Compatibility of a Jacobi structure and a Riemannian structure on a Lie algebroid

Compatibilité d'une structure de Jacobi et une structure Riemannienne sur une algebroide de Lie

Yacine Aït Amrane[1] and Ahmed Zeglaoui[2]

[1]Laboratoire Algèbre et Théorie des Nombres, Faculté de Mathématiques, USTHB, BP 32, El-Alia,
16111 Bab-Ezzouar, Alger, Algeria
yacinait@gmail.com
[2]Laboratoire de Géométrie, Analyse, Contrôle et Applications, Département de Mathématiques - Faculté des Sciences
Université de Saïda Dr Moulay Tahar, BP 138, Ennasr, 20006 Saïda, Algeria
ahmed.zeglaoui@gmail.com

ABSTRACT. In a preceding paper we introduced a notion of compatibility between a Jacobi structure and a Riemannian structure on a smooth manifold. We proved that in the case of fundamental examples of Jacobi structures : Poisson structures, contact structures and locally conformally symplectic structures, we get respectively Riemann-Poisson structures in the sense of M. Boucetta, $(1/2)$-Kenmotsu structures and locally conformally Kähler structures. In this paper we are generalizing this work to the framework of Lie algebroids.
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1. Introduction

In [3, 4], M. Boucetta introduced a notion of compatibility between a Poisson structure and a Riemannian structure on a smooth manifold, he called such a pair of compatible structures a Riemann-Poisson structure. In our preceding paper [1], we generalized this notion to a notion of compatibility between a Jacobi structure and a Riemannian structure and proved that in the case of a contact structure and of a locally conformally symplectic structure we get respectively a $(1/2)$-Kensmotsu structure and a locally conformally Kähler structure.

By now, most of these classical geometric structures have been considered on a Lie algebroid. For a Poisson structure on a Lie algebroid see for instance [6, 9]. For a Riemannian structure see [5]. For almost complex structures, Hermitian, locally conformally symplectic and locally conformally Kähler structures, see [7] and the references therein. For contact Riemannian, almost contact Riemannian and Kenmotsu structures, see [8].

The goal of the present paper is to generalize to the framework of Lie algebroids the results obtained by the authors on a smooth manifold [1]. We first introduce a notion of compatibility between a Poisson structure and a Riemannian structure on a Lie algebroid that generalizes that same notion introduced by Boucetta on a smooth manifold [3], and then we extend this notion of compatibility to a Jacobi structure on a Lie algebroid.

The paper is organized as follows. In section 2, we recall some basic notions about differential calculus on a Lie algebroid, [6, 9, 11], recall the notion of a Poisson structure on a Lie algebroid and establish some identities we will need later in the work. Then, we introduce the notion of contravariant Levi-Civita connection associated with a bivector field on a (pseudo-)Riemannian Lie algebroid and a notion of compatibility of the bivector field and the (pseudo-)Riemannian metric. The Lie algebroid equipped with such a pair of compatible bivector field and metric will be called (pseudo-)Riemannian Poisson Lie algebroid.

In section 3, we consider a pair of a bivector field and a vector field on a Lie algebroid and associate with it a skew algebroid structure on the dual bundle. We compute the torsion and the Jacobiator of this skew algebroid; this gives conditions for it to be an almost Lie algebroid and a Lie algebroid respectively. We investigate the cases of a contact Lie algebroid and a locally conformally symplectic Lie algebroid. We prove using tensors that a contact structure is a Jacobi structure that comes from an almost cosymplectic structure and that a locally conformally symplectic structure corresponds exactly to a Jacobi structure with a nondegenerate bivector field, and then we prove that in these two cases the skew algebroid structure on the dual bundle is a Lie algebroid structure.

In the last section, we propose a notion of compatibility of a triple composed of a bivector field, a vector field and a (pseudo-)Riemannian metric on a skew algebroid. We show under certain conditions that this induces a Jacobi structure on the skew algebroid. Then, after having recalled the notions of contact Riemannian Lie algebroid and (1/2)-Kenmotsu Lie algebroid, we prove that the triple associated with the contact Riemannian Lie algebroid is compatible if and only if the Riemannian contact Lie algebroid is (1/2)-Kenmotsu. Finally, we show that in the case of a Jacobi structure associated with a locally conformally symplectic structure and a somehow associated metric, the triple is compatible if and only if the locally conformally symplectic structure is locally conformally Kähler.

2. Riemannian Poisson Lie algebroids

Let M be a smooth manifold. For a vector bundle A over M we will denote by $\Gamma(A)$ the space of sections of A.

2.1. *Differential calculus on Lie algebroids*

An anchored vector bundle on M is a vector bundle A on M together with a vector bundle morphism $\rho : A \to TM$. The vector bundle morphism ρ, called the anchor, induces a map form $\Gamma(A)$ to $\Gamma(TM)$ that is $C^\infty(M)$-linear. A skew algebroid on M is an anchored vector bundle (A, ρ) on M endowed with a skew symmetric $\mathbb{R}$-bilinear map

$$[.,.] : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$$

that satisfies the Leibniz identity

$$[a, \varphi b] = \varphi[a, b] + (\rho(a) \cdot \varphi)b, \qquad \forall \varphi \in C^\infty(M),\ \forall a, b \in \Gamma(A).$$

An almost Lie algebroid on the base manifold M is a skew algebroid $(A, \rho, [.,.])$ such that the anchor ρ satisfies the property :

$$\rho([a, b]) = [\rho(a), \rho(b)], \qquad \forall a, b, \in \Gamma(A).$$

The bracket on the right hand side of the equality being the usual Lie bracket. A Lie algebroid on the base manifold M is a skew algebroid $(A, \rho, [.,.])$ such that the braket $[.,.]$ satisfies the Jacobi identity :

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0, \qquad \forall a,b,c \in \Gamma(A).$$

It is well known that the Leibniz identity and the Jacobi identity together imply that the anchor map is a Lie algebra morphism. Hence, a Lie algebroid is necessarily an almost Lie algebroid.

For a vector bundle A on M, we will denote by $A^\star$ the dual vector bundle. For a non negative integer k, denote by $\Gamma(\wedge^k A)$ (resp. $\Gamma(\wedge^k A^\star)$) the space of A-multivector fields of degree k (resp. the space of A-forms of degree k). Denote by $\Gamma(\wedge^\bullet A) := \bigoplus \Gamma(\wedge^k A)$ the space of A-multivector fields and by $\Gamma(\wedge^\bullet A^\star) := \bigoplus \Gamma(\wedge^k A^\star)$ that of A-forms. Equipped with the exterior product "$\wedge$", they are graded algebras. We can clearly define the interior product similarly to its definition for differential forms and multivector fields on a smooth manifold.

Let $(A, \rho, [.,.])$ be a skew algebroid on M. We define the Lie A-derivative with respect to a section $a \in \Gamma(A)$ to be the unique graded endomorphism $\mathcal{L}^\rho_a$ of degree 0 of the graded algebra $\Gamma(\wedge^\bullet A^\star)$ which satisfies the following properties :

$$\mathcal{L}^\rho_a \varphi := \mathcal{L}_{\rho(a)}\varphi = \rho(a)\cdot\varphi$$

for any $\varphi \in C^\infty(M)$, and, for $k \in \mathbb{N}^\star$,

$$\mathcal{L}^\rho_a \eta(a_1,\dots,a_k) = \rho(a)\cdot\eta(a_1,\dots,a_k) - \sum_{i=1}^{k} \eta(a_1,\dots,[a,a_i],\dots,a_k)$$

for any $\eta \in \Gamma(\wedge^k A^\star)$. The Lie A-derivative with respect to $a \in \Gamma(A)$ can be extended to the dual algebra $\Gamma(\wedge^\bullet A)$ as follows :

$$\langle \mathcal{L}^\rho_a P, \eta\rangle = \rho(a)\cdot\langle P,\eta\rangle - \langle P, \mathcal{L}^\rho_a \eta\rangle\,,$$

where $\langle .,.\rangle$ is the duality bracket between $\Gamma(\wedge^\bullet A)$ and $\Gamma(\wedge^\bullet A^\star)$. We clearly have $\mathcal{L}^\rho_a b = [a,b]$ for $a, b \in \Gamma(A)$.

Likewise, we define the exterior A-differential d_ρ, analoguous to the exterior differential of differential forms, associated with the skew algebroid $(A, \rho, [.,.])$ as follows : for any $\varphi \in C^\infty(M)$,

$$d_\rho\varphi(a) = \mathcal{L}^\rho_a\varphi = \rho(a)\cdot\varphi,$$

and, for $k \geq 1$ and $\eta \in \Gamma(\wedge^k A^\star)$, we set

$$\begin{aligned} d_\rho\eta(a_0,\dots,a_k) \;=\; & \sum_{i=0}^{k}(-1)^i \rho(a_i)\cdot\eta(a_0,\dots,\widehat{a_i},\dots,a_k) \\ & + \sum_{0\leq i<j\leq k}(-1)^{i+j}\eta([a_i,a_j],a_0,\dots,\widehat{a_i},\dots,\widehat{a_j},\dots,a_k). \end{aligned} \tag{2.1}$$

The Lie A-derivative, the exterior A-differential and the interior product satisfy the Cartan formula, i.e., for $a \in \Gamma(A)$, we have :

$$\mathcal{L}^\rho_a = i_a \circ d_\rho + d_\rho \circ i_a. \tag{2.2}$$

Finally, recall that the Schouten-Nijenhuis bracket [.,.] on A is the unique $\mathbb{R}$-bilinear map $\Gamma(\wedge^\bullet A) \times \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^\bullet A)$ which satisfies the following properties :

(i) for $\varphi, \psi \in C^\infty(M)$,
$$[\varphi, \psi] = 0,$$

(ii) for $a \in \Gamma(A)$ and $P \in \Gamma(\wedge^k A)$,
$$[a, P] = \mathcal{L}^\rho_a P,$$

(iii) for $P \in \Gamma(\wedge^k A)$ and $Q \in \Gamma(\wedge^l A)$;
$$[P, Q] = (-1)^{kl}[Q, P]$$

(iv) for $P \in \Gamma(\wedge^k A)$, $Q \in \Gamma(\wedge^l A)$ and $R \in \Gamma(\wedge^\bullet A)$,
$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(k+1)l} Q \wedge [P, R].$$

Recall also that if the skew algebroid $(A, \rho, [.,.])$ is an almost Lie algebroid then for any $\varphi \in C^\infty(M)$ we have $d_\rho(d_\rho \varphi) = 0$, and that, the skew algebroid $(A, \rho, [.,.])$ is a Lie algebroid if and only if $d_\rho \circ d_\rho = 0$, which is also equivalent to the Schouten-Nijenhuis bracket satisfying the graded Jacobi identity
$$(-1)^{kl}[[Q, R], P] + (-1)^{lr}[[R, P], Q] + (-1)^{rk}[[P, Q], R] = 0,$$
for any $P \in \Gamma(\wedge^k A)$, any $Q \in \Gamma(\wedge^l A)$ and any $R \in \Gamma(\wedge^r A)$.

2.2. *Poisson Lie algebroids*

A Poisson skew (resp. almost Lie, Lie) algebroid $(A, \rho, [.,.], \Pi)$ is a skew (resp. almost Lie, Lie) algebroid $(A, \rho, [.,.])$ equipped with an A-bivector field Π that satisfies $[\Pi, \Pi] = 0$. We also say that Π is a Poisson A-tensor.

Let $(A, \rho, [.,.])$ be a skew algebroid on M. To any A-bivector field Π, we associate vector bundle morphisms $\sharp_\Pi : A^\star \longrightarrow A$ and $\rho_\Pi : A^\star \longrightarrow TM$ defined by
$$\beta\left(\sharp_\Pi(\alpha)\right) := \Pi(\alpha, \beta) \qquad \text{and} \qquad \rho_\Pi = \rho \circ \sharp_\Pi, \tag{2.3}$$
and a bracket $[.,.]_\Pi$ on $\Gamma(A^\star)$ defined by
$$[\alpha, \beta]_\Pi := \mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta - \mathcal{L}^\rho_{\sharp_\Pi(\beta)}\alpha - d_\rho(\Pi(\alpha, \beta)), \tag{2.4}$$
called the Koszul bracket associated with Π. Therefore, with a bivector field Π on A we associate a skew algebroid structure $(\rho_\Pi, [.,.]_\Pi)$ on the dual bundle $A^\star$ of A.

Proposition 2.1. *Let* $\Pi \in \Gamma(\wedge^2 A)$. *For any* $Q \in \Gamma(\wedge^k A)$, *we have*
$$d_{\rho_\Pi} Q = -\left[\Pi, Q\right].$$

Proof. The proof is similar to that in the classical case in [12, Prop.4.3]. □

In particular with $Q = \Pi$, using the formula (2.1) for the differential d_{ρ_Π} we get the identity

$$[\Pi, \Pi](\alpha, \beta, \gamma) = -\oint \rho_\Pi(\alpha) \cdot \Pi(\beta, \gamma) + \oint \Pi\left([\alpha, \beta]_\Pi, \gamma\right), \tag{2.5}$$

where the symbol $\oint$ means the cyclic sum in α, β, γ. Using this identity we get the following theorem that gives the torsion of the skew algebroid $(A^\star, \rho_\Pi, [.,.]_\Pi)$, i.e. the default for the skew algebroid $(A^\star, \rho_\Pi, [.,.]_\Pi)$ to be an almost Lie algebroid.

Theorem 2.2. *We have the identity*

$$\gamma\left(\sharp_\Pi\left([\alpha, \beta]_\Pi\right) - [\sharp_\Pi(\alpha), \sharp_\Pi(\beta)]\right) = \frac{1}{2}\left[\Pi, \Pi\right](\alpha, \beta, \gamma). \tag{2.6}$$

Therefore, if $(A, \rho, [.,.], \Pi)$ *is a Poisson almost Lie algebroid, the skew algebroid* $(A^\star, \rho_\Pi, [.,.]_\Pi)$ *associated with is an almost Lie algebroid.*

Proof. For $\alpha, \beta, \gamma \in \Gamma(A^\star)$, put

$$\Phi(\alpha, \beta, \gamma) := \gamma\left(\sharp_\Pi([\alpha, \beta]_\Pi) - [\sharp_\Pi(\alpha), \sharp_\Pi(\beta)]\right).$$

Clearly we have $\Phi(\alpha, \beta, \gamma) = -\Phi(\beta, \alpha, \gamma)$, i.e. Φ is skew symmetric in the two first variables. Now, since $\gamma\left(\sharp_\Pi([\alpha, \beta]_\Pi)\right) = -[\alpha, \beta]_\Pi\left(\sharp_\Pi(\gamma)\right)$, by (2.4) we get

$$\gamma\left(\sharp_\Pi([\alpha, \beta]_\Pi\right) = -\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta\left(\sharp_\Pi(\gamma)\right) + \mathcal{L}^\rho_{\sharp_\Pi(\beta)}\alpha\left(\sharp_\Pi(\gamma)\right) + d_\rho(\Pi(\alpha, \beta))\left(\sharp_\Pi(\gamma)\right),$$

and therefore

$$\begin{array}{rcl}\Phi(\alpha, \beta, \gamma) & = & \rho_\Pi(\alpha) \cdot \Pi(\beta, \gamma) + \rho_\Pi(\beta) \cdot \Pi(\gamma, \alpha) + \rho_\Pi(\gamma) \cdot \Pi(\alpha, \beta) \\ & & -\alpha\left([\sharp_\Pi(\beta), \sharp_\Pi(\gamma)]\right) - \beta\left([\sharp_\Pi(\gamma), \sharp_\Pi(\alpha)]\right) - \gamma\left([\sharp_\Pi(\alpha), \sharp_\Pi(\beta)]\right)\end{array}$$

from which we deduce, on one hand, that $\Phi(\alpha, \gamma, \beta) = -\Phi(\alpha, \beta, \gamma)$, and on the other hand, using the identity (2.5) and the definition of the mapping Φ, that

$$\Phi(\alpha, \beta, \gamma) = -[\Pi, \Pi](\alpha, \beta, \gamma) + \Phi(\alpha, \beta, \gamma) + \Phi(\beta, \gamma, \alpha) + \Phi(\gamma, \alpha, \beta).$$

Hence, since Φ is an alternating map, $[\Pi, \Pi](\alpha, \beta, \gamma) = 2\Phi(\alpha, \beta, \gamma)$. □

Remark 2.3. A symplectic form on a skew (resp. almost Lie, Lie) algebroid $(A, \rho, [.,.])$, or a symplectic A-form, is a nondegenerate A-2-form $\Omega \in \Gamma(\wedge^2 A^\star)$ such that $d_\rho\Omega = 0$. We call $(A, \rho, [.,.], \Omega)$ a symplectic skew (resp. almost Lie, Lie) algebroid. Let $(A, \rho, [.,.])$ be a skew algebroid, $\Omega \in \Gamma(\wedge^2 A^\star)$ be a nondegenerate 2-form on A, $\sharp_\Omega$ be the inverse isomorphism of the vector bundle isomorphism $\flat_\Omega : A \to A^\star$, $\flat_\Omega(a) = -i_a\Omega$, and Π be the A-bivector field defined by

$$\Pi(\alpha, \beta) = \Omega(\sharp_\Omega(\alpha), \sharp_\Omega(\beta)).$$

We have $\sharp_\Pi = \sharp_\Omega$, and using the formulae (2.5) and (2.6) we get the identity

$$[\Pi, \Pi](\alpha, \beta, \gamma) = 2d_\rho\Omega(\sharp_\Omega(\alpha), \sharp_\Omega(\beta), \sharp_\Omega(\gamma)).$$

Hence, Π is a Poisson A-tensor if and only if Ω is a symplectic A-form.

Denote by J_Π the Jacobiator of the skew algebroid $(A^\star, \rho_\Pi, [.,.]_\Pi)$, i.e.,

$$J_\Pi(\alpha,\beta,\gamma) = \oint [[\alpha,\beta]_\Pi,\gamma]_\Pi,$$

for $\alpha,\beta,\gamma \in \Gamma(A^\star)$. The following result gives for a Lie algebroid $(A,\rho,[.,.])$ together with an A-bivector field Π the Jacobiator of the skew algebroid $(A^\star, \rho_\Pi, [.,.]_\Pi)$, i.e. the default for $(A^\star, \rho_\Pi, [.,.]_\Pi)$ to be a Lie algebroid.

Theorem 2.4. *Assume that (A,ρ,Π) is a Lie algebroid. Then*

$$J_\Pi(\alpha,\beta,\gamma) = \oint \mathcal{L}^\rho_{\sharp_\Pi([\alpha,\beta]_\Pi)-[\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]}\gamma - \oint d_\rho\left(\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma) + \Pi(\beta,\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\gamma)\right).$$

Proof. First, by (2.4), for $\alpha,\beta,\gamma \in \Gamma(A^\star)$, we have

$$[d_\rho(\Pi(\alpha,\beta)),\gamma]_\Pi = \mathcal{L}^\rho_{\sharp_\Pi(d_\rho(\Pi(\alpha,\beta)))}\gamma - \mathcal{L}^\rho_{\sharp_\Pi(\gamma)}\left(d_\rho(\Pi(\alpha,\beta))\right) - d_\rho\left(\Pi(d_\rho(\Pi(\alpha,\beta)),\gamma)\right).$$

Since $(A,\rho,[.,.])$ is a Lie algebroid, hence an almost Lie algebroid, we have

$$\mathcal{L}^\rho_{\sharp_\Pi(\gamma)}\left(d_\rho(\Pi(\alpha,\beta))\right) = d_\rho(\mathcal{L}^\rho_{\sharp_\Pi(\gamma)}\left(\Pi(\alpha,\beta)\right)) = d_\rho(\rho_\Pi(\gamma)\cdot\Pi(\alpha,\beta)),$$

and since

$$\Pi(d_\rho(\Pi(\alpha,\beta)),\gamma) = -d_\rho(\Pi(\alpha,\beta))(\sharp_\Pi(\gamma)) = -\rho_\Pi(\gamma)\cdot\Pi(\alpha,\beta),$$

we deduce that

$$[d_\rho(\Pi(\alpha,\beta)),\gamma]_\Pi = \mathcal{L}^\rho_{\sharp_\Pi(d_\rho(\Pi(\alpha,\beta)))}\gamma.$$

Consequently, by (2.4), we get

$$\begin{aligned}[][[\alpha,\beta]_\Pi,\gamma]_\Pi = \;& \mathcal{L}^\rho_{\sharp_\Pi([\alpha,\beta]_\Pi)}\gamma - \mathcal{L}^\rho_{\sharp_\Pi(\gamma)}\left(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta\right) + \mathcal{L}^\rho_{\sharp_\Pi(\gamma)}\left(\mathcal{L}^\rho_{\sharp_\Pi(\beta)}\alpha\right) \\ & - d_\rho\left[\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma) + \Pi(\gamma,\mathcal{L}^\rho_{\sharp_\Pi(\beta)}\alpha)\right].\end{aligned} \tag{2.7}$$

Since $(A,\rho,[.,.])$ is a Lie algebroid we have

$$\mathcal{L}^\rho_{\sharp_\Pi(\gamma)}\left(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta\right) - \mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\left(\mathcal{L}^\rho_{\sharp_\Pi(\gamma)}\beta\right) = \mathcal{L}^\rho_{[\sharp_\Pi(\gamma),\sharp_\Pi(\alpha)]}\beta.$$

Therefore, taking the cyclic sum in α,β,γ on the two sides of the equality (2.7) we get the result. □

Corollary 2.5. *If $(A,\rho,[.,.],\Pi)$ is a Poisson Lie algebroid, then $(A^\star,\rho_\Pi,[.,.]_\Pi)$ is a Lie algebroid.*

Proof. We need to prove that $J_\Pi = 0$. Since $[\Pi,\Pi] = 0$, by Theorem 2.2 and the theorem above we have

$$J_\Pi(\alpha,\beta,\gamma) = -\oint d_\rho\left(\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma) + \Pi(\beta,\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\gamma)\right).$$

For any α,β,γ we have

$$\begin{aligned}\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma) &= -\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta(\sharp_\Pi(\gamma)) \\ &= -\rho_\Pi(\alpha)(\beta(\sharp_\Pi(\gamma))) + \beta([\sharp_\Pi(\alpha),\sharp_\Pi(\gamma)]) \\ &= \rho_\Pi(\alpha)\cdot\Pi(\beta,\gamma) - \beta([\sharp_\Pi(\gamma),\sharp_\Pi(\alpha)]).\end{aligned}$$

Substituting in the expression of J_Π, we get

$$J_\Pi(\alpha,\beta,\gamma) = -2d_\rho\left[\oint \rho_\Pi(\alpha)\cdot\Pi(\beta,\gamma) - \oint \alpha([\sharp_\Pi(\beta),\sharp_\Pi(\gamma)])\right].$$

Since $[\Pi,\Pi]$=0, by (2.5) and Theorem 2.2, it follows that $J_\Pi = 0$. □

When A is the tangent Lie algebroid TM, the Lie algebroid $(A^\star, \rho_\Pi, [.,.]_\Pi)$ is the cotangent Lie algebroid of the Poisson manifold (M, Π).

2.3. *Riemannian Poisson Lie algebroids*

Contravariant A-connections associated with an A-bivector field

Let $(A, \rho, [.,.])$ be a skew algebroid. An affine connection on $(A, \rho, [.,.])$, or an affine A-connection, is an $\mathbb{R}$-bilinear map

$$\begin{array}{rccc} \nabla: & \Gamma(A)\times\Gamma(A) & \longrightarrow & \Gamma(A) \\ & (a,b) & \mapsto & \nabla_a b \end{array}$$

verifying $\nabla_{\varphi a} b = \varphi\nabla_a b$ and $\nabla_a(\varphi b) = \varphi\nabla_a b + (\rho(a).\varphi)\, b$, for any $\varphi \in C^\infty(M)$ and $a, b \in \Gamma(A)$. The A-torsion T^∇ of the connection ∇ is defined by

$$T^\nabla(a,b) = \nabla_a b - \nabla_b a - [a,b]$$

for any $a, b \in \Gamma(A)$. The A-connection ∇ is called symmetric, or torsion free, if its A-torsion is null.

Let Π be an A-bivector field. An affine connection on the skew algebroid $(A^\star, \rho_\Pi, [.,.]_\Pi)$ associated with Π will be called a contravariant A-connection associated with Π. If Π is nondegenerate, the formula

$$D_\alpha\beta = {\sharp_\Pi}^{-1}\left(\nabla_{\sharp_\Pi(\alpha)}(\sharp_\Pi(\beta))\right)$$

establishes a one-one correspondence between the affine A-connections ∇ and the contravariant A-connections D associated with the A-bivector field Π.

A (pseudo-)Riemannian skew (resp. almost Lie, Lie) algebroid is a quadruple $(A, \rho, [.,.], g)$ where $(A, \rho, [.,.])$ is a skew (resp. almost Lie, Lie) algebroid and g is a (pseudo-)Riemannian metric on its underlying vector bundle A.

Let $(A, \rho, [.,.], g)$ be a pseudo-Riemannian skew algebroid. There is a unique affine A-connection ∇ which is symmetric, i.e.

$$\nabla_a b - \nabla_b a = [a,b],$$

and compatible with g, i.e.

$$\rho(a)\cdot g(b,c) = g\left(\nabla_a b, c\right) + g\left(b, \nabla_a c\right).$$

It is entirely characterized by the Koszul formula :

$$\begin{aligned} 2g\left(\nabla_a b, c\right) &= \rho(a)\cdot g(b,c) + \rho(b)\cdot g(a,c) - \rho(c)\cdot g(a,b) \\ &\quad - g\left([b,c],a\right) - g\left([a,c],b\right) + g\left([a,b],c\right), \end{aligned} \tag{2.8}$$

for any $a, b, c \in \Gamma(A)$. We call ∇ the Levi-Civita A-connection associated with g, or the Levi-Civita connection of the pseudo-Riemannian skew algebroid $(A, \rho, [.,.], g)$.

Let $\flat_g$ be the vector bundle isomorphism $\Gamma(A) \to \Gamma(A^\star)$, $a \longmapsto g(a,.)$, and $\sharp_g$ be the inverse isomorphism. The cometric $g^\star$ of g is the pseudo-Riemannian metric on the dual bundle $A^\star$ defined by

$$g^\star(\alpha,\beta) = g(\sharp_g(\alpha), \sharp_g(\beta)), \qquad \alpha, \beta \in \Gamma(A^\star).$$

Let Π be a bivector field on A and let $(A^\star, \rho_\Pi, [.,.]_\Pi)$ be the associated skew algebroid. The Levi-Civita connection D of the pseudo-Riemannian skew algebroid $(A^\star, \rho_\Pi, [.,.]_\Pi, g^\star)$ will be called the contravariant Levi-Civita A-connection associated with the pair (Π, g).

Let $(A, \rho, [.,.], g)$ be a pseudo-Riemannian skew algebroid and let Π ba a bivector field on A. We say that the metric g is compatible with the bivector field Π or that the pair (Π, g) is compatible if $D\Pi = 0$, i.e., if

$$\rho_\Pi(\alpha) \cdot \Pi(\beta, \gamma) = \Pi(D_\alpha \beta, \gamma) + \Pi(\beta, D_\alpha \gamma),$$

for every $\alpha, \beta, \gamma \in \Gamma(A^\star)$.

With the pair (Π, g) we associate the vector bundle endomorphisms J of A and $J^\star$ of $A^\star$ defined respectively by

$$g\left(J\sharp_g(\alpha), \sharp_g(\beta)\right) = \Pi\left(\alpha, \beta\right) \qquad \text{and} \qquad g^\star\left(J^\star\alpha, \beta\right) = \Pi\left(\alpha, \beta\right). \tag{2.9}$$

Since $Dg^\star = 0$, we have, for any $\alpha, \beta, \gamma \in \Gamma(A^\star)$,

$$\begin{aligned} g^\star(\alpha, D_\gamma J^\star(\beta)) &= g^\star(\alpha, D_\gamma(J^\star\beta)) - g^\star(\alpha, J^\star(D_\gamma\beta)) \\ &= \rho_\Pi(\gamma) \cdot g^\star(\alpha, J^\star\beta) - g^\star(D_\gamma\alpha, J^\star\beta) - g^\star(\alpha, J^\star(D_\gamma\beta)), \end{aligned}$$

and therefore, by the second equality in (2.9),

$$g^\star(\alpha, D_\gamma J^\star(\beta)) = -\rho_\Pi\left(\gamma\right) \cdot \Pi\left(\alpha, \beta\right) + \Pi\left(D_\gamma\alpha, \beta\right) + \Pi\left(\alpha, D_\gamma\beta\right). \tag{2.10}$$

Hence $g^\star(\alpha, D_\gamma J^\star(\beta)) = -D_\gamma\Pi(\alpha, \beta)$, from which we deduce that the pair (Π, g) is compatible if and only if $DJ^\star = 0$, or equivalently $DJ = 0$.

Proposition 2.6. *Let $(A, \rho, [.,.], g)$ be a pseudo-Riemannian skew algebroid and let Π be an A-bivector field. If the pair (Π, g) is compatible then $(A, \rho, [.,.], \Pi)$ is a Poisson skew algebroid.*

Proof. Let $\alpha, \beta, \gamma \in \Gamma\left(A^\star\right)$. Taking the cyclic sum in α, β, γ on the two sides of the formula (2.10), taking in account that D is symmetric and using (2.5) we get the identity :

$$\left[\Pi, \Pi\right]\left(\alpha, \beta, \gamma\right) = g^\star\left(\alpha, D_\gamma J^\star\left(\beta\right)\right) + g^\star\left(\beta, D_\alpha J^\star\left(\gamma\right)\right) + g^\star\left(\gamma, D_\beta J^\star\left(\alpha\right)\right)$$

from which the result follows. ☐

A (pseudo-)Riemannian Poisson skew (resp. almost Lie, Lie) algebroid $(A, \rho, [.,.], \Pi, g)$ is a skew (resp. almost Lie, Lie) algebroid $(A, \rho, [.,.])$ equipped with a (pseudo-)Riemannian metric g and an A-bivector field Π that are compatible. This definition is justified by the proposition above (i.e. Π is necessarily Poisson).

Example. 1. When $A = TM$, we get the notion of (pseudo-)Riemannian Poisson manifold defined by M. Boucetta, see [3, 4].

2. Let $(A, \rho, [.,.])$ be a skew algebroid, Ω be a nondegenerate A-form and Π be the A-bivector field associated with in Remark 2.3. Assume that g is a pseudo-Riemannian metric on A associated with Ω, i.e.,

$$g^\star(\alpha, \beta) = g(\sharp_\Omega(\alpha), \sharp_\Omega(\beta)),$$

for any $\alpha, \beta \in \Gamma(A^\star)$. If D is the contravariant Levi-Civita A-connection associated with the pair (Π, g), then

$$\sharp_\Omega(D_\alpha\beta) = \nabla_{\sharp_\Omega(\alpha)}\left(\sharp_\Omega(\beta)\right),$$

where ∇ is the (covariant) Levi-Civita A-connection associated with g. Hence,

$$D\Pi(\alpha,\beta,\gamma) = \nabla\Omega(\sharp_\Omega(\alpha),\sharp_\Omega(\beta),\sharp_\Omega(\gamma))$$

for any $\alpha,\beta,\gamma \in \Gamma(A^\star)$. Therefore, $(A,\rho,[.,.],\Pi,g)$ is Riemannian Poisson if and only if $(A,\rho,[.,.],\Omega,g)$ is Kähler.

3. Let $(\mathfrak{g},[.,.])$ be a real Lie algebra of finite dimension seen as a Lie algebroid over a point ($\rho = 0$). A $\mathfrak{g}$-bivector field Π is just a skew symmetric bilinear form on the dual space $\mathfrak{g}^\star$ of $\mathfrak{g}$. Hence, the associated Koszul bracket $[.,.]_\Pi$ is defined by

$$[\alpha,\beta]_\Pi(a) = \alpha\left([\sharp_\Pi(\beta),a]\right) - \beta\left([\sharp_\Pi(\alpha),a]\right),$$

for any $\alpha,\beta \in \mathfrak{g}^\star$ and any $a \in \mathfrak{g}$. Let $\langle .,. \rangle$ be a (pseudo-)Riemannian metric on the Lie algebroid $(\mathfrak{g},[.,.])$, i.e. a scalar product on the vector space $\mathfrak{g}$, and let $\langle .,. \rangle^\star$ be its cometric. The contravariant Levi-Civita $\mathfrak{g}$-connection associated with the pair $(\Pi,\langle .,. \rangle)$ is defined by

$$2\left\langle D_\alpha\beta,\gamma\right\rangle^\star = \langle[\alpha,\beta]_\Pi,\gamma\rangle^\star - \langle[\beta,\gamma]_\Pi,\alpha\rangle^\star - \langle[\alpha,\gamma]_\Pi,\beta\rangle^\star.$$

The pair $(\Pi,\langle .,. \rangle)$ is compatible if and only if

$$\Pi\left(D_\alpha\beta,\gamma\right) + \Pi\left(\beta,D_\alpha\gamma\right) = 0,$$

for every $\alpha,\beta,\gamma \in \mathfrak{g}^\star$. So, if this last identity is satisfied, we call the quadruple $(\mathfrak{g},[.,.],\Pi,\langle .,. \rangle)$ a (pseudo-)Riemannian Poisson Lie algebra.

3. Jacobi structure on a Lie algebroid

3.1. *Jacobi structure on an almost Lie algebroid*

Let $(A,\rho,[.,.])$ be a skew algebroid on M. Let Π be an A-bivector field and ξ be an A-vector field. We say that the pair (Π,ξ) is a Jacobi structure on $(A,\rho,[.,.])$ if the following identities are satisfied

$$[\Pi,\Pi] = 2\xi\wedge\Pi \qquad \text{and} \qquad [\xi,\Pi] = \mathcal{L}^\rho_\xi\Pi = 0.$$

A Jacobi structure (Π,ξ) for which $\xi = 0$ is just a Poisson structure.

If $(A,\rho,[.,.])$ is an almost Lie algebroid the pushforward of multivectors $\rho_\star : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^\bullet TM)$ is compatible with the Schouten-Nijenhuis bracket and in particular we have

$$\rho_\star[\Pi,\Pi] = [\rho_\star\Pi,\rho_\star\Pi] \qquad \text{and} \qquad \rho_\star[\xi,\Pi] = [\rho_\star\xi,\rho_\star\Pi].$$

Therefore, a Jacobi structure (Π,ξ) on the almost Lie algebroid $(A,\rho,[.,.])$ induces a Jacobi structure $(\rho_\star\Pi,\rho_\star\xi)$ on the manifold M. The corresponding Jacobi bracket on the smooth functions $C^\infty(M)$ is given by

$$\{\varphi,\psi\} = \rho_\star\Pi\left(d\varphi,d\psi\right) + \varphi\mathcal{L}_{\rho_\star\xi}\psi - \psi\mathcal{L}_{\rho_\star\xi}\varphi,$$

i.e.

$$\{\varphi,\psi\} = \Pi\left(d_\rho\varphi,d_\rho\psi\right) + \varphi\mathcal{L}^\rho_\xi\psi - \psi\mathcal{L}^\rho_\xi\varphi.$$

When $\xi = 0$, the A-Poisson tensor Π induces a Poisson tensor $\rho_\star\Pi$ on M, the corresponding Poisson bracket on the smooth functions $C^\infty(M)$ being given by $\{\varphi,\psi\} = \Pi(d_\rho\varphi,d_\rho\psi)$.

3.2. *Skew algebroids associated with the pair* (Π, ξ)

Let $(A, \rho, [.,.])$ be a skew algebroid. Let Π be an A-bivector field and ξ be an A-vector field. With the pair (Π, ξ) we associate the vector bundle morphisms $\sharp_{\Pi,\xi} : A^\star \to A$ and $\rho_{\Pi,\xi} : A^\star \to TM$ defined by

$$\sharp_{\Pi,\xi}(\alpha) = \sharp_\Pi(\alpha) + \alpha(\xi)\xi \qquad \text{and} \qquad \rho_{\Pi,\xi} = \rho \circ \sharp_{\Pi,\xi}. \tag{3.1}$$

For any section λ of the vector bundle $A^\star$, consider the bracket $[.,.]^\lambda_{\Pi,\xi}$ on $\Gamma(A^\star)$ defined by

$$[\alpha, \beta]^\lambda_{\Pi,\xi} = [\alpha, \beta]_\Pi + \alpha(\xi)\left(\mathcal{L}^\rho_\xi \beta - \beta\right) - \beta(\xi)\left(\mathcal{L}^\rho_\xi \alpha - \alpha\right) - \Pi(\alpha, \beta)\lambda. \tag{3.2}$$

The pair $\left(\rho_{\Pi,\xi}, [.,.]^\lambda_{\Pi,\xi}\right)$ induces a skew algebroid structure on the dual bundle of A. We call $\left(A^\star, \rho_{\Pi,\xi}, [.,.]^\lambda_{\Pi,\xi}\right)$ the skew algebroid associated with the triple (Π, ξ, λ). In the particular case where $\xi = 0$ and $\lambda = 0$, we get the skew algebroid $(A^\star, \rho_\Pi, [.,.]_\Pi)$ associated in §2.2 with the A-bivector field Π.

Theorem 3.1. *Assume that* (Π, ξ) *is a Jacobi structure on the skew algebroid* $(A, \rho, [.,.])$ *and let* $\lambda \in \Gamma(A^\star)$. *We have*

$$\sharp_{\Pi,\xi}\left([\alpha, \beta]^\lambda_{\Pi,\xi}\right) - [\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)] = \Pi(\alpha, \beta)\left(\xi - \sharp_{\Pi,\xi}(\lambda)\right),$$

for any $\alpha, \beta \in \Gamma(A^\star)$.

Proof. The proof is the same as in the classical case of the tangent algebroid $A = TM$, [1, Th. 2.1]. Use Theorem 2.2. ☐

Corollary 3.2. *Assume that* $(A, \rho, [.,.])$ *in the theorem above is an almost Lie algebroid. If* $\xi - \sharp_{\Pi,\xi}(\lambda) \in \ker \rho$, *then the skew algebroid* $\left(A^\star, \rho_{\Pi,\xi}, [.,.]^\lambda_{\Pi,\xi}\right)$ *is an almost Lie algebroid, i.e., we have*

$$\rho_{\Pi,\xi}\left([\alpha, \beta]^\lambda_{\Pi,\xi}\right) = [\rho_{\Pi,\xi}(\alpha), \rho_{\Pi,\xi}(\beta)],$$

for any A-forms α and β. The converse is also true if $\Pi \neq 0$.

Proof. Applying ρ to the identity in the theorem above, we get the following identity

$$\rho_{\Pi,\xi}\left([\alpha, \beta]^\lambda_{\Pi,\xi}\right) - [\rho_{\Pi,\xi}(\alpha), \rho_{\Pi,\xi}(\beta)] = \Pi(\alpha, \beta)\rho\left(\xi - \sharp_{\Pi,\xi}(\lambda)\right)$$

that gives the torsion of the skew algebroid $\left(A^\star, \rho_{\Pi,\xi}, [.,.]^\lambda_{\Pi,\xi}\right)$, i.e. the default for it to be an almost Lie algebroid. ☐

The following result gives for a Lie algebroid $(A, \rho, [.,.])$ the Jacobiator $J^\lambda_{\Pi,\xi}$,

$$J^\lambda_{\Pi,\xi}(\alpha, \beta, \gamma) := \oint [[\alpha, \beta]^\lambda_{\Pi,\xi}, \gamma]^\lambda_{\Pi,\xi},$$

of the skew algebroid $(A^\star, \rho_{\Pi,\xi}, [.,.]^\lambda_{\Pi,\xi})$.

Theorem 3.3. *Assume that* $(A,\rho,[.,.])$ *is a Lie algebroid. We have*

$$\begin{array}{rl} J^\lambda_{\Pi,\xi}(\alpha,\beta,\gamma)= & \oint \mathcal{L}^\rho_{(\frac{1}{2}[\Pi,\Pi]-\xi\wedge\Pi)(\alpha,\beta,\cdot)}\gamma+\Big[(2\xi\wedge\Pi-[\Pi,\Pi])(\alpha,\beta,\gamma)+\oint\gamma(\xi)\mathcal{L}^\rho_\xi\Pi(\alpha,\beta)\Big]\lambda \\ & +d_\rho\left((2\xi\wedge\Pi-[\Pi,\Pi])(\alpha,\beta,\gamma)\right)-\oint\gamma(\xi)\mathcal{L}^\rho_\xi([.,.]_\Pi)(\alpha,\beta) \\ & +\oint(\mathcal{L}^\rho_\xi\gamma-\gamma)\mathcal{L}^\rho_\xi\Pi(\alpha,\beta)+\oint\Pi(\alpha,\beta)\left(\mathcal{L}^\rho_\xi\gamma-[\lambda,\gamma]^\lambda_{\Pi,\xi}\right), \end{array}$$

where $\mathcal{L}^\rho_\xi([.,.]_\Pi)(\alpha,\beta)=\mathcal{L}^\rho_\xi([\alpha,\beta]_\Pi)-[\mathcal{L}^\rho_\xi\alpha,\beta]_\Pi-[\alpha,\mathcal{L}^\rho_\xi\beta]_\Pi$.

Proof. A long but direct calculation using (2.4),(2.5),(3.1) and (3.2) gives

$$\begin{array}{rl} J^\lambda_{\Pi,\xi}(\alpha,\beta,\gamma)= & J_\Pi(\alpha,\beta,\gamma)-\oint\gamma(\xi)\mathcal{L}^\rho_\xi([.,.]_\Pi)(\alpha,\beta)+\oint(\mathcal{L}^\rho_\xi\gamma-\gamma)\mathcal{L}^\rho_\xi\Pi(\alpha,\beta) \\ & -\oint\gamma(\xi)[\alpha,\beta]_\Pi+\Big((2\xi\wedge\Pi-[\Pi,\Pi])(\alpha,\beta,\gamma)+\oint\gamma(\xi)\mathcal{L}^\rho_\xi\Pi(\alpha,\beta)\Big)\lambda \\ & +\oint\Pi(\alpha,\beta)[\gamma,\lambda]^\lambda_{\Pi,\xi}. \end{array}$$

By Theorem 2.4, we have

$$J_\Pi(\alpha,\beta,\gamma)=\oint\mathcal{L}^\rho_{\sharp_\Pi([\alpha,\beta]_\Pi)-[\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]}\gamma-\oint d_\rho\left(\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma)+\Pi(\beta,\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\gamma)\right).$$

On one hand, since by (2.6) we have

$$\begin{array}{rl} \sharp_\Pi([\alpha,\beta]_\Pi)-[\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]= & \left(\frac{1}{2}[\Pi,\Pi]-\xi\wedge\Pi\right)(\alpha,\beta,.)+\alpha(\xi)\sharp_\Pi(\beta)-\beta(\xi)\sharp_\Pi(\alpha) \\ & +\Pi(\alpha,\beta)\xi, \end{array}$$

it comes that

$$\begin{array}{rl} \mathcal{L}^\rho_{\sharp_\Pi([\alpha,\beta]_\Pi)-[\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]}\gamma= & \mathcal{L}^\rho_{\left(\frac{1}{2}[\Pi,\Pi]-\xi\wedge\Pi\right)(\alpha,\beta,.)}\gamma+\alpha(\xi)\mathcal{L}^\rho_{\sharp_\Pi(\beta)}\gamma-\beta(\xi)\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\gamma \\ & +\gamma(\xi)d_\rho(\Pi(\alpha,\beta))+\Pi(\beta,\gamma)d_\rho(\alpha(\xi))+\Pi(\gamma,\alpha)d_\rho(\beta(\xi)) \\ & +\Pi(\alpha,\beta)\mathcal{L}^\rho_\xi\gamma. \end{array}$$

Taking the cyclic sum in α,β,γ on the two sides we get

$$\begin{array}{rl} \oint\mathcal{L}^\rho_{\sharp_\Pi([\alpha,\beta]_\Pi)-[\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]}\gamma= & \oint\mathcal{L}^\rho_{\left(\frac{1}{2}[\Pi,\Pi]-\xi\wedge\Pi\right)(\alpha,\beta,.)}\gamma+\oint\gamma(\xi)[\alpha,\beta]_\Pi \\ & +2d_\rho\left((\xi\wedge\Pi)(\alpha,\beta,\gamma)\right)+\oint\Pi(\alpha,\beta)\mathcal{L}^\rho_\xi\gamma. \end{array}$$

On the other hand, we have

$$\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma)=-\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta(\sharp_\Pi(\gamma))=\rho_\Pi(\alpha)\cdot\Pi(\beta,\gamma)-\beta([\sharp_\Pi(\gamma),\sharp_\Pi(\alpha)]),$$

and then, by (2.6),

$$\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma)=\rho_\Pi(\alpha)\cdot\Pi(\beta,\gamma)-\Pi([\gamma,\alpha]_\Pi,\beta)+\frac{1}{2}[\Pi,\Pi](\gamma,\alpha,\beta).$$

Hence,

$$\begin{array}{rl} \Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma)+\Pi(\beta,\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\gamma)= & 2\rho_\Pi(\alpha)\cdot\Pi(\beta,\gamma)-\Pi([\gamma,\alpha]_\Pi,\beta)-\Pi([\alpha,\beta]_\Pi,\gamma) \\ & +[\Pi,\Pi](\alpha,\beta,\gamma), \end{array}$$

and taking the cyclic sum on the two sides we get

$$\begin{array}{rl} \oint(\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma)+\Pi(\beta,\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\gamma))= & 2\oint\left(\rho_\Pi(\alpha)\cdot\Pi(\beta,\gamma)-\Pi([\alpha,\beta]_\Pi,\gamma)\right) \\ & +3[\Pi,\Pi](\alpha,\beta,\gamma), \end{array}$$

and consequently, by (2.5),

$$\oint (\Pi(\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\beta,\gamma)+\Pi(\beta,\mathcal{L}^\rho_{\sharp_\Pi(\alpha)}\gamma)) = [\Pi,\Pi](\alpha,\beta,\gamma).$$

Substituting in the expression of J_Π above we get

$$\begin{aligned} J_\Pi(\alpha,\beta,\gamma) = \; & \oint \mathcal{L}^\rho_{\left(\frac{1}{2}[\Pi,\Pi]-\xi\wedge\Pi\right)(\alpha,\beta,.)}\gamma + \oint \gamma(\xi)[\alpha,\beta]_\Pi + \oint \Pi(\alpha,\beta)\mathcal{L}^\rho_\xi\gamma \\ & + d_\rho\left((2\xi\wedge\Pi - [\Pi,\Pi])(\alpha,\beta,\gamma)\right). \end{aligned}$$

In remains only to substitute this in the the expression of $J^\lambda_{\Pi,\xi}$ above to get the result. □

Corollary 3.4. *Assume that* (Π,ξ) *is a Jacobi structure on a Lie algebroid* $(A,\rho,[.,.])$. *If* λ *satisfies the property :*

$$\mathcal{L}^\rho_\xi\alpha = [\lambda,\alpha]^\lambda_{\Pi,\xi} \quad \textit{for any A-1-form } \alpha,$$

then $(A^\star,\rho_{\Pi,\xi},[.,.]^\lambda_{\Pi,\xi})$ *is a Lie algebroid.*

Proof. We need only to prove that $J^\lambda_{\Pi,\xi}=0$. From the theorem above we see that

$$J^\lambda_{\Pi,\xi}(\alpha,\beta,\gamma) = -\oint \gamma(\xi)\mathcal{L}^\rho_\xi([.,.]_\Pi)(\alpha,\beta).$$

A direct calculation gives

$$\mathcal{L}^\rho_\xi([.,.]_\Pi)(\alpha,\beta) = \mathcal{L}^\rho_{[\xi,\sharp_\Pi(\alpha)]-\sharp_\Pi(\mathcal{L}^\rho_\xi\alpha)}\beta - \mathcal{L}^\rho_{[\xi,\sharp_\Pi(\beta)]-\sharp_\Pi(\mathcal{L}^\rho_\xi\beta)}\alpha - d_\rho(\mathcal{L}^\rho_\xi\Pi(\alpha,\beta)).$$

Since $\mathcal{L}^\rho_\xi\Pi=0$, then $\mathcal{L}^\rho_\xi([.,.]_\Pi)(\alpha,\beta)=0$. □

3.3. *Contact Lie algebroids*

In this paragraph we use the global language of tensors to generalize to the Lie algebroid framework the results in [10] about the contravariant characterization of almost cosymplectic manifolds and contact manifolds.

Almost cosymplectic structures on a skew algebroid

Let $(A,\rho,[.,.])$ be a skew algebroid with the underlying vector bundle A of odd rank $2m+1$. By an almost cosymplectic structure on A we mean a pair (Ω,η) of a 1-form η and a 2-form Ω on A such that the $(2m+1)$-form $\eta\wedge\Omega^m$ is everywhere nonzero. It is then clear that Ω is of rank $2m$, and, as in the classical case, that there is a unique vector field ξ on A such that

$$i_\xi\Omega = 0 \qquad \text{and} \qquad i_\xi\eta = 1. \tag{3.3}$$

We call the A-vector field ξ the Reeb section or the fundamental A-vector field. With the almost cosymplectic structure (Ω,η) we associate a pair (Π,ξ) where ξ is the Reeb section and Π is the A-bivector field, called the fundamental A-bivector field, defined by

$$\Pi(\alpha,\beta) = \Omega(\sharp_{\Omega,\eta}(\alpha),\sharp_{\Omega,\eta}(\beta)),$$

where $\sharp_{\Omega,\eta}$ is the inverse isomorphism of the vector bundle isomorphism $\flat_{\Omega,\eta}: A\to A^\star$ defined by

$$\flat_{\Omega,\eta}(a) = -i_a\Omega + \eta(a)\eta.$$

Lemma 3.5. *We have* $\sharp_{\Pi,\xi} = \sharp_{\Omega,\eta}$.

Proof. Let $\alpha, \beta \in \Gamma(A^\star)$ and let a, b such that $\alpha = \flat_{\Omega,\eta}(a)$ and $\beta = \flat_{\Omega,\eta}(b)$. Notice that $\alpha(\xi) = \flat_{\Omega,\eta}(a)(\xi) = i_\xi\Omega(a) + \eta(a)\eta(\xi) = \eta(a)$ and likewise $\beta(\xi) = \eta(b)$. Therefore, we have

$$\begin{aligned}\beta(\sharp_{\Pi,\xi}(\alpha)) &= \Pi(\alpha,\beta) + \alpha(\xi)\beta(\xi)\\ &= \Omega(a,b) + \eta(a)\eta(b)\\ &= \flat_{\Omega,\eta}(b)(a)\\ &= \beta\left(\sharp_{\Omega,\eta}(\alpha)\right).\end{aligned}$$

Thus $\sharp_{\Pi,\xi} = \sharp_{\Omega,\eta}$. □

Proposition 3.6. *Let* (Ω,η) *be an almost cosymplectic structure on* A *and let* (Π,ξ) *be the associated fundamental pair. Let* a, b, c *be* A*-vector fields and let* $\alpha = \flat_{\Omega,\eta}(a)$*,* $\beta = \flat_{\Omega,\eta}(b)$ *and* $\gamma = \flat_{\Omega,\eta}(c)$*. We have*

1. $\left(\frac{1}{2}[\Pi,\Pi] - \xi\wedge\Pi\right)(\alpha,\beta,\gamma) = \Big(d_\rho\Omega + \eta\wedge\Big(d_\rho\eta - \Omega - \mathcal{L}^\rho_\xi\Omega\Big)\Big)(a,b,c)$.
2. $\mathcal{L}^\rho_\xi\Pi(\alpha,\beta) = \Big(\eta\wedge\mathcal{L}^\rho_\xi\eta - \mathcal{L}^\rho_\xi\Omega\Big)(a,b)$.

Proof. Notice that since we have $\alpha(\xi) = \eta(a)$, $\beta(\xi) = \eta(b)$, $\gamma(\xi) = \eta(c)$ and, by the lemma above, $\sharp_{\Omega,\eta} = \sharp_{\Pi,\xi}$, it comes that $\sharp_\Pi(\alpha) = a - \eta(a)\xi$, $\sharp_\Pi(\beta) = b - \eta(b)\xi$ and $\sharp_\Pi(\gamma) = c - \eta(c)\xi$. Let us now do some computations. First, we have

$$\begin{aligned}[\sharp_\Pi(\alpha),\sharp_\Pi(\beta)] &= [a,b] - \eta(a)[\xi,b] + \eta(b)[\xi,a]\\ &\quad (\rho(b)(\eta(a)) - \rho(a)(\eta(b)) + \eta(a)\rho(\xi)(\eta(b)) - \eta(b)\rho(\xi)(\eta(a)))\,\xi.\end{aligned}$$

Applying η and since $\eta(\xi) = 1$, we get

$$\begin{aligned}\eta\left([\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]\right) &= \eta\left([a,b]\right) - \rho(a)\left(\eta(b)\right) + \rho(b)\left(\eta(a)\right)\\ &\quad \eta(a)\mathcal{L}^\rho_\xi\eta(b) - \eta(b)\mathcal{L}^\rho_\xi\eta(a),\end{aligned}$$

and therefore

$$\eta\left([\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]\right) = \Big(\eta\wedge\mathcal{L}^\rho_\xi\eta - d_\rho\eta\Big)(a,b). \tag{3.4}$$

Also, using $i_\xi\Omega = 0$, we get

$$\begin{aligned}\Omega\left([\sharp_\Pi(\alpha),\sharp_\Pi(\beta)],\sharp_\Pi(\gamma)\right) &= -i_{\sharp_\Pi(\gamma)}\Omega\left([\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]\right)\\ &= -i_c\Omega\left([\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]\right)\\ &= (\gamma - \eta(c)\eta)\left([\sharp_\Pi(\alpha),\sharp_\Pi(\beta)]\right),\end{aligned}$$

and then, using (2.6) and the relation (3.4) above, we get

$$\begin{aligned}\Omega\left([\sharp_\Pi(\alpha),\sharp_\Pi(\beta)],\sharp_\Pi(\gamma)\right) &= -\tfrac{1}{2}\left[\Pi,\Pi\right](\alpha,\beta,\gamma) + \Pi\left([\alpha,\beta]_\Pi,\gamma\right)\\ &\quad -\eta(c)\Big(\eta\wedge\mathcal{L}^\rho_\xi\eta - d_\rho\eta\Big)(a,b).\end{aligned}$$

On the other hand we clearly have

$$\rho\left(\sharp_\Pi(\alpha)\right)\cdot\Omega\left(\sharp_\Pi(\beta),\sharp_\Pi(\gamma)\right) = \rho_\Pi(\alpha)\cdot\Pi(\beta,\gamma).$$

Using the two relations above and the identity (2.5), we get

$$d_\rho\Omega\left(\sharp_\Pi(\alpha),\sharp_\Pi(\beta),\sharp_\Pi(\gamma)\right)=\frac{1}{2}\left[\Pi,\Pi\right](\alpha,\beta,\gamma)-\eta\wedge d_\rho\eta(a,b,c).$$

Also, observe that since $i_\xi\Omega=0$, using the Cartan formula we have

$$d_\rho\Omega(a,b,c)=d_\rho\Omega\left(\sharp_\Pi(\alpha),\sharp_\Pi(\beta),\sharp_\Pi(\gamma)\right)+\eta\wedge\mathcal{L}^\rho_\xi\Omega(a,b,c).$$

Finally, from the two identities above it follows that

$$\frac{1}{2}\left[\Pi,\Pi\right](\alpha,\beta,\gamma)=\left(d_\rho\Omega+\eta\wedge\left(d_\rho\eta-\mathcal{L}^\rho_\xi\Omega\right)\right)(a,b,c),$$

and it remains only to notice that $\xi\wedge\Pi(\alpha,\beta,\gamma)=\eta\wedge\Omega(a,b,c)$ to get the first assertion of the proposition. Let us now prove the second one. On one hand, we have

$$\mathcal{L}^\rho_\xi\Pi(\alpha,\beta)=\rho(\xi)\cdot\Pi(\alpha,\beta)-\Pi\left(\mathcal{L}^\rho_\xi\alpha,\beta\right)-\Pi\left(\alpha,\mathcal{L}^\rho_\xi\beta\right). \tag{3.5}$$

On the other hand, we have

$$\Pi\left(\mathcal{L}^\rho_\xi\alpha,\beta\right)=-\mathcal{L}^\rho_\xi\alpha\left(\sharp_\Pi(\beta)\right)=\rho(\xi)\cdot\Pi(\alpha,\beta)+\alpha\left([\xi,\sharp_\Pi(\beta)]\right),$$

and, since we have $\Pi(\alpha,\beta)=\Omega(a,b)$, $\sharp_\Pi(\beta)=b-\eta(b)\xi$ and $\alpha(\xi)=\eta(a)$, it follows that

$$\begin{aligned}\Pi\left(\mathcal{L}^\rho_\xi\alpha,\beta\right) &= \rho(\xi)\cdot\Omega(a,b)+\alpha\left([\xi,b]\right)-\eta(a)\rho(\xi)\left(\eta(b)\right)\\ &= \rho(\xi)\cdot\Omega(a,b)+\alpha\left([\xi,b]\right)-\eta(a)\eta([\xi,b])-\eta(a)\mathcal{L}^\rho_\xi\eta(b).\end{aligned}$$

Since $\alpha=\flat_{\Omega,\eta}(a)$, then $\alpha\left([\xi,b]\right)-\eta(a)\eta([\xi,b])=-i_a\Omega([\xi,b])$, and therefore

$$\Pi\left(\mathcal{L}^\rho_\xi\alpha,\beta\right)=\rho(\xi)\cdot\Omega(a,b)-\Omega\left(a,[\xi,b]\right)-\eta(a)\mathcal{L}^\rho_\xi\eta(b).$$

Interchanging α and β we also get

$$\Pi\left(\alpha,\mathcal{L}^\rho_\xi\beta\right)=-\Pi(\mathcal{L}^\rho_\xi\beta,\alpha)=\rho(\xi)\cdot\Omega(a,b)-\Omega\left([\xi,a],b\right)+\eta(b)\mathcal{L}^\rho_\xi\eta(a).$$

Substituting in (3.5), we get

$$\mathcal{L}^\rho_\xi\Pi(\alpha,\beta)=-\rho(\xi)\cdot\Omega(a,b)+\Omega\left(a,[\xi,b]\right)+\Omega\left([\xi,a],b\right)+\eta(a)\mathcal{L}^\rho_\xi\eta(b)-\eta(b)\mathcal{L}^\rho_\xi\eta(a)$$

which proves the second assertion of the proposition. □

Contact Lie algebroids

Let $(A,\rho,[.,.])$ be a skew algebroid with the underlying vector bundle A of odd rank $2m+1$. A contact structure on A is an A-form η of degree 1 such that the A-form $\eta\wedge(d_\rho\eta)^m$ of degree $2m+1$ does not vanish. So, a contact structure on A is an almost cosymplectic structure (Ω,η) on A with $\Omega=d_\rho\eta$. If $(A,\rho,[.,.])$ is a skew (resp. almost Lie, Lie) algebroid and η is a contact structure on A we call $(A,\rho,[.,.],\eta)$ a contact skew (resp. almost Lie, Lie) algebroid.

Let (Ω,η) be an almost cosymplectic structure on a skew algebroid $(A,\rho,[.,.])$ and let ξ be the associated Reeb section. By the relations (3.3) and the Cartan formula (2.2) we have :

$$\mathcal{L}^\rho_\xi\Omega=i_\xi\left(d_\rho\Omega\right)\qquad\text{and}\qquad\mathcal{L}^\rho_\xi\eta=i_\xi\left(d_\rho\eta\right). \tag{3.6}$$

If $(A, \rho, [.,.])$ is a Lie algebroid and η is a contact structure on A, i.e. an almost cosymplectic structure (Ω, η) on A such that $\Omega = d_\rho \eta$, since we have $d_\rho (d_\rho \eta) = 0$ and $i_\xi \Omega = 0$, it follows that

$$\mathcal{L}^\rho_\xi \Omega = \mathcal{L}^\rho_\xi (d_\rho \eta) = 0 \qquad \text{and} \qquad \mathcal{L}^\rho_\xi \eta = 0. \tag{3.7}$$

The following theorem says that on a Lie algebroid the contact structures are precisely the Jacobi structures (Π, ξ) such that $\xi \wedge \Pi^m$ is everywhere nonzero.

Theorem 3.7. *Let $(A, \rho, [.,.])$ be a Lie algebroid. Let (Ω, η) be an almost cosymplectic structure on A and (Π, ξ) be the associated fundamental pair. The pair (Ω, η) is a contact structure if and only if the pair (Π, ξ) is a Jacobi structure.*

Proof. Assume that $\Omega = d_\rho \eta$. That (Π, ξ) is a Jacobi structure is a direct consequence of Proposition 3.6 and the relations (3.7) above. Inversely, assume that (Π, ξ) is a Jacobi structure on A. Since $\mathcal{L}^\rho_\xi \Pi = 0$, by the second assertion of Proposition 3.6, it follows that

$$\mathcal{L}^\rho_\xi \Omega = \eta \wedge \mathcal{L}^\rho_\xi \eta. \tag{3.8}$$

By the second relation in (3.6), for any $a \in \Gamma(A)$ we have

$$\mathcal{L}^\rho_\xi \eta(a) = i_\xi (d_\rho \eta)(a) = \eta \wedge i_\xi (d_\rho \eta)(\xi, a) = \eta \wedge \mathcal{L}^\rho_\xi \eta(\xi, a),$$

and then, by (3.8) and the first relation in (3.6), it follows that

$$\mathcal{L}^\rho_\xi \eta(a) = \mathcal{L}^\rho_\xi \Omega(\xi, a) = i_\xi (d_\rho \Omega)(\xi, a) = 0.$$

Hence, $\mathcal{L}^\rho_\xi \eta = 0$, and by (3.8) again, it follows that $\mathcal{L}^\rho_\xi \Omega = 0$. Now, since we also have $[\Pi, \Pi] = 2\xi \wedge \Pi$, by the first assertion of Proposition 3.6, it follows that $d_\rho \Omega + \eta \wedge (d_\rho \eta - \Omega) = 0$, and hence, for $a, b \in \Gamma(A)$, that

$$d_\rho \Omega(\xi, a, b) + \eta \wedge (d_\rho \eta - \Omega)(\xi, a, b) = 0.$$

On the other hand, since $d_\rho \Omega(\xi, a, b) = i_\xi (d_\rho \Omega)(a, b) = \mathcal{L}^\rho_\xi \Omega(a, b) = 0$ and $i_\xi (d_\rho \eta) = \mathcal{L}^\rho_\xi \eta = 0$, and by the relations (3.3), we have

$$d_\rho \Omega(\xi, a, b) + \eta \wedge (d_\rho \eta - \Omega)(\xi, a, b) = (d_\rho \eta - \Omega)(a, b).$$

Therefore $(d_\rho \eta - \Omega)(a, b) = 0$. □

Remark 3.8. 1. If $(A, \rho, [.,.])$ in the theorem above is just a skew algebroid, we see from the proof that (Π, ξ) being Jacobi still implies (Ω, η) is contact.

2. With a contact Lie algebroid $(A, \rho, [.,.], \eta)$ we naturally associate a Lie algebroid structure on the dual bundle $A^\star$ of A as follows. Let (Π, ξ) be the Jacobi structure associated with the contact structure η. Put $\flat_\eta = \flat_{\Omega,\eta}$ and $\sharp_\eta = \sharp_{\Omega,\eta} (= \flat_\eta^{-1})$ with $\Omega = d_\rho \eta$. By Lemma 3.5, we have $\sharp_{\Pi,\xi} = \sharp_\eta$, so $\sharp_{\Pi,\xi}$ is an isomorphism satisfying $\sharp_{\Pi,\xi}(\eta) = \xi$, and hence, by Theorem 3.1, an isomorphim satisfying

$$\sharp_{\Pi,\xi} \left([\alpha, \beta]^\eta_{\Pi,\xi} \right) = \left[\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta) \right].$$

Therefore, the skew algebroid $(A^\star, \rho_{\Pi,\xi}, [.,.]^\eta_{\Pi,\xi})$ is a Lie algebroid isomorphic to the Lie algebroid $(A, \rho, [.,.])$. If we put $\rho_\eta = \rho \circ \sharp_\eta$ and $[.,.]_\eta = [.,.]^\eta_{\Pi,\xi}$, the Lie algebroid $(A^\star, \rho_\eta, [.,.]_\eta)$ may be called the dual Lie algebroid of the contact Lie algebroid $(A, \rho, [.,.], \eta)$.

3.4. *Locally conformally symplectic Lie algebroids*

Let $(A, \rho, [.,.])$ be a skew (resp. almost Lie, Lie) algebroid on M. A locally conformally symplectic structure on $(A, \rho, [.,.])$ is a pair (Ω, θ) of a closed 1-form θ and a nondegenerate 2-form Ω on A such that

$$d_\rho \Omega + \theta \wedge \Omega = 0.$$

We also say that $(A, \rho, [.,.], \Omega, \theta)$ is a locally conformally symplectic skew (resp. almost Lie, Lie) algebroid. In case θ is exact, i. e. $\theta = d_\rho f$ for some smooth function $f \in C^\infty(M)$, we say that $(A, \rho, [.,.], \Omega, \theta)$ is a conformally symplectic skew (resp. almost Lie, Lie) algebroid, it is equivalent to $(A, \rho, [.,.], e^f \Omega)$ being a symplectic skew (resp. almost Lie, Lie) algebroid.

Let Ω be a nondegenerate 2-form on A and let $\theta \in \Gamma(A^\star)$. With the pair (Ω, θ) we associate a contravariant pair (Π, ξ) as follows :

$$\Pi(\alpha, \beta) = \Omega\left(\sharp_\Omega(\alpha), \sharp_\Omega(\beta)\right) \quad \text{and} \quad \xi = \sharp_\Omega(\theta),$$

where $\sharp_\Omega$ is the inverse isomorphism of the vector bundle isomorphism $\flat_\Omega : A \to A^\star$, $\flat_\Omega(a) = -i_a\Omega$. Clearly Π is nondegenerate and $\sharp_\Pi = \sharp_\Omega$. So we have a one-to-one correspondence between such pairs (Ω, θ) and the pairs (Π, ξ) with Π a nondegenerate A-bivector field and ξ an A-vector field. We recover the pair (Ω, θ) by the relations

$$\Omega(a, b) = \Pi(\sharp_\Pi^{-1}(a), \sharp_\Pi^{-1}(a)) \quad \text{and} \quad \theta = \sharp_\Pi^{-1}(\xi).$$

We have the following :

Proposition 3.9. *Let $(A, \rho, [.,.])$ be a skew algebroid. Let (Ω, θ) and (Π, ξ) be two pairs as above. Let a, b, c be A-vector fields and let $\alpha = \flat_\Omega(a)$, $\beta = \flat_\Omega(b)$ and $\gamma = \flat_\Omega(c)$. We have*

1. $\left(\frac{1}{2}[\Pi, \Pi] - \xi \wedge \Pi\right)(\alpha, \beta, \gamma) = (d_\rho\Omega + \theta \wedge \Omega)(a, b, c)$.
2. $\mathcal{L}^\rho_\xi \Pi(\alpha, \beta) = -\mathcal{L}^\rho_\xi \Omega(a, b)$.

Proof. Using the identity (2.6), we get

$$\Omega([a, b], c) = \gamma([a, b]) = -\frac{1}{2}[\Pi, \Pi](\alpha, \beta, \gamma) + \Pi([\alpha, \beta]_\Pi, \gamma).$$

We also have $\rho(a) \cdot \Omega(b, c) = \rho_\Pi(\alpha) \cdot \Pi(\beta, \gamma)$. Therefore, by a direct calculation using the identity (2.5), we get

$$d_\rho\Omega(a, b, c) = \frac{1}{2}[\Pi, \Pi](\alpha, \beta, \gamma). \tag{3.9}$$

On the other hand, notice that $\theta(a) = -i_\xi\Omega(a) = i_a\Omega(\xi) = -\alpha(\xi)$, likewise $\theta(b) = -\beta(\xi)$ and $\theta(c) = -\gamma(\xi)$, thus $\theta \wedge \Omega(a, b, c) = -\xi \wedge \Pi(\alpha, \beta, \gamma)$. Hence, with (3.9), we get the first assertion of the proposition. For the second assertion, is suffices to notice that

$$\Pi(\mathcal{L}^\rho_\xi\alpha, \beta) = -\mathcal{L}^\rho_\xi\alpha(b) = -\rho(\xi)(\alpha(b)) + \alpha(\mathcal{L}^\rho_\xi b) = \rho(\xi) \cdot \Omega(a, b) - \Omega(a, \mathcal{L}^\rho_\xi b).$$

□

The following theorem shows that the locally conformally symplectic structures are precisely the Jacobi structures (Π, ξ) with a nondegenerate underlying bivector field Π.

Theorem 3.10. *The pair (Ω, θ) is a locally conformally symplectic structure if and only if the pair (Π, ξ) is Jacobi.*

Proof. From the first assertion of Proposition 3.9 we deduce that the identity $d_\rho\Omega + \theta\wedge\Omega = 0$ is satisfied if and only if the identity $[\Pi, \Pi] = 2\xi \wedge \Pi$ is, and if one of the two is satisfied then, using the Cartan formula, we get

$$\mathcal{L}^{\rho}_{\xi}\Omega = d_\rho(i_\xi\Omega) + i_\xi d_\rho\Omega = -d_\rho\theta - i_\xi(\theta\wedge\Omega) = -d_\rho\theta,$$

and then, with the assertion 2. of Proposition 3.9, that $\mathcal{L}^{\rho}_{\xi}\Pi = 0$ if and only if $d_\rho\theta = 0$. □

Remark 3.11. Let $(A, \rho, [.,.], \Omega, \theta)$ be a locally conformally symplectic skew (resp. almost Lie, Lie) algebroid and let (Π, ξ) be the associated Jacobi structure. Since we have $\sharp_{\Pi,\xi}(\theta) = \sharp_\Pi(\theta) + \theta(\xi)\xi = \sharp_\Pi(\theta) = \xi$, by Theorem 3.1, we have

$$\sharp_{\Pi,\xi}\left([\alpha, \beta]^{\theta}_{\Pi,\xi}\right) = [\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)]\,.$$

On the other hand, the vector bundle morphism $\sharp_{\Pi,\xi}$ is an ismorphism. Indeed, since we have $\sharp_{\Pi,\xi}(\alpha) = \sharp_\Pi(\alpha + \alpha(\xi)\xi)$ and since the bivector field Π is nondegenerate, then $\sharp_{\Pi,\xi}(\alpha) = 0$ implies $\alpha = -\alpha(\xi)\theta$, thus $\alpha(\xi) = -\alpha(\xi)\theta(\xi) = 0$, and therefore $\alpha = 0$. Hence, just as in the contact situation, Remark 3.8(2), the skew algebroid $\left(A^\star, \rho_{\Pi,\xi}, [.,.]^{\theta}_{\Pi,\xi}\right)$ is a skew (resp. almost Lie, Lie) algebroid isomorphic to the skew (resp. almost Lie, Lie) algebroid $(A, \rho, [.,.])$. If we put $\rho_{\Omega,\theta} = \rho_{\Pi,\xi}$ and $[.,.]_{\Omega,\theta} = [.,.]^{\theta}_{\Pi,\xi}$, the skew (resp. almost Lie, Lie) algebroid $(A^\star, \rho_{\Omega,\theta}, [.,.]^{\theta}_{\Omega,\theta})$ may be called the dual skew (resp. almost Lie, Lie) algebroid of the locally conformally symplectic skew (resp. almost Lie, Lie) algebroid $(A, \rho, [.,.], \Omega, \theta)$.

4. Compatibility of a Jacobi structure and a Riemannian metric

4.1. *Compatibility of the triple* (Π, ξ, g)

Let g be a pseudo-Riemannian metric on a skew algebroid $(A, \rho, [.,.])$. Let Π be an A-bivector field and ξ be an A-vector field. With the triple (Π, ξ, g) we associate the A-1-form λ defined by

$$\lambda := g(\xi, \xi)\flat_g(\xi) - \flat_g(J\xi),$$

and we use the notation $[.,.]^{g}_{\Pi,\xi}$ instead of $[.,.]^{\lambda}_{\Pi,\xi}$. The Levi-Civita contravariant A-connection associated with the triple (Π, ξ, g) is the Levi-Civita connection $\mathcal{D}$ of the pseudo-Riemannian skew algebroid $(A^\star, \rho_{\Pi,\xi}, [.,.]^{g}_{\Pi,\xi})$. When $\xi = 0$, it is just the Levi-Civita contravariant connection associated with the pair (Π, g) in § 2.3

Proposition 4.1. *Assume that the vector bundle morphism $\sharp_{\Pi,\xi}$ is a skew algebroid morphism, i.e. we have*

$$\sharp_{\Pi,\xi}\left([\alpha, \beta]^{g}_{\Pi,\xi}\right) = [\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)]\,.$$

Assume also that $\sharp_{\Pi,\xi}$ is an isometry. Then

$$\sharp_{\Pi,\xi}\left(\mathcal{D}_\alpha\beta\right) = \nabla_{\sharp_{\Pi,\xi}(\alpha)}\left(\sharp_{\Pi,\xi}(\beta)\right),$$

where ∇ is the Levi-Civita connection of $(A, \rho, [.,.], g)$.

Proof. Since $\sharp_{\Pi,\xi}$ is an isometry we have $g(\sharp_{\Pi,\xi}(\mathcal{D}_\alpha\beta), \sharp_{\Pi,\xi}(\gamma)) = g^\star(\mathcal{D}_\alpha\beta, \gamma)$. Now, using the Koszul formula (2.8) for the Levi-Civita connections $\mathcal{D}$ and ∇, it follows that

$$g\left(\sharp_{\Pi,\xi}\left(\mathcal{D}_\alpha\beta\right), \sharp_{\Pi,\xi}(\gamma)\right) = g\left(\nabla_{\sharp_{\Pi,\xi}(\alpha)}\left(\sharp_{\Pi,\xi}(\beta)\right), \sharp_{\Pi,\xi}(\gamma)\right),$$

for any $\alpha, \beta, \gamma \in \Gamma(A^\star)$. □

We say that the metric g is compatible with the pair (Π, ξ) or that the triple (Π, ξ, g) is compatible if

$$\mathcal{D}\Pi(\alpha, \beta, \gamma) = \frac{1}{2}\left(\gamma(\xi)\Pi(\alpha, \beta) - \beta(\xi)\Pi(\alpha, \gamma) - J^\star\gamma(\xi)g^\star(\alpha, \beta) + J^\star\beta(\xi)g^\star(\alpha, \gamma)\right), \tag{4.1}$$

for every $\alpha, \beta, \gamma \in \Gamma(A^\star)$. The formula (4.1) can also be written in the form

$$\left(\mathcal{D}_\alpha J^\star\right)\beta = \frac{1}{2}\left(\Pi(\alpha, \beta)\flat_g(\xi) - \beta(\xi)J^\star\alpha - g^\star(\alpha, \beta)J^\star\flat_g(\xi) + J^\star\beta(\xi)\alpha\right), \tag{4.2}$$

for every $\alpha, \beta \in \Gamma(A^\star)$.

The compatibility in the case ξ is the zero A-vector field means that $(A, \rho, [.,.], \Pi, g)$ is a pseudo-Riemannian Poisson skew algebroid, and Riemannian Poisson if moreover the metric g is positive definite.

The following result generalizes the proposition 2.6.

Theorem 4.2. *Let $(A, \rho, [.,.], g)$ be a pseudo-Riemannian skew algebroid, Π an A-bivector field and ξ an A-vector field. Assume that $\mathcal{L}^\rho_\xi\Pi = 0$ and (Π, ξ, g) is compatible. Then, (Π, ξ) is a Jacobi structure if and only if $(\xi - \sharp_\Pi(\lambda)) \wedge \Pi = 0$.*

Proof. Taking the cyclic sum on the two sides of the equality (4.1) we get

$$\oint \mathcal{D}\Pi(\alpha, \beta, \gamma) = \xi \wedge \Pi(\alpha, \beta, \gamma). \tag{4.3}$$

On the other hand, we have

$$\begin{aligned}\mathcal{D}\Pi(\alpha, \beta, \gamma) &= \rho_{\Pi,\xi}(\alpha)\cdot\Pi(\beta, \gamma) - \Pi(\mathcal{D}_\alpha\beta, \gamma) - \Pi(\beta, \mathcal{D}_\alpha\gamma)\\ &= \rho_\Pi(\alpha)\cdot\Pi(\beta, \gamma) - \Pi(\mathcal{D}_\alpha\beta, \gamma) + \Pi(\mathcal{D}_\alpha\gamma, \beta) + \alpha(\xi)(\rho(\xi)\cdot\Pi(\beta, \gamma).\end{aligned}$$

Hence, the connection $\mathcal{D}$ being torsion free,

$$\oint \mathcal{D}\Pi(\alpha, \beta, \gamma) = \oint\left(\rho_\Pi(\alpha)\cdot\Pi(\beta, \gamma) - \Pi([\alpha, \beta]^\lambda_{\Pi,\xi}, \gamma) + \alpha(\xi)(\rho(\xi)\cdot\Pi(\beta, \gamma))\right).$$

Since

$$\begin{aligned}\Pi([\alpha, \beta]^\lambda_{\Pi,\xi}, \gamma) = \ &\Pi([\alpha, \beta]_\Pi, \gamma) + \alpha(\xi)\Pi(\mathcal{L}^\rho_\xi\beta, \gamma) + \beta(\xi)\Pi(\gamma, \mathcal{L}^\rho_\xi\alpha)\\ &-\alpha(\xi)\Pi(\beta, \gamma) - \beta(\xi)\Pi(\gamma, \alpha) - \Pi(\alpha, \beta)\Pi(\lambda, \gamma),\end{aligned}$$

it follows using (2.5) that

$$\oint \mathcal{D}\Pi(\alpha, \beta, \gamma) = \left(2\xi \wedge \Pi - [\Pi, \Pi]\right)(\alpha, \beta, \gamma) + \left(\sharp_\Pi(\lambda) \wedge \Pi\right)(\alpha, \beta, \gamma) + \oint \alpha(\xi)\mathcal{L}^\rho_\xi\Pi(\beta, \gamma),$$

and, by (4.3) and since $\mathcal{L}^\rho_\xi\Pi = 0$, that $2\xi \wedge \Pi - [\Pi, \Pi] = (\xi - \sharp_\Pi(\lambda)) \wedge \Pi$. □

4.2. $(1/2)$-*Kenmostsu Lie algebroids*

For the notions of almost contact pseudo-Riemannian structures, contact pseudo-Riemannian structures and Kenmotsu structures on a smooth manifold, see [2]. For the definition of these same structures on a Lie algebroid, see [8].

Almost contact and contact Riemannian Lie algebroids

Let $(A, \rho, [.,.])$ be a skew algebroid. Let (Φ, ξ, η) be a triple consisting of an A-1-form η, an A-vector field ξ and an A-(1,1)-tensor field Φ. The triple (Φ, ξ, η) is an almost contact structure on A if $\Phi^2 = -\mathrm{Id}_A + \eta \otimes \xi$ and $\eta(\xi) = 1$. From what it follows, in a similar manner as in the case of almost contact manifolds [2, Th.4.1], that $\Phi(\xi) = 0$ and $\eta \circ \Phi = 0$.

We say that a metric g is associated with the triple (Φ, ξ, η) if the following identity is verified

$$g(\Phi(a), \Phi(b)) = g(a, b) - \eta(a)\eta(b). \tag{4.4}$$

Let $(A, \rho, [.,.])$ be a skew (resp. almost Lie, Lie) algebroid. We say that $(A, \rho, [.,.], \Phi, \xi, \eta, g)$ is an almost contact (pseudo-)Riemannian skew (resp. almost Lie, Lie) algebroid if the triple (Φ, ξ, η) is an almost contact structure on A and g is an associated (pseudo-)Riemannian metric.

Let $(A, \rho, [.,.], \Phi, \xi, \eta, g)$ be an almost contact pseudo-Riemannian skew algebroid. Notice that if we set $b = \xi$ in (4.4), we deduce that

$$g(a, \xi) = \eta(a),$$

for any $a \in \Gamma(A)$, i.e., $\flat_g(\xi) = \eta$, and in particular that $g(\xi, \xi) = 1$. Also, using (4.4), $\Phi^2 = -\mathrm{Id}_A + \eta \otimes \xi$ and $\eta \circ \Phi = 0$, we get that $g(\Phi(a), b) = -g(a, \Phi(b))$, for any $a, b \in \Gamma(A)$. So, the map $\Pi : \Gamma(A^\star) \times \Gamma(A^\star) \to C^\infty(M)$ defined by

$$\Pi(\alpha, \beta) = g(\sharp_g(\alpha), \Phi(\sharp_g(\beta)))$$

is a bivector field on A. Let us call it the fundamental bivector field of the almost contact pseudo-Riemannian skew algebroid $(A, \rho, [.,.], \Phi, \xi, \eta, g)$. We have the following result

Proposition 4.3. *The vector bundle morphism $\sharp_{\Pi,\xi}$ is an isometry. Therefore, if in addition $\sharp_{\Pi,\xi}$ is a skew algebroid morphism, then*

$$\sharp_{\Pi,\xi}\left(\mathcal{D}_\alpha \beta\right) = \nabla_{\sharp_{\Pi,\xi}(\alpha)}\left(\sharp_{\Pi,\xi}(\beta)\right),$$

for every $\alpha, \beta \in \Gamma(A^\star)$.

Proof. Let us prove that $\sharp_{\Pi,\xi}$ is an isometry. Let $\alpha \in \Gamma(A^\star)$. Recall that by definition, we have $\sharp_{\Pi,\xi}(\alpha) = \sharp_\Pi(\alpha) + \alpha(\xi)\xi$. Since we have on one hand $\alpha(\xi) = g(\sharp_g(\alpha), \xi) = \eta(\sharp_g(\alpha))$ and on the other hand, for any $\beta \in \Gamma(A^\star)$,

$$\beta(\sharp_\Pi(\alpha)) = g(\sharp_g(\alpha), \Phi(\sharp_g(\beta))) = -g(\Phi(\sharp_g(\alpha)), \sharp_g(\beta)) = -\beta(\Phi(\sharp_g(\alpha))),$$

i.e. $\sharp_\Pi(\alpha) = -\Phi(\sharp_g(\alpha))$, we deduce that

$$\sharp_{\Pi,\xi}(\alpha) = -\Phi(\sharp_g(\alpha)) + \eta(\sharp_g(\alpha))\xi. \tag{4.5}$$

Let $\alpha, \beta \in \Gamma(A^\star)$. From the formula (4.5) and the fact that $g(\Phi(a), \xi) = \eta \circ \Phi(a) = 0$ and $g(\xi, \xi) = 1$, we deduce that

$$g(\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)) = g(\Phi(\sharp_g(\alpha)), \Phi(\sharp_g(\beta))) + \eta(\sharp_g(\alpha))\eta(\sharp_g(\beta)).$$

By using the formula (4.4), we get

$$g(\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)) = g(\sharp_g(\alpha), \sharp_g(\beta)) = g^*(\alpha, \beta).$$

The second claim is a direct consequence of Proposition 4.1. □

Let $(A, \rho, [.,.])$ be a skew (resp. almost Lie, Lie) algebroid. Assume that η is a contact form on $(A, \rho, [.,.])$. We say that $(A, \rho, [.,.], \eta, g)$ is a contact pseudo-Riemannian skew (resp. almost Lie, Lie) algebroid, or that the metric g is associated with the contact form η, if there exists a vector bundle endomorphism Φ of A such that (Φ, ξ, η, g) is an almost contact pseudo-Riemannian structure on A and that

$$g(a, \Phi(b)) = d_\rho\eta(a, b). \tag{4.6}$$

If in addition g is positive definite, we say that $(A, \rho, [.,.], \eta, g)$ is a contact Riemannian skew (resp. almost Lie, Lie) algebroid.

Theorem 4.4. *Assume that $(A, \rho, [.,.], \eta, g)$ is a contact pseudo-Riemannian Lie algebroid. We have*

$$\sharp_\eta (\mathcal{D}_\alpha\beta) = \nabla_{\sharp_\eta(\alpha)} (\sharp_\eta(\beta)),$$

for every $\alpha, \beta \in \Gamma(A^\star)$.

Proof. Let (Π, ξ) be the Jacobi structure associated with the contact form η, then $\sharp_\eta = \sharp_{\Pi,\xi}$. Let (Φ, ξ, η, g) be the almost contact pseudo-Riemannian structure associated with the contact pseudo-Riemannian structure (η, g) on A. By Remark 3.8(2) and the proposition above, we need only to prove that Π is the fundamental bivector field associated with (Φ, ξ, η, g) and that $\eta = \lambda$. Let $\alpha \in \Gamma(A^\star)$ and put $a = \sharp_\eta(\alpha)$. By using (4.6), we have

$$\sharp_g(\alpha) = \sharp_g(\flat_\eta(a)) = -\sharp_g(i_a d\eta) + \eta(a)\xi = \Phi(a) + \eta(a)\xi.$$

Therefore, applying Φ,

$$\Phi(\sharp_g(\alpha)) = \Phi^2(a) = -a + \eta(a)\xi = -\sharp_\eta(\alpha) + \alpha(\xi)\xi = -\sharp_\Pi(\alpha).$$

We deduce that $\Pi(\alpha, \beta) = g(\sharp_g(\alpha), \Phi(\sharp_g(\beta)))$ for any $\alpha, \beta \in \Gamma(A^\star)$, and hence, that $\Phi = -J$. Therefore, $J\xi = 0$, and since $g(\xi, \xi) = 1$ it follows that $\lambda = \flat_g(\xi) = \eta$. □

$(1/2)$-Kenmostsu Lie algebroids

Let $(A, \rho, [.,.])$ be a skew (resp. almost Lie, Lie) algebroid. An almost contact Riemannian structure (Φ, ξ, η, g) on A is said to be $(1/2)$-Kenmotsu if we have

$$(\nabla_a\Phi)(b) = \frac{1}{2}(g(\Phi(a), b)\xi - \eta(b)\Phi(a)),$$

for any $a, b \in \Gamma(A)$. We say also that $(A, \rho, [.,.], \Phi, \xi, \eta, g)$ is a $(1/2)$-Kenmotsu skew (resp. almost Lie, Lie) algebroid.

Lemma 4.5. *Let $(A, \rho, [.,.])$ be a skew algebroid. Assume that (Φ, ξ, η, g) is an almost contact pseudo-Riemannian structure on A and let Π be the associated fundamental bivector field. If the fiber bundle morphism $\sharp_{\Pi,\xi}$ is a skew algebroid morphism, then*

$$\sharp_{\Pi,\xi}\left((\mathcal{D}_\alpha J^*)\beta\right) = -\left(\nabla_{\sharp_{\Pi,\xi}(\alpha)}\Phi\right)(\sharp_{\Pi,\xi}(\beta)),$$

for every $\alpha, \beta \in \Gamma(A^\star)$.

Proof. By using the formula (4.5) and the fact that we have $\sharp_g \circ J^* = J \circ \sharp_g$, we deduce that $\sharp_{\Pi,\xi}(J^*\alpha) = -\Phi(\sharp_g(J^*\alpha)) + \eta(\sharp_g(J^*\alpha))\xi = -\Phi(\sharp_\Pi(\alpha)) = -\Phi(\sharp_{\Pi,\xi}(\alpha))$. Therefore

$$\sharp_{\Pi,\xi} \circ J^* = -\Phi \circ \sharp_{\Pi,\xi}. \tag{4.7}$$

Hence, with Proposition 4.3, we have

$$\begin{aligned}
\sharp_{\Pi,\xi}\left((\mathcal{D}_\alpha J^*)\,\beta\right) &= \sharp_{\Pi,\xi}\left(\mathcal{D}_\alpha\left(J^*\beta\right)\right) - \left(\sharp_{\Pi,\xi} \circ J^*\right)\left(\mathcal{D}_\alpha\beta\right),\\
&= \nabla_{\sharp_{\Pi,\xi}(\alpha)}\left(\sharp_{\Pi,\xi}\left(J^*\beta\right)\right) + \Phi\left(\sharp_{\Pi,\xi}\left(\mathcal{D}_\alpha\beta\right)\right),\\
&= -\nabla_{\sharp_{\Pi,\xi}(\alpha)}\left(\Phi\left(\sharp_{\Pi,\xi}(\beta)\right)\right) + \Phi\left(\nabla_{\sharp_{\Pi,\xi}(\alpha)}\sharp_{\Pi,\xi}(\beta)\right),\\
&= -\left(\nabla_{\sharp_{\Pi,\xi}(\alpha)}\Phi\right)(\sharp_{\Pi,\xi}(\beta)).
\end{aligned}$$

□

Proposition 4.6. *Under the same hypotheses of the above lemma, the compatibility of the triple (Π, ξ, g) is equivalent to*

$$(\nabla_a\Phi)(b) = \frac{1}{2}\left(g(\Phi(a), b)\xi - \eta(b)\Phi(a)\right),$$

for any $a, b \in \Gamma(A)$, and if moreover the metric g is positive definite, then the triple (Π, ξ, g) is compatible if and only if the almost contact Riemannian skew algebroid $(A, \rho, [.,.], \Phi, \xi, \eta, g)$ is $(1/2)$-Kenmotsu.

Proof. Since we have $J^*\flat_g(\xi) = \flat_g(J\xi) = -\flat_g(\Phi\xi) = 0$ and

$$J^*\beta(\xi) = J^*\beta(\sharp_g(\eta)) = \eta(\sharp_g(J^*\beta)) = \eta(J\sharp_g(\beta)) = -\eta(\Phi(\sharp_g(\beta))) = 0,$$

then the formula (4.2) becomes

$$(\mathcal{D}_\alpha J^*)\,\beta = \frac{1}{2}\left(\Pi(\alpha,\beta)\eta - \beta(\xi)J^*\alpha\right).$$

Applying $\sharp_{\Pi,\xi}$ which by Proposition 4.3 is an isometry and hence an isomorphism, this last formula is equivalent to

$$\sharp_{\Pi,\xi}\left((\mathcal{D}_\alpha J^*)\,\beta\right) = \frac{1}{2}\left(\Pi(\alpha,\beta)\sharp_{\Pi,\xi}(\eta) - \beta(\xi)\sharp_{\Pi,\xi}(J^*\alpha)\right).$$

Now, by Formula (4.5), we have $\sharp_{\Pi,\xi}(\eta) = \xi$, and if we put $a = \sharp_{\Pi,\xi}(\alpha)$ and $b = \sharp_{\Pi,\xi}(\beta)$, then we have $\beta(\xi) = \eta(b)$, also using (4.7), we have $\sharp_{\Pi,\xi}(J^*\alpha) = -\Phi(a)$ and

$$\begin{aligned}
\Pi(\alpha,\beta) &= g(\sharp_g(\alpha), \Phi(\sharp_g(\beta)))\\
&= -g(\sharp_g(\alpha), \sharp_g(J^*\beta))\\
&= -g^*(\alpha, J^*\beta)\\
&= g(a, \Phi(b)).
\end{aligned}$$

It remains to use the lemma above. □

Theorem 4.7. *Assume that* $(A, \rho, [.,.], \eta, g)$ *is a contact Riemannian Lie algebroid and let* (Φ, ξ, η, g) *be the associated almost contact Riemannian structure. Assume that* (Π, ξ) *is the Jacobi structure associated with the contact form* η*. Then the triple* (Π, ξ, g) *is compatible if and only if* $(A, \rho, [.,.], \Phi, \xi, \eta, g)$ *is* $(1/2)$*-Kenmotsu.*

Proof. We have proved, see the proof of Theorem 4.4, that Π is the bivector field of the proposition above and that $\lambda = \eta$. □

4.3. *Locally conformally Kähler Lie algebroids*

For the notions of almost Hermitian, Hermitian, Kähler Lie algebroids see [7]. In this paragraph $(A, \rho, [.,.])$ is a skew algebroid.

Riemannian metric associated with a locally conformally symplectic structure

Assume that $\Omega \in \Gamma(\wedge^2 A^\star)$ is a nondegenerate 2-form and let $\theta \in \Gamma(A^\star)$. Assume that the pair (Π, ξ) is associated with the pair (Ω, θ). We say that a pseudo-Riemannian metric g is associated with the pair (Ω, θ) if $\sharp_{\Omega,\theta} := \sharp_{\Pi,\xi}$ is an isometry, i.e. if

$$g\left(\sharp_{\Omega,\theta}(\alpha), \sharp_{\Omega,\theta}(\beta)\right) = g^*(\alpha, \beta), \tag{4.8}$$

for every $\alpha, \beta \in \Gamma(A^\star)$.

If $\theta = 0$, then $\xi = 0$ and $\sharp_{\Omega,\theta} = \sharp_\Omega$, and if J and J^* are the fields of endomorphisms defined by the formulae (2.9), then

$$\begin{aligned} g(\sharp_{\Omega,\theta}(\alpha), \sharp_{\Omega,\theta}(\beta)) &= g(\sharp_\Omega(\alpha), \sharp_\Omega(\beta)) \\ &= g^*(\flat_g(\sharp_\Omega(\alpha)), \flat_g(\sharp_\Omega(\beta))) \\ &= g^*(J^*\alpha, J^*\beta), \end{aligned}$$

for any $\alpha, \beta \in \Gamma(A^\star)$. Hence, in the case $\theta = 0$, the relation (4.8) is equivalent to

$$g^*\left(J^*\alpha, J^*\beta\right) = g^*(\alpha, \beta).$$

If moreover g is positive definite, this last identity means that the pair (Ω, g) is an almost Hermitian structure on A and that J is the associated almost complex structure, i.e., we have

$$g(Ja, Jb) = g(a, b) \qquad \text{and} \qquad \Omega(a, b) = g(Ja, b),$$

for every $a, b \in \Gamma(A)$.

Theorem 4.8. *Assume that* (Ω, θ) *is a locally conformally symplectic structure on* A *and that* g *is an associated metric. We have*

$$\sharp_{\Omega,\theta}\left(\mathcal{D}_\alpha \beta\right) = \nabla_{\sharp_{\Omega,\theta}(\alpha)}\left(\sharp_{\Omega,\theta}(\beta)\right),$$

for every $\alpha, \beta \in \Gamma(A^\star)$.

Proof. By Proposition 4.1 and Remark 3.11, we need only to prove that $\lambda = \theta$. On one hand, we have $\sharp_{\Pi,\xi}(\theta) = \xi$. On the other hand, for any $\alpha \in \Gamma(A^\star)$, we have

$$\begin{aligned} g(\sharp_{\Pi,\xi}(\lambda), \sharp_{\Pi,\xi}(\alpha)) &= g(\sharp_g(\lambda), \sharp_g(\alpha)) \\ &= g(\xi,\xi)\alpha(\xi) + g(\xi, J\sharp_g(\alpha)) \\ &= g(\xi,\xi)\alpha(\xi) + g(\xi, \sharp_\Pi(\alpha)) \\ &= g(\xi, \sharp_{\Pi,\xi}(\alpha)). \end{aligned}$$

Since $\sharp_{\Pi,\xi}$ is an isometry, hence an isomorphism, then $\sharp_{\Pi,\xi}(\lambda) = \xi$. □

Corollary 4.9. *Under the same hypotheses of the theorem above, we have*

$$\mathcal{D}\Pi(\alpha,\beta,\gamma) = \nabla\Omega(\sharp_{\Omega,\theta}(\alpha), \sharp_{\Omega,\theta}(\beta), \sharp_{\Omega,\theta}(\gamma)).$$

Proof. We have $\Omega\left(\xi, \sharp_\Pi(\alpha)\right) = -i_{\sharp_\Pi(\alpha)}\Omega(\xi) = \alpha(\xi)$ and likewise $\Omega(\xi, \sharp_\Pi(\beta)) = \beta(\xi)$. Consequently

$$\Omega(\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)) = \Pi(\alpha,\beta). \tag{4.9}$$

It suffices now to compute $\nabla\Omega(\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta), \sharp_{\Pi,\xi}(\gamma))$ and use the theorem above. □

Locally conformally Kähler Lie algebroids

Recall, see [7], that if Ω is a nondegenerate 2-form on A and g an associated Riemannian metric, the almost Hermitian structure (Ω, g) is Hermitian if the associated almost complex structure is integrable, and Kähler if moreover Ω is closed. Recall also that if (Ω, g) is almost Hermitian, then it is Kähler if and only if the 2-form Ω is parallel for the Levi-Civita connection of g.

If (Ω,θ) is a locally conformally symplectic structure and (Ω, g) a Hermitian structure, we say that the triple (Ω,θ,g) is a locally conformally Kähler structure.

We shall prove that if (Ω,θ) is a locally conformally symplectic structure on A and that (Π,ξ) is the associated Jacobi structure, if g is a Riemannian metric associated with Ω and with (Ω,θ), the compatibility of the triple (Π,ξ,g) induces a locally conformally Kähler structure on A.

Lemma 4.10. *Assume that $\Omega \in \Gamma(\wedge^2 A)$ is a nondegenerate 2-form on A and let $\theta \in \Gamma(A^\star)$. Assume that (Π,ξ) is the pair associated with (Ω,θ). If the pseudo-Riemannian metric g is associated with the 2-form Ω and with the pair (Ω,θ), then we have*

$$J \circ \sharp_{\Pi,\xi} = \sharp_{\Pi,\xi} \circ J^*.$$

Proof. Since the metric g is assumed to be associated with Ω and using (4.9), we get

$$g(J\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)) = \Omega(\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)) = \Pi(\alpha,\beta) = g^*(J^*\alpha,\beta),$$

and since g is also assumed to be associated with the pair (Ω,θ), i.e. $\sharp_{\Pi,\xi}$ is an isometry, then

$$g(J\sharp_{\Pi,\xi}(\alpha), \sharp_{\Pi,\xi}(\beta)) = g(\sharp_{\Pi,\xi}(J^*\alpha), \sharp_{\Pi,\xi}(\beta)).$$

Finally, since $\sharp_{\Pi,\xi}$ is an isometry, hence an isomorphism, the result follows. □

Theorem 4.11. *Assume that $(A, \rho, [.,.])$ is an almost Lie algebroid. Let $(\Omega, d_\rho f)$ be a conformally symplectic structure on A and let (Π, ξ) be the associated Jacobi structure. If g is a Riemannian metric associated with Ω and with $(\Omega, d_\rho f)$, then the triple (Π, ξ, g) is compatible if and only if the triple $(\Omega, d_\rho f, g)$ is a conformally Kähler structure.*

Proof. We need to prove that the triple (Π, ξ, g) is compatible if and only if the pair $(e^f\Omega, e^f g)$ is compatible, i.e., if and only if the A-2-form $e^f\Omega$ is parallel with respect to the Levi-Civita connection ∇^f associated with the metric $g^f = e^f g$. As the connections ∇ and ∇^f are related by the formula

$$\nabla^f_a b = \nabla_a b + \frac{1}{2}\left(d_\rho f(a)b + d_\rho f(b)a - g(a,b)\sharp_g(d_\rho f)\right),$$

we deduce that

$$\begin{aligned}\nabla^f\Omega(a,b,c) &= \nabla\Omega(a,b,c) - d_\rho f(a)\Omega(b,c)\\ &\quad -\tfrac{1}{2}d_\rho f(b)\Omega(a,c) + \tfrac{1}{2}d_\rho f(c)\Omega(a,b)\\ &\quad +\tfrac{1}{2}\left(g(a,b)\Omega(\sharp_g(d_\rho f),c) - g(a,c)\Omega(\sharp_g(d_\rho f),b)\right),\end{aligned}$$

and then that

$$\nabla^f(e^f\Omega)(a,b,c) = e^f\left(d_\rho f(a)\Omega(b,c) + \nabla^f\Omega(a,b,c)\right) = e^f\Lambda_f(a,b,c),$$

where we have set

$$\begin{aligned}\Lambda_f(a,b,c) &= \nabla\Omega(a,b,c) - \frac{1}{2}\left(d_\rho f(b)\Omega(a,c) - d_\rho f(c)\Omega(a,b)\right)\\ &\quad +\tfrac{1}{2}\left(g(a,b)\Omega(\sharp_g(d_\rho f),c) - g(a,c)\Omega(\sharp_g(d_\rho f),b)\right).\end{aligned}$$

It follows that $\nabla^f(e^f\Omega) = 0$ if and only if $\Lambda_f = 0$, hence that the pair $(e^f\Omega, e^f g)$ is compatible if and only if

$$\begin{aligned}\nabla\Omega(a,b,c) &= \frac{1}{2}\left(d_\rho f(b)\Omega(a,c) - d_\rho f(c)\Omega(a,b) - g(a,b)\Omega(\sharp_g(d_\rho f),c)\right.\\ &\quad \left.+g(a,c)\Omega(\sharp_g(d_\rho f),b)\right).\end{aligned}$$

Let us prove now that this last identity is equivalent to the formula (4.1). Let $\alpha, \beta, \gamma \in \Gamma(A^\star)$ be such that $a = \sharp_{\Pi,\xi}(\alpha)$, $b = \sharp_{\Pi,\xi}(\beta)$ and $c = \sharp_{\Pi,\xi}(\gamma)$. By Corollary 4.9, we have $\nabla\Omega(a,b,c) = \mathcal{D}\Pi(\alpha,\beta,\gamma)$. On the other hand, setting $\theta = d_\rho f$, we have $d_\rho f(b) = \theta(b) = \theta(\sharp_\Pi(\beta)) + \beta(\xi)\theta(\xi) = -\beta(\sharp_\Pi(\theta)) = -\beta(\xi)$ and likewise $d_\rho f(c) = -\gamma(\xi)$. Also, by (4.9), we have $\Omega(a,b) = \Pi(\alpha,\beta)$ and $\Omega(a,c) = \Pi(\alpha,\gamma)$. Finally, since the metric g is associated with Ω, it follows that

$$\begin{aligned}\Omega(\sharp_g(d_\rho f),b) &= -\Omega(b,\sharp_g(\theta))\\ &= -g(Jb,\sharp_g(\theta))\\ &= -\theta(Jb)\\ &= \Omega(\xi,Jb)\\ &= \Omega(\sharp_{\Pi,\xi}(\theta), J\sharp_{\Pi,\xi}(\beta)),\end{aligned}$$

and since g is associated with Ω and with (Ω,θ), by using the lemma above and (4.9), we get

$$\Omega(\sharp_g(d_\rho f),b) = \Omega(\sharp_{\Pi,\xi}(\theta),\sharp_{\Pi,\xi}(J^*\beta)) = \Pi(\theta,J^*\beta) = J^*\beta(\sharp_\Pi(\theta)) = J^*\beta(\xi)$$

and likewise $\Omega(\sharp_g(d_\rho f),c) = J^*\gamma(\xi)$. □

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