde{B}$ is associative and non-degenerate entails that $\mathfrak{J}_{\bar{0}}\mathfrak{J}_{\bar{1}} = \{0\}$. Moreover, since $B$ is associative and non-degenerate, then $\mathfrak{J}_{\bar{1}}\mathfrak{J}_{\bar{1}} = \{0\}$ and finally $\mathfrak{J}^2 = \{0\}$. ☐

**Proposition 3.5.** *Let* $(\mathfrak{J}, B, \omega)$ *be a symplectic metric Jordan superalgebra such that* $\deg B \neq \deg \omega$. *Then,* $\dim \mathfrak{J}$ *is even and* $\dim \mathfrak{J} \geq 4$.

*Proof.* Since $\deg B \neq \deg \omega$, there exists an odd invertible skew-supersymmetric superderivation $D$ of $(\mathfrak{J}, B)$ such that,

$$\omega(x, y) = B(D(x), y), \quad x, y \in \mathfrak{J}.$$

Then $\tilde{D} = [D, D]$ is an even invertible skew-supersymmetric superderivation of $(\mathfrak{J}, B)$. Consequently, the metric Jordan superalgebra $(\mathfrak{J}, B)$ has simultaneously an even and an odd symplectic structures. By Proposition 3.3, we get the result. ☐

We are now ready to prove our main result of this section.

**Theorem 3.2.** *Let* $(\mathfrak{J}, B, \omega)$ *be a symplectic metric Jordan superalgebra such that* $\deg B \neq \deg \omega$. *Then,* $\mathfrak{J}$ *is obtained by a sequence of (s.m.d.e.2).*

*Proof.* Assume that $\omega$ is odd and $B$ is even. Let $\Delta$ be the odd invertible skew-supersymmetric superderivation of $(\mathfrak{J}, B)$ defined by (3.1) such that

$$\omega(x, y) = B(\Delta(x), y), \quad \text{for} \quad x, y \in \mathfrak{J}.$$

Since the field $\mathbb{K}$ is algebraically closed, $\Delta$ is invertible and $\Delta(\mathrm{Ann}(\mathfrak{J})) = \mathrm{Ann}(\mathfrak{J})$. There exist therefore $\lambda \in \mathbb{K}\backslash\{0\}$ and $e^* \in \mathrm{Ann}(\mathfrak{J})\backslash\{0\}$ such that $\Delta(e^*) = \lambda e^*$. Set $e^* = e_0^* + e_1^*$, where $e_0^* \in \big(\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{0}}\big)\backslash\{0\}$ and $e_1^* \in \big(\mathrm{Ann}(\mathfrak{J}) \cap \mathfrak{J}_{\bar{1}}\big)\backslash\{0\}$, such that $D(e_0^*) = \lambda e_1^*$ and $D(e_1^*) = \lambda e_0^*$. Let $\mathcal{I} = \mathbb{K}e_0^* \oplus \mathbb{K}e_1^*$ and let $\mathcal{I}^\perp$ be its orthogonal with respect to $B$. The fact that $B$ is even and $\omega$ is skew-supersymmetric give that

$$B(e_0^*, e_0^*) = -B(e_1^*, e_1^*) = 0.$$

So, $\mathcal{I} \subset \mathcal{I}^\perp$. Moreover, the fact that $e_i^* \in \mathfrak{J}_{\bar{i}}$, $i \in \{0, 1\}$ and $B$ is even and non-degenerate says that, there exist $e_0 \in \mathfrak{J}_{\bar{0}}$ and $e_1 \in \mathfrak{J}_{\bar{1}}$ such that $B(e_0^*, e_0) = 1 = B(e_1^*, e_1)$ and $B(e_0, e_0) = 0$. Consider the $\mathbb{Z}_2$-graded vector space $N = \mathcal{I} \oplus \mathbb{K}e_0 \oplus \mathbb{K}e_1$. By Proposition 3.5, we have $N \neq \mathfrak{J}$. Since $B_{|N\times N}$ is non-degenerate, then $\mathfrak{J} = N \oplus N^\perp$ where $N^\perp$ is the orthogonal of $N$ with respect to $B$. Furthermore, since $B_{|N^\perp \times N^\perp}$ is non-degenerate, then $\mathcal{I}^\perp = \mathcal{I} \oplus N^\perp$. Let $x, y \in N^\perp$, we have

$$\begin{cases} x.y = \alpha(x, y) + \phi(x, y)e_0^* + \psi(x, y)e_1^*, \\ e_0.x = D(x) + f(x)e_0^* + g(x)e_1^*, \\ e_1.x = \overline{D}(x) + F(x)e_0^* + G(x)e_1^*, \end{cases}$$

where $\alpha(x, y), D(x), \overline{D}(x) \in N^\perp$ and $\phi(x, y), \psi(x, y), f(x), g(x), F(x), G(x) \in \mathbb{K}$. It is clear that $N^\perp$ is endowed with the multiplication $\alpha$ and the bilinear form $\widetilde{B} := B_{|N^\perp\times N^\perp}$ is an even metric Jordan superalgebra. The fact that $B(e_1, x.y) = B(e_1.x, y)$ and $B(e_0, x.y) = B(e_0.x, y)$, implies respectively that $\psi(x,y) = -\widetilde{B}(\overline{D}(x),y)$ and $\phi(x,y) = \widetilde{B}(D(x),y)$. Further, by the associativity of $B$, we have $B(e_0^*, e_0.e_0) = 0$, $B(e_1^*, e_0.e_1) = 0$ and $B(e_1, e_0.e_1) = 0$. So, we get

$$e_0.e_1 = x_1 \quad \text{and} \quad e_0.e_0 = ke_0^* + x_0, \quad \text{where} \quad x_0 \in N_{\bar{0}}^{\perp}, x_1 \in N_{\bar{1}}^{\perp} \quad \text{and} \quad k \in \mathbb{K}.$$

Besides we get by the associativity of $B$,

$$f(x) = \widetilde{B}(x_0, x), \quad g(x) = -\widetilde{B}(x_1, x), \quad F(x) = \widetilde{B}(x_1, x) \quad \text{and} \quad G(x) = 0, \quad x \in N^{\perp}.$$

It is clear that $D$ and $\overline{D}$ are two endomorphisms on $N^{\perp}$ which are respectively even and odd. Let $x, y \in N^{\perp}$. We have,

$$B(y, e_1.x) = (-1)^{|y|}B(e_1.y, x) \quad \text{and} \quad B(y, e_0.x) = B(e_0.y, x).$$

Thus, $\widetilde{B}(\overline{D}(y), x) = (-1)^{|y|}\widetilde{B}(y, \overline{D}(x))$ and $\widetilde{B}(D(y), x) = \widetilde{B}(y, D(x))$. So, $\overline{D}$ and $D$ are supersymmetric with respect to $\widetilde{B}$. By a simple computation using the super-Jordan identity, we can check that $D$ and $\overline{D}$ satisfy the conditions of Lemma 3.2. Thus, $(\mathfrak{J}, B)$ is a double extension of $(N^{\perp}, \widetilde{B})$ by the two dimensional Jordan superalgebra $\mathbb{K}e_0 \oplus \mathbb{K}e_1$. The fact that $\Delta$ is skew-supersymmetric and $\Delta(\mathcal{J}) \subset \mathcal{J}$ entails that $\Delta(\mathcal{J}^{\perp}) \subset \mathcal{J}^{\perp}$. Let $x \in \mathcal{J}^{\perp}$. We have,

$$\begin{cases} \Delta(x) = \mu(x)e_0^* + \delta(x) + \mu'(x)e_1^*, \\ \Delta(e_0) = \gamma e_1^* + \xi e_1 + c_1, \\ \Delta(e_1) = \gamma' e_0^* + \xi' e_0 + c_0, \end{cases}$$

where $\mu(x), \mu'(x), \gamma, \gamma', \xi, \xi' \in \mathbb{K}$, $(c_0, c_1) \in N_{\bar{0}}^{\perp} \times N_{\bar{1}}^{\perp}$ and $\delta$ is an endomorphism on $N^{\perp}$. The fact that $\Delta$ is skew-supersymmetric with respect to $B$, gives that $\delta$ is odd and skew-supersymmetric with respect to $\widetilde{B}$. Since $\Delta$ is skew-supersymmetric, then so is $\delta$. Once more, by the associativity of $B$ we get:

$$\gamma' = -\gamma, \quad \xi' = -\lambda, \quad \xi = \lambda, \quad \mu(x) = -\widetilde{B}(c_1, x) \quad \text{and} \quad \mu'(x) = -\widetilde{B}(c_0, x).$$

Let $x, y \in N^{\perp}$. By the equality $\Delta(x.y) = \Delta(x).y + (-1)^x x.\Delta(y)$, we obtain

$$L_{c_0} := [\delta, \overline{D}] - \lambda D \quad \text{and} \quad L_{c_1} := [\delta, D] + \lambda \overline{D}.$$

Furthermore, $\Delta(e_0.e_0)$ and $\Delta(e_0.e_1)$ give the other identities of $(4.1)$ and Proposition $3.4$ allows to conclude. The case where the associative scalar product is odd and the symplectic form is even can be dealt in a similar manner. □


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