Morphology of the connected components of the boolean sum of two digraphs (≤ 5)-hypomorphic up to complementation

Forme des composantes connexes de la somme booléenne de deux digraphes (≤ 5)-hypomorphes à complémentaire près

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ABSTRACT. Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs, (≤ 5)-hypomorphic up to complementation, and $U := G \dot{+} G'$ be the boolean sum of G and G'. The case where U and $\overline{U}$ are both connected was studied by the authors and B.Chaari giving the form of the pair $\{G, G'\}$. In this paper we study the case where U is not connected and give the morphology of the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ whenever $\mathcal{C}$ is a connected component of U.
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Dedicated with warmth and admiration to Maurice Pouzet
at the occasion of his 78th birthday

1. Introduction and definitions

A *directed graph* or simply *digraph* G consists of a finite and nonempty set V of vertices together with a prescribed collection E of ordered pairs of distinct vertices, called the set of the *edges* of G. Such a digraph is denoted by $(V(G), E(G))$ or simply (V, E). The cardinality of a set V is denoted $|V|$. Given a digraph $G = (V, E)$, to each nonempty subset X of V associate the *subdigraph* $(X, E \cap (X \times X))$ of G induced by X denoted by $G_{\restriction X}$. Given a proper subset X of V, $G_{\restriction V \setminus X}$ is also denoted by $G - X$, and by $G - v$ whenever $X = \{v\}$. With each digraph $G = (V, E)$ associate its *dual* $G^* = (V, E^*)$ and its *complement* $\overline{G} = \left(V, \overline{E}\right)$ defined as follows: Given $x \neq y \in V$, $(x, y) \in E^*$ if $(y, x) \in E$, and $(x, y) \in \overline{E}$ if $(x, y) \notin E$.

Let $G = (V, E)$ be a digraph, for $x \neq y \in V$, $x \longrightarrow_G y$ or $y \longleftarrow_G x$ (or simply $x \longrightarrow y$ if there is no confusion) means $(x, y) \in E$ and $(y, x) \notin E$; $x \underline{\quad}_G y$ (or simply $x \underline{\quad} y$) means $(x, y) \in E$ and $(y, x) \in E$; $x \ldots_G y$ (or $x \ldots y$ or $x \; _G \, y$) means $(x, y) \notin E$ and $(y, x) \notin E$. For $X, Y \subseteq V$, $X \longrightarrow_G Y$ means $x \longrightarrow_G y$ for each $(x, y) \in X \times Y$. Similarly, $X \underline{\quad}_G Y$ and $X \ldots_G Y$ (or $X \; _G Y$) are defined in the same way. If $X = \{x\}$ or $Y = \{y\}$, we can replace X by x and Y by y. A subset I of V is an

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interval of G if for $a \neq b \in I$ and $x \in V \setminus I$, $(a, x) \in E$ if and only if $(b, x) \in E$, and $(x, a) \in E$ if and only if $(x, b) \in E$. Two distinct vertices x and y of G form a *directed pair* or an *oriented pair* of G if either $x \longrightarrow_G y$ or $x \longleftarrow_G y$. Otherwise, $\{x, y\}$ is a *neutral pair*; it is *full* if $x \underline{}_G y$, and *void* if $x \ldots_G y$. The *nature* of the pair $\{x, y\}$ in G is one of the three possibilities: oriented, full or void. We set $G(x, y) := G_{\restriction\{x,y\}}$. Two interesting kinds of digraphs are symmetric digraphs and tournaments. A digraph $G = (V, E)$ is a *symmetric digraph* or *graph* (resp. *tournament*) whenever for $x \neq y \in V$, $x \underline{}_G y$ or $x \ldots_G y$ (resp. $x \longrightarrow_G y$ or $y \longrightarrow_G x$). If $G = (V, E)$ is a symmetric digraph, each edge (x, y) of G is identified with the pair $\{x, y\}$ and is called an *edge* of G. For instance, given a set V, $(V, \emptyset)$ is the *empty graph* on V whereas $(V, [V]^2)$ is the *complete graph* on V, where $[V]^2$ is the set of pairs $\{x, y\}$ of distinct elements of V. A subset of vertices of a symmetric digraph is *homogeneous* if it is either a clique or an idependent set.

Given two digraphs $G = (V, E)$ and $G' = (V', E')$, a bijection f from V onto V' is an *isomorphism* from G onto G' provided that for any $x, y \in V$, $(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$. The digraphs G and G' are *isomorphic*, which is denoted by $G \simeq G'$, if there is an isomorphism from one onto the other, otherwise $G \not\simeq G'$. A digraph H *embeds* into G, or H is *embeddable* in G, if H is isomorphic to an induced subdigraph of G.

Let G and G' be two digraphs on the same vertex set V. They are *equal up to complementation* if $G' = G$ or $G' = \overline{G}$. Let k be an integer with $0 < k < |V|$, the digraphs G and G' are k*-hypomorphic* (resp. $(-k)$*-hypomorphic*) if for every k-element (resp. $(|V| - k)$-element) subset X of V, the induced subdigraphs $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic. The digraphs G and G' are $(\leq k)$*-hypomorphic* if they are t-hypomorphic for each integer $t \leq k$. A digraph G is k*-reconstructible* (resp. $(-k)$*-reconstructible*) if any digraph k-hypomorphic (resp. $(-k)$-hypomorphic) to G is isomorphic to G. A digraph G is $(\leq k)$*-reconstructible* if any digraph $(\leq k)$-hypomorphic to G is isomorphic to G. The digraphs G and G' are *isomorphic up to complementation* (resp. *hemimorphic*) if G' is isomorphic to G or $\overline{G}$ (resp. to G or G^*). The digraphs G' and G are *hereditarily isomorphic* [9] if for each nonempty subset X of V, $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic. They are *hereditarily isomorphic up to complementation* [3] if they are hereditarily isomorphic, or G' and $\overline{G}$ are hereditarily isomorphic. Let k be a positive integer, the digraphs G and G' are k*-hypomorphic up to complementation* (resp. k*-hemimorphic*) if for every k-element subset X of V, $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic up to complementation (resp. hemimorphic). The digraphs G and G' are $(\leq k)$*-hypomorphic up to complementation* (resp. $(\leq k)$*-hemimorphic*) if they are t-hypomorphic up to complementation (resp. t*-hemimorphic*) for each integer $t \leq k$. A digraph G is k*-reconstructible up to complementation* (resp. k*-half-reconstructible*) if any digraph k-hypomorphic up to complementation (resp. k-hemimorphic) to G is isomorphic up to complementation (resp. hemimorphic) to G. A digraph G is $(\leq k)$*-reconstructible up to complementation* (resp. $(\leq k)$*-half-reconstructible*) if any digraph $(\leq k)$-hypomorphic up to complementation (resp. $(\leq k)$-hemimorphic) to G is isomorphic up to complementation (resp. hemimorphic) to G.

We define the symmetric digraph P_n in the following manner, $V(P_n) = \{0, 1, \cdots, n-1\}$, and for $i \neq j \in \{0, 1, \ldots, n-1\}$, $\{i, j\}$ is an edge of P_n when $|i - j| = 1$. Thus $P_n := 0 \underline{} 1 \underline{} \ldots \underline{} n-2 \underline{} n-1$. A *path* is a symmetric digraph isomorphic to P_n. A *cycle* is a symmetric digraph isomorphic to $C_n := (V(P_n), E(P_n) \cup \{\{0, n-1\}\})$ for some integer $n \geq 3$.

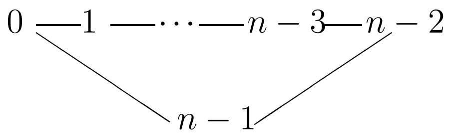

Figure 1. C_n

We define the digraph $\overrightarrow{P_n}$ by, $V(\overrightarrow{P_n}) = \{0, 1, \ldots, n-1\}$, and for $i \neq j \in \{0, 1, \ldots, n-1\}$, $i \longrightarrow_{\overrightarrow{P_n}} j$ when $j = i + 1$. Thus $\overrightarrow{P_n} := 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-2 \longrightarrow n-1$. We call *directed path* or *oriented path* a digraph isomorphic to $\overrightarrow{P_n}$, and *directed cycle* or *oriented cycle* a digraph isomorphic to $\overrightarrow{C_n} := (V(\overrightarrow{P_n}), E(\overrightarrow{P_n}) \cup \{(n-1, 0)\})$ for some integer $n \geq 3$.

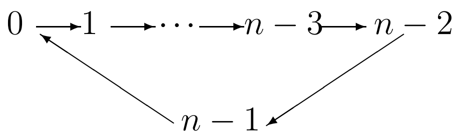

Figure 2. $\overrightarrow{C_n}$

We define $\overrightarrow{P_n^f}$ (resp. $\overrightarrow{C_n^f}$) obtained from $\overrightarrow{P_n}$ (resp. $\overrightarrow{C_n}$) by switching the void pairs by the full pairs. Thus $\overrightarrow{P_n^f} = \overline{\left(\overrightarrow{P_n}\right)^*}$ and $\overrightarrow{C_n^f} = \overline{\left(\overrightarrow{C_n}\right)^*}$.

A 3-*consecutivity* is every digraph isomorphic to $\overrightarrow{P_3} = (\{0, 1, 2\}, \{(0, 1), (1, 2)\})$ or to $\overrightarrow{P_3^f} = (\{0, 1, 2\}, \{(0, 1), (1, 2), (0, 2), (2, 0)\})$.

A 3-*cycle* is a tournament isomorphic to $\overrightarrow{C_3} := (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$.

A *total order*, or a *chain*, is a tournament T such that for $x, y, z \in V(T)$, if $x \longrightarrow_T y$ and $y \longrightarrow_T z$ then $x \longrightarrow_T z$. Given a total order $O = (V, E)$, for $x, y \in V$, $x < y$ means $x \longrightarrow_O y$. Thus, a total order on n vertices can be denoted by $v_0 < v_1 < \cdots < v_{n-1}$, we say that v_0 is the first element of the chain, and v_{n-1} its last element.

A *near-chain* is a digraph obtained from a chain (with first element a and last element b) by replacing the oriented pair $\{a, b\}$ by a neutral pair.

A *flag* is a digraph isomorphic to $(\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 1)\})$ or to $(\{0, 1, 2\}, \{(1, 0), (1, 2), (2, 1)\})$. A *peak* is a digraph isomorphic to $(\{0, 1, 2\}, \{(0, 1), (0, 2)\})$ or to $(\{0, 1, 2\}, \{(1, 0), (2, 0)\})$ or to $(\{0, 1, 2\}, \{(0, 1), (0, 2), (1, 2), (2, 1)\})$ or to $(\{0, 1, 2\}, \{(1, 0), (2, 0), (1, 2), (2, 1)\})$.

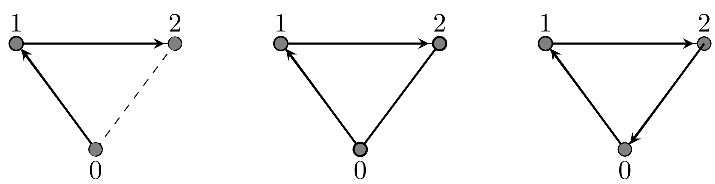

Figure 3. 3-*consecutivity and* 3-*cycle*

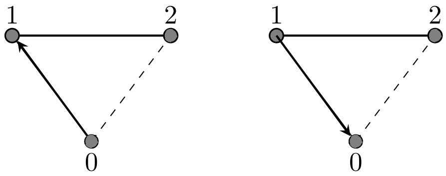

Figure 4. *Flags*

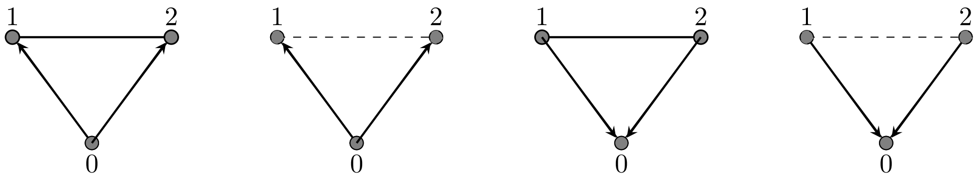

Figure 5. *Peaks*

Let G be a digraph, the *positive degree* (resp. *negative degree*) of a vertex x of G, denoted $d^+_G(x)$ (resp. $d^-_G(x)$), is the number of $y \in V(G)$ such that $x \longrightarrow_G y$ (resp. $y \longrightarrow_G x$). The *type* of G is $\tau(G) := (e, e')$ where e and e' are respectively the number of full pairs of G and $\overline{G}$. Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs and $a, b \in V$. We say that $\{a, b\}$ *have the same nature in* G and G' if and only if $G_{\restriction \{a,b\}} \simeq G'_{\restriction \{a,b\}}$. Let $G = (V, E)$ be a symmetric digraph, the *degree* of a vertex x of G, denoted $d_G(x)$, is the number of $y \in V(G)$ such that $x \underline{\hspace{1.5em}}_G y$.

Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs, 2-hypomorphic up to complementation. *The boolean sum* $G \dot{+} G'$ of G and G' is the symmetric digraph $U = (V, E(U))$ defined by $\{x, y\} \in E(U)$ if and only if $(x, y) \in E$ and $(x, y) \notin E'$, or $(x, y) \notin E$ and $(x, y) \in E'$. Clearly $\overline{U} = \overline{G} \dot{+} G'$ and $\overline{U} = G \dot{+} \overline{G'}$. Denote $\mathfrak{D}_{G,G'}$ the binary relation on V such that: for $x \in V$, $x \, \mathfrak{D}_{G,G'} \, x$; and for $x \neq y \in V$, $x \, \mathfrak{D}_{G,G'} \, y$ if there is a sequence $x = x_0, x_1, \ldots, x_m = y$ of elements of V satisfying $(x_i, x_{i+1}) \in E$ if and only if $(x_i, x_{i+1}) \notin E'$, for each $i \in \{0, 1, \ldots, m-1\}$. The relation $\mathfrak{D}_{G,G'}$ is an equivalence relation called *the difference relation*, its classes are called *difference classes*, this relation was introduced by Lopez [6]. Then clearly, $\mathcal{C}$ is a connected component of $U := G \dot{+} G'$ if and only if $\mathcal{C}$ is an equivalence class of $\mathfrak{D}_{G,G'}$, and thus $\mathfrak{D}_{G,G'}$ and $\mathfrak{D}_{\overline{G},G'}$ have only one class if and only if U and $\overline{U}$ are connected.

In 1970, R.Fraïssé conjectured the $(\leq k)$-reconstruction of digraphs (having a large number of vertices), k is a sufficiently large integer. In 1972, G.Lopez gave a positive answer to this conjecture by proving that the digraphs are (≤ 6)-reconstructible and that the value 6 is sharp.

In 1999, P.Ille conjectured the $(\leq k)$-reconstruction up to complementation of digraphs (having a large number of vertices), k is a sufficiently large integer. The case of symmetric digraphs was solved by J.Dammak, G.Lopez, M.Pouzet and H.Si Kaddour [4, 5], they proved that, the symmetric digraphs on v vertices are t-reconstructible up to complementation for every $4 \leq t \leq v - 3$ and that the value 4 is sharp. In fact, the case $t = v - 3$ was solved in [5]. For digraphs, a partially answer, Theorem 1.1, was obtained in [1].

Theorem 1.1. *(Theorem 1.3 of [1]) Let G and G' be two digraphs on the same set V of $n \geq 4$ vertices such that G and G' are (≤ 5)-hypomorphic up to complementation. Let $U := G \dot{+} G'$. If U and $\overline{U}$ are connected, then G' and G are hereditarily isomorphic up to complementation; more precisely one of the following holds:*

1) G and G' are two total orders.

2) $G \simeq \overrightarrow{P_n}$ or $G \simeq \overrightarrow{C_n}$, and $G' = G^*$.

3) $G \simeq \overrightarrow{P_n}$ or $G \simeq \overrightarrow{C_n}$, and $G' = \overline{G^*}$.

4) $G \simeq \overrightarrow{P_n^f}$ or $G \simeq \overrightarrow{C_n^f}$, and $G' = G^*$.

5) $G \simeq \overrightarrow{P_n^f}$ or $G \simeq \overrightarrow{C_n^f}$, and $G' = \overline{G^*}$.

Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs, (≤ 5)-hypomorphic up to complementation, and $U := G \dot{+} G'$ be the boolean sum of G and G'. The case where U and $\overline{U}$ are connected was studied by the authors and B.Chaari giving the form of the pair $\{G, G'\}$, see Theorem 1.1 (Theorem 1.3 of [1]). In this paper we look to the case where U is not connected. Let $\mathcal{C}$ be a connected component of U, if $\overline{\mathcal{C}}$ is connected, the form of the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is given by Theorem 1.1. Whenever $\overline{\mathcal{C}}$ is not connected, we will give the form of the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$, and deduce immediately the isomorphism up to complementation between $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$.

We consider the following digraphs:
$\alpha_4 = (\{v_0, v_1, v_2, v_3\}, \{(v_0, v_1), (v_0, v_3), (v_3, v_0), (v_1, v_2), (v_2, v_1), (v_3, v_2)\})$.
$\beta = (\{v_0, v_1, v_2, v_3\}, \{(v_0, v_1), (v_0, v_3), (v_3, v_0), (v_1, v_2), (v_2, v_1), (v_2, v_3)\})$.
$\gamma_4^+ = (\{v_0, v_1, v_2, v_3\}, \{(v_0, v_1), (v_0, v_3), (v_3, v_0), (v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2)\})$.
$\gamma_4^- = (\{v_0, v_1, v_2, v_3\}, \{(v_1, v_0), (v_0, v_2), (v_2, v_0), (v_1, v_3), (v_3, v_1), (v_2, v_3), (v_3, v_2)\})$.
$\lambda_4^+ = (\{v_0, v_1, v_2, v_3\}, \{(v_0, v_1), (v_0, v_3), (v_3, v_0), (v_1, v_2), (v_2, v_1)\})$.
$\lambda_4^- = (\{v_0, v_1, v_2, v_3\}, \{(v_1, v_0), (v_0, v_2), (v_2, v_0), (v_1, v_3), (v_3, v_1)\})$.

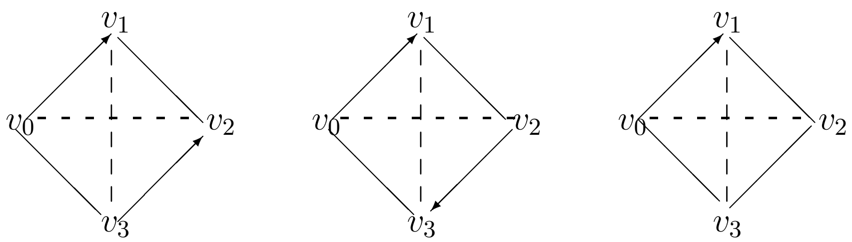

Figure 6. α_4, β_4, γ_4^+.

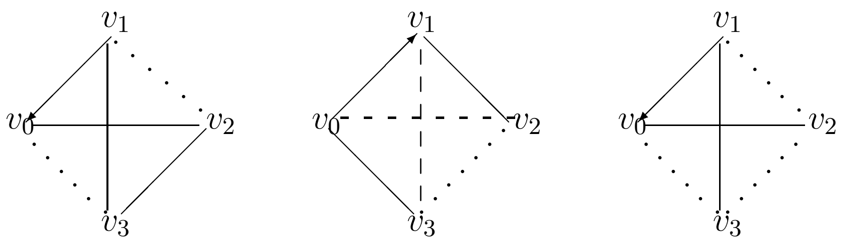

Figure 7. γ_4^-, λ_4^+, λ_4^-.

Note that $\gamma_4^- \simeq \gamma_4^+$, $\lambda_4^+ \simeq \lambda_4^-$, $\alpha_4 \not\simeq \overline{\alpha_4}$ and $\beta_4 \not\simeq \overline{\beta_4}$.

We consider the following symmetric digraphs:
$M_2 = (\{v_0, v_1\}, \{\{v_0, v_1\}\})$, $M_2' = (\{v_0, v_1\}, \{\})$.
$M_3 = (\{v_0, v_1, v_2\}, \{\{v_0, v_1\}, \{v_0, v_2\}\})$.
$M_3' = (\{v_0, v_1, v_2\}, \{\{v_0, v_2\}, \{v_1, v_2\}\})$.

$M_4 = (\{v_0, v_1, v_2, v_3\}, \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\})$.
$M_4'' = (\{v_0, v_1, v_2, v_3\}, \{\{v_0, v_2\}, \{v_0, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}\})$.
$D_4 = (\{v_0, v_1, v_2, v_3\}, \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_2, v_3\}\})$.
$D_4' = (\{v_0, v_1, v_2, v_3\}, \{\{v_0, v_2\}, \{v_0, v_3\}, \{v_1, v_2\}\})$.

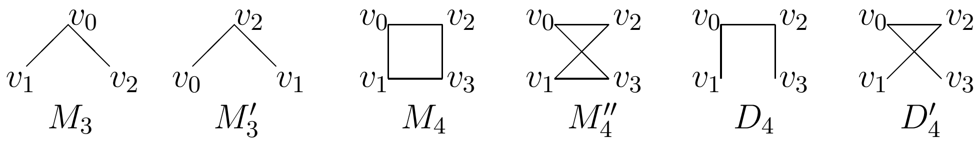

Figure 8. M_3, M_3', M_4, M_4'', D_4, D_4'.

Now we state our main result.

Theorem 1.2. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 5)-hypomorphic up to complementation and let $U := G \dot{+} G'$. If U is not connected and $\mathcal{C}$ is a connected component of U whose complement is not connected, then one of the following assertions holds:*

1) Let $v_0, v_1, v_2 \in V(\mathcal{C})$. If $G_{\restriction\{v_0,v_1,v_2\}}$ is a flag then $|V(\mathcal{C})| \in \{3, 4\}$.

- *i) If $|V(\mathcal{C})| = 3$ then $G'_{\restriction\mathcal{C}}$ is a flag and $G'_{\restriction\mathcal{C}} = \overline{G}_{\restriction\mathcal{C}}$.*
- *ii) If $|V(\mathcal{C})| = 4$ then the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is one of the following pairs:*
  $\{\alpha_4, \overline{\alpha_4}\}$, $\{\beta_4, \overline{\beta_4}\}$, $\{\gamma_4^+, \gamma_4^-\}$, $\{\lambda_4^+, \lambda_4^-\}$, $\{(\alpha_4)^*, (\overline{\alpha_4})^*\}$, $\{(\beta_4)^*, (\overline{\beta_4})^*\}$, $\{(\gamma_4^+)^*, (\gamma_4^-)^*\}$, $\{(\lambda_4^+)^*, (\lambda_4^-)^*\}$.

2) Let $v_0, v_1 \in V(\mathcal{C})$. If $\{v_0, v_1\}$ is a neutral pair in G reversed in G' and no flag is embeddable in $G_{\restriction V(\mathcal{C})}$, then $|V(\mathcal{C})| \leq 4$, $G_{\restriction V(\mathcal{C})}$ is a symmetric digraph and the following assertions hold.

- *i) If $|V(\mathcal{C})| = 2$ then the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is $\{M_2, M_2'\}$ or $\{\overline{M_2}, \overline{M_2'}\}$. So $G'_{\restriction V(\mathcal{C})} \simeq \overline{G}_{\restriction V(\mathcal{C})}$.*
- *ii) If $|V(\mathcal{C})| = 3$ then the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is $\{M_3, M_3'\}$ or $\{\overline{M_3}, \overline{M_3'}\}$. So $G'_{\restriction V(\mathcal{C})} \simeq G_{\restriction V(\mathcal{C})}$.*
- *iii) If $|V(\mathcal{C})| = 4$ then the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is one of the following pairs: $\{D_4, D_4'\}$, $\{\overline{D_4}, \overline{D_4'}\}$, $\{M_4, M_4''\}$, $\{\overline{M_4}, \overline{M_4''}\}$. So $G'_{\restriction V(\mathcal{C})} \simeq G_{\restriction V(\mathcal{C})}$.*

3) If every neutral pair in $G_{\restriction V(\mathcal{C})}$ is not reversed in $G'_{\restriction V(\mathcal{C})}$, then $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$ are (≤ 4)-hypomorphic, $\mathcal{C}$ is an interval of G and G' and the following assertions hold.

- *i) If $G_{\restriction\mathcal{C}}$ is a tournament, then $G_{\restriction\mathcal{C}}$ is a diamond-free tournament and $G'_{\restriction V(\mathcal{C})} \simeq {G^*}_{\restriction V(\mathcal{C})}$.*
- *ii) If $G_{\restriction\mathcal{C}}$ is not a tournament and $G_{\restriction\mathcal{C}}$ has no 3-directed cycle, then $G_{\restriction\mathcal{C}}$ is either a chain or a near-chain or a $\overrightarrow{P_n}$ or a $\overrightarrow{P_n^f}$ or a $\overrightarrow{C_n}$ or a $\overrightarrow{C_n^f}$, and $G'_{\restriction V(\mathcal{C})} \simeq G_{\restriction V(\mathcal{C})} \simeq {G^*}_{\restriction V(\mathcal{C})}$.*
- *iii) If $G_{\restriction\mathcal{C}}$ has a 3-directed cycle and $G_{\restriction\mathcal{C}}$ is not a tournament, then $G_{\restriction\mathcal{C}}$ embeds neither peaks, nor diamonds, nor adjacent neutral pairs, and $G'_{\restriction V(\mathcal{C})} \simeq {G^*}_{\restriction V(\mathcal{C})}$.*

2. Ingredients for the proof of Theorem 1.2

Let m be an integer, $m \geq 1$, $S = (\{0, 1, \ldots, m-1\}, E)$ be a digraph and for $i \in \{0, 1, \ldots, m-1\}$, $G_i = (V_i, E_i)$ be a digraph such that the V_i's are nonempty and pairwise disjoint. The *lexicographic sum over S of the G_i's* or simply the S-*sum* of the G_i's, is the digraph denoted by $S(G_0, G_1, \ldots, G_{m-1})$ and defined on the union of the V_i's as follows: given $x \in V_i$ and $y \in V_j$, where $i, j \in \{0, 1, \ldots, m-1\}$, (x, y) is an edge of $S(G_0, G_1, \ldots, G_{m-1})$ if either $i = j$ and $(x, y) \in E_i$, or $i \neq j$ and $(i, j) \in E(S)$: this digraph replaces each vertex i of S by G_i. For that, we say that the vertex i of S is *dilated by G_i*.

Remark 2.1. *Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs, (≤ 5)-hypomorphic up to complementation. Let $U := G \dot{+} G'$. We assume that U is not connected. Let $\mathcal{C}$ be a connected component of U.*
If $\overline{\mathcal{C}}$ is connected, then Theorem 1.1 gives the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$. So we can assume that $\overline{\mathcal{C}}$ is not connected. Let $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ be the connected components of $\overline{\mathcal{C}}$ ($k \geq 2$). Then $U_{\restriction \mathcal{C}} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph. Each $\mathcal{C}_i$ is called a subclass of $\mathcal{C}$.
Since for all $i \in \{0, 1, \cdots, k-1\}$, $\overline{\mathcal{C}_i}$ is connected then for every distinct vertices v_0 and v_1 of $V(\mathcal{C}_i)$, there are n vertices $v_0 = x_0, x_1, \cdots, x_{n-1} = v_1$ of $V(\mathcal{C}_i)$, such that $x_i \ldots_U x_{i+1}$ for all $i \in \{0, 1, \cdots, n-2\}$.

Remark 2.2. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 3)-hypomorphic up to complementation. Let $U := G \dot{+} G'$ and $a, b, c \in V$. If $G_{\restriction \{a,b,c\}}$ is a peak or a flag or a 3-homogeneous set, then $U_{\restriction \{a,b,c\}}$ is the complete or the empty graph.*

Lemma 2.3. *(Lemma 4.3 of [1]) Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 3)-hypomorphic up to complementation. Let $U := G \dot{+} G'$ and $a, b, c \in V$.*

1) If $E(U_{\restriction \{a,b,c\}})$ or $E(\overline{U}_{\restriction \{a,b,c\}})$ is the set $\{\{a,b\},\{b,c\}\}$, then $\{a,b\}$ is an oriented pair in G if and only if $\{b,c\}$ is an oriented pair in G.

2) If $E(U_{\restriction \{a,b,c\}})$ or $E(\overline{U}_{\restriction \{a,b,c\}})$ is the set $\{\{a,b\}\}$ and $\{a,b\}$ is an oriented pair in G, then $\{a,b\}$ is an interval of $G_{\restriction \{a,b,c\}}$ and $G'_{\restriction \{a,b,c\}}$.

3) If $E(U_{\restriction \{a,b,c\}})$ or $E(\overline{U}_{\restriction \{a,b,c\}})$ is the set $\{\{a,b\}\}$ and $\{a,b\}$ is a neutral pair in G, then $\{a,b\}$ is not an interval of $G_{\restriction \{a,b,c\}}$, and $\{b,c\}$ is an oriented pair in G if and only if $\{a,c\}$ is an oriented pair in G. Moreover if $c \longrightarrow_G a$ (resp. $c \underline{\quad}_G a$) then $b \longrightarrow_G c$ (resp. $c \ldots_G b$).

Lemma 2.4. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 4)-hypomorphic up to complementation. Let $U := G \dot{+} G'$, U not connected and let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. Let $v_0, v_1, v_2 \in V(\mathcal{C})$. If $\{v_0, v_1\}$ and $\{v_1, v_2\}$ are two neutral pairs having the same nature in G then $\{v_0, v_1\}$ and $\{v_1, v_2\}$ are two neutral pairs without the same nature in G'.*

Proof. W.l.o.g we can assume that $v_0 \underline{\quad}_G v_1$ and $v_1 \underline{\quad}_G v_2$.
From Remark 2.1, $U_{\restriction \mathcal{C}} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ ($k \geq 2$). By contradiction, we assume that ($G'_{\restriction \{v_0,v_1\}} = v_0 \ldots v_1$ and $G'_{\restriction \{v_1,v_2\}} = v_1 \ldots v_2$) or ($G'_{\restriction \{v_0,v_1\}} = v_0 \underline{\quad} v_1$ and $G'_{\restriction \{v_1,v_2\}} = v_1 \underline{\quad} v_2$).
• Case 1. ($v_0 \underline{\quad}_G v_1$, $v_1 \underline{\quad}_G v_2$) and ($v_0 \ldots_{G'} v_1$, $v_1 \ldots_{G'} v_2$).
Let $v_3 \in V(U) \setminus V(\mathcal{C})$, then $v_3 \ldots_U \{v_0, v_1, v_2\}$. According to the nature of $\{v_1, v_3\}$ in G, we have the following subcases:

Case 1.1. $\{v_1, v_3\}$ is an oriented pair in G. W.l.o.g. we assume that $v_3 \longrightarrow_G v_1$, so $v_3 \longrightarrow_{G'} v_1$. We have $U_{\restriction\{v_0,v_1,v_3\}} = \{v_0 \text{——} v_1\} \ldots v_3$ (resp. $U_{\restriction\{v_1,v_2,v_3\}} = \{v_1 \text{——} v_2\} \ldots v_3$) and $\{v_0, v_1\}$ (resp. $\{v_1, v_2\}$) is a neutral pair in G. So from 3) of Lemma 2.3, applied to $\{v_0, v_1, v_3\}$ (resp. $\{v_1, v_2, v_3\}$), we have $v_0 \longrightarrow_G v_3$ and $v_0 \longrightarrow_{G'} v_3$ (resp. $v_2 \longrightarrow_G v_3$ and $v_2 \longrightarrow_{G'} v_3$). We have $G'_{\restriction\{v_0,v_1,v_2,v_3\}} \not\simeq G_{\restriction\{v_0,v_1,v_2,v_3\}}$ because their types are different. If σ is an isomorphism from $\overline{G}_{\restriction\{v_0,v_1,v_2,v_3\}}$ into $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$, as v_3 is the only vertex in $\{v_0, v_1, v_2, v_3\}$ not adjacent to a neutral pair, then $\sigma(v_3) = v_3$. From $d^+_{G_{\restriction\{v_0,v_1,v_2,v_3\}}}(v_3) = 2$ and $d^+_{G'_{\restriction\{v_0,v_1,v_2,v_3\}}}(v_3) = 1$, we get a contradiction.

Case 1.2. $\{v_1, v_3\}$ is a neutral pair in G. W.l.o.g. we assume that $v_1 \underset{G}{\text{——}} v_3$, so $v_1 \underset{G'}{\text{——}} v_3$. We have $U_{\restriction\{v_0,v_1,v_3\}} = \{v_0 \text{——} v_1\} \ldots v_3$ (resp. $U_{\restriction\{v_1,v_2,v_3\}} = \{v_1 \text{——} v_2\} \ldots v_3$) and $\{v_0, v_1\}$ (resp. $\{v_1, v_2\}$) is a neutral pair in G, so from 3) of Lemma 2.3, applied to $\{v_0, v_1, v_3\}$ (resp. $\{v_1, v_2, v_3\}$), we have $v_0 \ldots_G v_3$ and $v_0 \ldots_{G'} v_3$ (resp. $v_2 \ldots_G v_3$ and $v_2 \ldots_{G'} v_3$). $\tau(G_{\restriction\{v_0,v_1,v_2,v_3\}}) = (3 + i, 3 - i)$ and $\tau(G'_{\restriction\{v_0,v_1,v_2,v_3\}}) = (1 + j, 5 - j)$ with $i, j \in \{0, 1\}$. This implies $1 + j = 3 - i$, so $i + j = 2$ and thus $i = j = 1$. We deduce that $\{v_0, v_2\}$ is a full pair in G and G'. So $G'_{\restriction\{v_0,v_1,v_2\}} \not\simeq G_{\restriction\{v_0,v_1,v_2\}}$ and $G'_{\restriction\{v_0,v_1,v_2\}} \not\simeq \overline{G}_{\restriction\{v_0,v_1,v_2\}}$, which contradicts the 3-hypomorphy up to complementation.

• Case 2. ($v_0 \underset{G}{\text{——}} v_1$, $v_1 \underset{G}{\text{——}} v_2$) and ($v_0 \underset{G'}{\text{——}} v_1$, $v_1 \underset{G'}{\text{——}} v_2$).

We have $v_0 \underset{\overline{U}}{\text{——}} v_1$ and $v_1 \underset{\overline{U}}{\text{——}} v_2$, then there is $i \in \{0, 1, \cdots, k-1\}$ such that $\{v_0, v_1, v_2\} \subseteq V(\mathcal{C}_i)$. Let $v_3 \in V(\mathcal{C}_j)$ with $j \in \{0, 1, \cdots, k-1\} \setminus \{i\}$ and $G_1 := \overline{G'}$. We have $\overline{U} := G \dot{+} G_1$, $v_3 \ldots_{\overline{U}} \{v_0, v_1, v_2\}$, ($v_0 \underset{G}{\text{——}} v_1$, $v_1 \underset{G}{\text{——}} v_2$) and ($v_0 \ldots_{G_1} v_1$, $v_1 \ldots_{G_1} v_2$). By exchanging G' by $G_1 = \overline{G'}$ we come back to case 1. □

As an immediate consequence of Lemma 2.4, we have:

Corollary 2.5. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 4)-hypomorphic up to complementation. Let $U := G \dot{+} G'$, U not connected and let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. We have the following:*

1) Let $v_0, v_1, v_2 \in V(\mathcal{C})$. If $G_{\restriction\{v_0,v_1,v_2\}}$ is a symmetric digraph then $U_{\restriction\{v_0,v_1,v_2\}}$ is neither the complete graph nor the empty graph.

2) $G_{\restriction V(\mathcal{C})}$ does not embed a 3-homogeneous subset.

Lemma 2.6. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 4)-hypomorphic up to complementation. Let $U := G \dot{+} G'$, U not connected and let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. Then $G_{\restriction V(\mathcal{C})}$ does not embed a peak.*

Proof. From Remark 2.1, $U_{\restriction\mathcal{C}} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$, $k \geq 2$. By contradiction, w.l.o.g. we assume that there are $v_0, v_1, v_2 \in V(\mathcal{C})$ such that $G_{\restriction\{v_0,v_1,v_2\}} = \{v_0 \text{——} v_1\} \longrightarrow v_2$. From Remark 2.2, we have $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G}_{\restriction\{v_0,v_1,v_2\}}$ or $G'_{\restriction\{v_0,v_1,v_2\}} = G_{\restriction\{v_0,v_1,v_2\}}$.

• Case 1. $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G}_{\restriction\{v_0,v_1,v_2\}}$.

Let $v_3 \in V(U) \setminus V(\mathcal{C})$, then $v_3 \ldots_U \{v_0, v_1, v_2\}$. We have $U_{\restriction\{v_0,v_2,v_3\}} = \{v_0 \text{——} v_2\} \ldots v_3$ (resp. $U_{\restriction\{v_1,v_2,v_3\}} = \{v_1 \text{——} v_2\} \ldots v_3$) and $\{v_0, v_2\}$ (resp. $\{v_1, v_2\}$) is an oriented pair in G, so from 2) of Lemma 2.3, applied to $\{v_0, v_2, v_3\}$ (resp. $\{v_1, v_2, v_3\}$), we have $\{v_0, v_2\}$ (resp. $\{v_1, v_2\}$) is an interval of $G_{\restriction\{v_0,v_2,v_3\}}$ (resp. $G_{\restriction\{v_1,v_2,v_3\}}$), so $\{v_0, v_1\}$ is an interval of $G_{\restriction\{v_0,v_1,v_3\}}$ and $\{v_0, v_1\}$ is a neutral edge of G, which contradicts 3) of Lemma 2.3.

• Case 2. $G'_{\upharpoonright\{v_0,v_1,v_2\}} = G_{\upharpoonright\{v_0,v_1,v_2\}}$.
We have $\overline{U}_{\upharpoonright\{v_0,v_1,v_2\}}$ is a complete graph, then there is $i \in \{0,1,\cdots,k-1\}$ such that $\{v_0,v_1,v_2\} \subseteq V(\mathcal{C}_i)$. Let $v_3 \in \mathcal{C}_j$ with $j \in \{0,1,\cdots,k-1\} \setminus \{i\}$ and $G_1 := \overline{G}$. We have $\overline{U} = G_1 \dot{+} G'$, $G'_{\upharpoonright\{v_0,v_1,v_2\}} = \overline{G_1}_{\upharpoonright\{v_0,v_1,v_2\}}$ and $v_3 \ldots_{\overline{U}} \{v_0,v_1,v_2\}$. By exchanging G by $G_1 = \overline{G}$ we come back to case 1. □

A *diamond* is a tournament on 4 vertices admitting only one interval of cardinality 3. The *center* of a diamond δ is the unique vertex $a \in V(\delta)$ satisfying $a \longrightarrow (V(\delta) - \{a\})$ or $a \longleftarrow (V(\delta) - \{a\})$. Up to isomorphism, there are exactly two diamonds δ^+ and $\delta^- = (\delta^+)^*$, where δ^+ is the tournament defined on $\{0,1,2,3\}$ by $\delta^+_{\upharpoonright\{0,1,2\}} = \overrightarrow{C_3}$ and $\{0,1,2\} \rightarrow 3$. A tournament isomorphic to δ^+ (resp. isomorphic to δ^-) is said to be a positive (resp. negative) diamond.

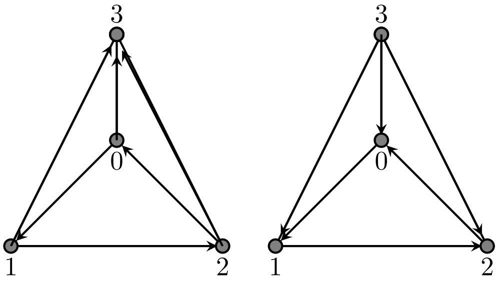

Figure 9. *Diamonds*

Lemma 2.7. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 5)-hypomorphic up to complementation. Let $U := G \dot{+} G'$, U not connected and let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. Then $G_{\upharpoonright V(\mathcal{C})}$ does not embed a diamond.*

Proof. From Remark 2.1, $U_{\upharpoonright\mathcal{C}} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ $(k \geq 2)$.

By contradiction, we assume w.l.o.g. that there are $v_0, v_1, v_2, v_3 \in V(\mathcal{C})$ such that $G_{\upharpoonright\{v_0,v_1,v_2\}} = \overrightarrow{C_3}$, with $v_0 \longrightarrow_G v_1 \longrightarrow_G v_2 \longrightarrow_G v_0$, and $\{v_0,v_1,v_2\} \longrightarrow_G v_3$. By the 4-hypomorphy up to complementation, we have $v_3 \longrightarrow_{G'} \{v_0,v_1,v_2\}$ or $v_3 \longleftarrow_{G'} \{v_0,v_1,v_2\}$, and $G'_{\upharpoonright\{v_0,v_1,v_2\}} \simeq \overrightarrow{C_3}$.
• Case 1. $v_3 \longrightarrow_{G'} \{v_0,v_1,v_2\}$.
Let $v_4 \in V(U) - V(\mathcal{C})$, then $v_4 \ldots_U \{v_0,v_1,v_2,v_3\}$. We have $U_{\upharpoonright\{v_0,v_3,v_4\}} = \{v_0 \text{——} v_3\} \ldots v_4$ (resp. $U_{\upharpoonright\{v_1,v_3,v_4\}} = \{v_1 \text{——} v_3\} \ldots v_4$, $U_{\upharpoonright\{v_2,v_3,v_4\}} = \{v_2 \text{——} v_3\} \ldots v_4$) and $\{v_0,v_3\}$ (resp. $\{v_1,v_3\}$, $\{v_2,v_3\}$) is an oriented pair in G, so from 2) of Lemma 2.3, we have $\{v_0,v_3\}$ (resp. $\{v_1,v_3\}$, $\{v_2,v_3\}$) is an interval of $G_{\upharpoonright\{v_0,v_3,v_4\}}$ (resp. $G_{\upharpoonright\{v_1,v_3,v_4\}}$, $G_{\upharpoonright\{v_2,v_3,v_4\}}$) and of $G'_{\upharpoonright\{v_0,v_3,v_4\}}$ (resp. $G'_{\upharpoonright\{v_1,v_3,v_4\}}$, $G'_{\upharpoonright\{v_2,v_3,v_4\}}$). Then $\{v_0,v_1,v_2,v_3\}$ is an interval of $G_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}}$ and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}}$.
According to the nature of $\{v_3,v_4\}$ in G, we have the following cases.

Case 1.1. $\{v_3,v_4\}$ is an oriented pair in G. W.l.o.g. we assume that $v_3 \longrightarrow_G v_4$, so $v_3 \longrightarrow_{G'} v_4$. As $\{v_0,v_1,v_2,v_3\}$ is an interval of $G_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}}$ and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}}$, $v_4 \longleftarrow_G \{v_0,v_1,v_2,v_3\}$ and $v_4 \longleftarrow_{G'} \{v_0,v_1,v_2,v_3\}$. Then $G_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}} = \{v_0,v_1,v_2\} < v_3 < v_4$, $\overline{G}_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}} = v_4 < v_3 < \{v_0,v_1,v_2\}$ and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}} = v_3 < \{v_0,v_1,v_2\} < v_4$. Thus $G'_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}} \not\simeq G_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}}$ and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}} \not\simeq \overline{G}_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}}$, which contradicts the 5-hypomorphy up to complementation.

Case 1.2. $\{v_3,v_4\}$ is a neutral pair in G. W.l.o.g. we assume that $v_3 \text{——}_G v_4$, so $v_3 \text{——}_{G'} v_4$. As $\{v_0,v_1,v_2,v_3\}$ is an interval of $G_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}}$ and $G'_{\upharpoonright\{v_0,v_1,v_2,v_3,v_4\}}$, $v_4 \text{——}_G \{v_0,v_1,v_2,v_3\}$ and

$v_4 \underline{\quad}_{G'} \{v_0, v_1, v_2, v_3\}$. So $G_{\restriction\{v_0,v_1,v_2,v_3,v_4\}} = v_4 \underline{\quad} \{\{v_0, v_1, v_2\} \longrightarrow v_3\}$, $G'_{\restriction\{v_0,v_1,v_2,v_3,v_4\}} = v_4 \underline{\quad} \{v_3 \longrightarrow \{v_0, v_1, v_2\}\}$. Thus $G'_{\restriction\{v_0,v_1,v_2,v_3,v_4\}} \not\simeq G_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3,v_4\}} \not\simeq \overline{G}_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$, which contradicts the 5-hypomorphy up to complementation.

• Case 2. $v_3 \longleftarrow_{G'} \{v_0, v_1, v_2\}$.

We have $v_3 \underline{\quad}_{\overline{U}} \{v_0, v_1, v_2\}$, then there is $i \in \{0, 1, \cdots, k-1\}$ such that $\{v_0, v_1, v_2, v_3\} \subseteq \mathcal{C}_i$. Let $v_4 \in \mathcal{C}_j$ with $j \in \{0, 1, \cdots, k-1\} \setminus \{i\}$ and $G_1 := \overline{G'}$. We have $\overline{U} := G \dot{+} G_1$, $v_3 \longrightarrow_{G_1} \{v_0, v_1, v_2\}$, $v_4 \ldots_{\overline{U}} \{v_0, v_1, v_2, v_3\}$. By exchanging G' by $G_1 = \overline{G'}$ we come back to case 1. □

Lemma 2.8. *Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs, (≤ 5)-hypomorphic up to complementation. Let $U := G \dot{+} G'$. We assume that U is not connected. Let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. Let $v_0, v_1, v_2 \in V(\mathcal{C})$. If $G_{\restriction\{v_0,v_1,v_2\}}$ is a 3-directed cycle then $G'_{\restriction\{v_0,v_1,v_2\}} = G^*_{\restriction\{v_0,v_1,v_2\}}$.*

Proof. From Remark 2.1, $\mathcal{C} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ $(k \geq 2)$. By contradiction, we assume that $G'_{\restriction\{v_0,v_1,v_2\}} = G_{\restriction\{v_0,v_1,v_2\}} = \overrightarrow{C_3}$. We have $U_{\restriction\{v_0,v_1,v_2\}}$ is an empty graph, so there is $i \in \{0, 1, \cdots, k-1\}$ such that $v_0, v_1, v_2 \in \mathcal{C}_i$. Let $v_3 \in \mathcal{C}_j$ with $j \in \{0, 1, \cdots, k-1\} \setminus \{i\}$, then $v_3 \underline{\quad}_U \{v_0, v_1, v_2\}$. We have $U_{\restriction\{v_0,v_1,v_3\}} = v_3 \underline{\quad} \{v_0 \ldots v_1\}$ (resp. $U_{\restriction\{v_0,v_2,v_3\}} = v_3 \underline{\quad} \{v_0 \ldots v_2\}$) and $\{v_0, v_1\}$ (resp. $\{v_0, v_2\}$) is an oriented pair in G, so from 2) of Lemma 2.3, we have $\{v_0, v_1\}$ (resp. $\{v_0, v_2\}$) is an interval of $G_{\restriction\{v_0,v_1,v_3\}}$ (resp. $G_{\restriction\{v_0,v_2,v_3\}}$) and of $G'_{\restriction\{v_0,v_1,v_3\}}$ (resp. $G'_{\restriction\{v_0,v_2,v_3\}}$). Then $\{v_0, v_1, v_2\}$ is an interval of $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$.
According to the nature of $\{v_0, v_3\}$ in G, we have the following cases:

• Case 1. $\{v_0, v_3\}$ is an oriented pair in G. W.l.o.g. we assume that $v_0 \longrightarrow_G v_3$, so $v_3 \longrightarrow_{G'} v_0$.
As $\{v_0, v_1, v_2\}$ is an interval of $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$, then $\{v_0, v_1, v_2\} \longrightarrow_G v_3$ and $v_3 \longrightarrow_{G'} \{v_0, v_1, v_2\}$ Then $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ is a diamond, which contradicts Lemma 2.7.

• Case 2. $\{v_0, v_3\}$ is a neutral pair in G. W.l.o.g. we assume that $v_0 \underline{\quad}_G v_3$, so $v_0 \ldots_{G'} v_3$.
As $\{v_0, v_1, v_2\}$ is an interval of $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$. Then $\{v_0, v_1, v_2\} \underline{\quad}_G v_3$ and $\{v_0, v_1, v_2\} \ldots_{G'} v_3$, which contradicts Lemma 2.4. □

Lemma 2.9. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 4)-hypomorphic up to complementation. Let $U := G \dot{+} G'$, U not connected and let $\mathcal{C}$ be a connected component of U.*

1) If $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$ are 2-hypomorphic, then $\mathcal{C}$ is an interval of G.

2) If $\overline{\mathcal{C}}$ is not connected and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ $(k \geq 2)$ then $\mathcal{C}_i$ is an interval of $G_{\restriction V(\mathcal{C}_i) \cup \{z\}}$ for all $i \in \{0, 1, \cdots, k-1\}$ and for all $z \in V(U) \setminus V(\mathcal{C})$.

Proof. 1) Let $v_0 \neq v_1 \in V(\mathcal{C})$ and $z \in V(U) \setminus V(\mathcal{C})$. As $\mathcal{C}$ is a connected component of U, there are n vertices $v_0 = x_0, x_1, \cdots, x_{n-1} = v_1$ of $V(\mathcal{C})$, such that $x_k \underline{\quad}_U x_{k+1}$ for all $k \in \{0, 1, \cdots, n-2\}$. As $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$ are 2-hypomorphic, then $\{x_k, x_{k+1}\}$ is an oriented pair in G. We have $U_{\restriction\{x_k,x_{k+1},z\}} = \{x_k \underline{\quad} x_{k+1}\} \ldots z$ and $\{x_k, x_{k+1}\}$ is an oriented pair in G, so from 2) of Lemma 2.3 applied to $\{x_k, x_{k+1}, z\}$, we have $\{x_k, x_{k+1}\}$ is an interval of $G_{\restriction\{x_k,x_{k+1},z\}}$ for all $k \in \{0, 1, \cdots, n-2\}$. Then $\{v_0, v_1\}$ is an interval of $G_{\restriction\{v_0,v_1,z\}}$. Thus $\mathcal{C}$ is an interval of G.
2) Let $i \in \{0, 1, \cdots, k-1\}$, $z \in V(U) \setminus V(\mathcal{C})$. Let $v_0 \neq v_1 \in V(\mathcal{C}_i)$. From Remark 2.1, there are n vertices $v_0 = x_0, x_1, \cdots, x_{n-1} = v_1$ of $V(\mathcal{C}_i)$, such that $x_k \ldots_U x_{k+1}$ for all $k \in \{0, 1, \ldots, n-2\}$.

Let $w \in V(\mathcal{C}_j)$ with $j \in \{0, 1, \cdots, k-1\} \setminus \{i\}$, thus $w \underline{\quad}_U \{x_k, x_{k+1}\}$ and $z \ldots_U \{x_k, x_{k+1}, w\}$ for all $k \in \{0, 1, \ldots, n-2\}$.
To prove that $V(\mathcal{C}_i)$ is an interval of $G_{\restriction V(\mathcal{C}_i) \cup \{z\}}$, it suffices to prove that for each $k \in \{0, 1, \ldots, n-2\}$, $\{x_k, x_{k+1}\}$ is an interval of $G_{\restriction \{x_k, x_{k+1}, z\}}$. To do so, according to the nature of $\{x_k, x_{k+1}\}$ in G, we consider the following cases:
• Case 1. $\{x_k, x_{k+1}\}$ is an oriented pair in G.
We have $U_{\restriction \{x_k, x_{k+1}, w\}} = w \underline{\quad} \{x_k \ldots x_{k+1}\}$ and $\{x_k, x_{k+1}\}$ is an oriented pair in G, then from 2) of Lemma 2.3 applied to $\{x_k, x_{k+1}, w\}$, $\{x_k, x_{k+1}\}$ is an interval of $G_{\restriction \{x_k, x_{k+1}, w\}}$ and $G'_{\restriction \{x_k, x_{k+1}, w\}}$, and from Lemma 2.4 applied to $\{x_k, x_{k+1}, w\}$, we have that $\{w, x_k\}$ and $\{w, x_{k+1}\}$ are oriented pairs in G reversed in G'. We have $U_{\restriction \{x_k, w, z\}} = \{x_k \underline{\quad} w\} \ldots z$ (resp. $U_{\restriction \{x_{k+1}, w, z\}} = \{x_{k+1} \underline{\quad} w\} \ldots z$) and $\{x_k, w\}$ (resp. $\{x_{k+1}, w\}$) is an oriented pair in G, then from 2) of Lemma 2.3, we have $\{x_k, w\}$ (resp. $\{x_{k+1}, w\}$) is an interval of $G_{\restriction \{x_k, w, z\}}$ (resp. $G_{\restriction \{x_{k+1}, w, z\}}$), so $\{x_k, x_{k+1}\}$ is an interval of $G_{\restriction \{x_k, x_{k+1}, z\}}$.
• Case 2. $\{x_k, x_{k+1}\}$ is a neutral pair in G. W.l.o.g. we can assume that $x_k \underline{\quad}_G x_{k+1}$, so $x_k \underline{\quad}_{G'} x_{k+1}$.
According to the nature of $\{x_k, w\}$ in G, we have the following subcases:

Case 2.1. $\{x_k, w\}$ is an oriented pair in G. W.l.o.g. we assume that $x_k \longrightarrow_G w$, so $w \longrightarrow_{G'} x_k$.
We have $U_{\restriction \{x_k, x_{k+1}, w\}} = \{x_k \ldots x_{k+1}\} \underline{\quad} w$ and $\{x_k, x_{k+1}\}$ is a neutral pair in G, then from 3) of Lemma 2.3 we have $w \longrightarrow_G x_{k+1}$, so $x_{k+1} \longrightarrow_{G'} w$.
We have $U_{\restriction \{x_k, w, z\}} = \{x_k \underline{\quad} w\} \ldots z$ (resp. $U_{\restriction \{x_{k+1}, w, z\}} = \{x_{k+1} \underline{\quad} w\} \ldots z$) and $\{x_k, w\}$ (resp. $\{x_{k+1}, w\}$) is an oriented pair in G, so from 2) of Lemma 2.3, we have $\{x_k, w\}$ (resp. $\{x_{k+1}, w\}$) is an interval of $G_{\restriction \{x_k, w, z\}}$ (resp. $G_{\restriction \{x_{k+1}, w, z\}}$), so $\{x_k, x_{k+1}\}$ is an interval of $G_{\restriction \{x_k, x_{k+1}, z\}}$.

Case 2.2. $\{x_k, w\}$ is a neutral pair in G. W.l.o.g. we can assume that $x_k \underline{\quad}_G w$, so $x_k \ldots_{G'} w$.
We have $U_{\restriction \{x_k, x_{k+1}, w\}} = \{x_k \ldots x_{k+1}\} \underline{\quad} w$ and $\{x_k, x_{k+1}\}$ is a neutral pair in G, then from 3) of Lemma 2.3 we have $x_{k+1} \ldots_G w$, so $x_{k+1} \underline{\quad}_{G'} w$.
According to the nature of $\{z, w\}$ in G, we have the following subcases:

Case 2.2.1. $\{z, w\}$ is an oriented pair in G. W.l.o.g. we can assume $w \longrightarrow_G z$, so $w \longrightarrow_{G'} z$.
We have $U_{\restriction \{x_k, w, z\}} = \{x_k \underline{\quad} w\} \ldots z$ (resp. $U_{\restriction \{x_{k+1}, w, z\}} = \{x_{k+1} \underline{\quad} w\} \ldots z$) and $\{x_k, w\}$ (resp. $\{x_{k+1}, w\}$) is a neutral pair in G, then from 3) of Lemma 2.3, applied to $\{z, w, x_k\}$ (resp. $\{z, w, x_{k+1}\}$), we have $x_k \longleftarrow_G z$ (resp. $x_{k+1} \longleftarrow_G z$). So $\{x_k, x_{k+1}\}$ is an interval of $G_{\restriction \{x_k, x_{k+1}, z\}}$.

Case 2.2.2. $\{z, w\}$ is a neutral pair in G. W.l.o.g. we can assume $w \underline{\quad}_G z$, so $w \underline{\quad}_{G'} z$.
We have $U_{\restriction \{x_k, w, z\}} = \{x_k \underline{\quad} w\} \ldots z$ (resp. $U_{\restriction \{x_{k+1}, w, z\}} = \{x_{k+1} \underline{\quad} w\} \ldots z$) and $\{x_k, w\}$ (resp. $\{x_{k+1}, w\}$) is a neutral pair in G, then from 3) of Lemma 2.3, applied to $\{z, w, x_k\}$ (resp. $\{z, w, x_{k+1}\}$), we have $x_k \ldots_G z$ (resp. $x_{k+1} \ldots_G z$). So $\{x_k, x_{k+1}\}$ is an interval of $G_{\restriction \{x_k, x_{k+1}, z\}}$. □

As an immediate consequence of Lemmas 2.3 and 2.9, we have the following.

Corollary 2.10. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 4)-hypomorphic up to complementation. Let $U := G \dot{+} G'$, U not connected and let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. Let $v_0, v_1 \in V(\mathcal{C})$ and $\mathcal{C} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ $(k \geq 2)$. If $\{v_0, v_1\}$ is a neutral pair in G reversed in G', then there are two subclasses $\mathcal{C}_i, \mathcal{C}_j$ of $\mathcal{C}$ $(i \neq j \in \{0, 1, \cdots, k-1\})$ such that $v_0 \in V(\mathcal{C}_i)$ and $v_1 \in V(\mathcal{C}_j)$.*

Lemma 2.11. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 4)-hypomorphic up to complementation. Let $U := G \dot{+} G'$, U not connected and let $\mathcal{C}$ be a connected* *component of U such that $\overline{\mathcal{C}}$ is not connected. Let $v_0, v_1, v_2 \in V(\mathcal{C})$. If $G_{\restriction\{v_0,v_1,v_2\}}$ is a 3−consecutivity, then $G'_{\restriction\{v_0,v_1,v_2\}} = G^*_{\restriction\{v_0,v_1,v_2\}}$.*

Proof. From Remark 2.1, $U_{\restriction\mathcal{C}} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ ($k \geq 2$). If $G_{\restriction\{v_0,v_1,v_2\}} = \overrightarrow{P_3} = v_0 \longrightarrow v_1 \longrightarrow v_2$, assume by contradiction that $G'_{\restriction\{v_0,v_1,v_2\}} = G_{\restriction\{v_0,v_1,v_2\}}$ or $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G}_{\restriction\{v_0,v_1,v_2\}}$ or $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G^*}_{\restriction\{v_0,v_1,v_2\}}$.

• Case 1. $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G}_{\restriction\{v_0,v_1,v_2\}}$.
Let $v_3 \in V(U) \setminus V(\mathcal{C})$, then $v_3 \ldots_U \{v_0, v_1, v_2\}$. We have $U_{\restriction\{v_0,v_1,v_3\}} = \{v_0 \text{---} v_1\} \ldots v_3$ (resp. $U_{\restriction\{v_1,v_2,v_3\}} = \{v_1 \text{---} v_2\} \ldots v_3$) and $\{v_0, v_1\}$ (resp. $\{v_1, v_2\}$) is an oriented pair in G, so from 2) of Lemma 2.3, we have $\{v_0, v_1\}$ (resp. $\{v_1, v_2\}$) is an interval of $G_{\restriction\{v_0,v_1,v_3\}}$ (resp. $G_{\restriction\{v_1,v_2,v_3\}}$), so $\{v_0, v_2\}$ is an interval of $G_{\restriction\{v_0,v_2,v_3\}}$. Since $U_{\restriction\{v_0,v_2,v_3\}} = \{v_0 \text{---} v_2\} \ldots v_3$, we get a contradiction with 3) of Lemma 2.3.

• Case 2. $G'_{\restriction\{v_0,v_1,v_2\}} = G_{\restriction\{v_0,v_1,v_2\}}$.
We have $\overline{U}_{\restriction\{v_0,v_1,v_2\}}$ is a complete graph, then there is $i \in \{0, 1, \cdots, k-1\}$ such that $\{v_0, v_1, v_2\} \subseteq V(\mathcal{C}_i)$. Let $v_3 \in V(\mathcal{C}_j)$ with $j \in \{0, 1, \cdots, k-1\} \setminus \{i\}$ and $G_1 := \overline{G}$. We have $\overline{U} = G_1 \dot{+} G'$, $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G_1}_{\restriction\{v_0,v_1,v_2\}}$ and $v_3 \ldots_{\overline{U}} \{v_0, v_1, v_2\}$. By exchanging G by $G_1 = \overline{G}$ we come back to case 1.

• Case 3. $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G^*}_{\restriction\{v_0,v_1,v_2\}}$.
We have $U_{\restriction\{v_0,v_1,v_2\}} = \{v_0 \text{---} v_2\} \ldots v_1$, then there is a subclass $\mathcal{C}_i$ of $\mathcal{C}$ with $i \in \{0, 1, \cdots, k-1\}$ such that $\{v_0, v_1, v_2\} \subseteq V(\mathcal{C}_i)$. Since $\{v_0, v_2\}$ is a neutral pair in G reversed in G', we get a contradiction with Corollary 2.10. □

Lemma 2.12. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 5)-hypomorphic up to complementation and let $U := G \dot{+} G'$. Assume that U is not connected and let $\mathcal{C}$ be a connected component of U whose complement is not connected and let $v_0, v_1, v_2 \in V(\mathcal{C})$. If $G_{\restriction\{v_0,v_1,v_2\}}$ is a flag, then $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G}_{\restriction\{v_0,v_1,v_2\}}$.*

Proof. We can assume $G_{\restriction\{v_0,v_1,v_2\}} = v_0 \longrightarrow v_1 \text{---} v_2$.
From Remark 2.1, $\mathcal{C} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ ($k \geq 2$). By contradiction, we assume that $G_{\restriction\{v_0,v_1,v_2\}} = G'_{\restriction\{v_0,v_1,v_2\}} = v_0 \longrightarrow v_1 \text{---} v_2$. We have $\overline{U}_{\restriction\{v_0,v_1,v_2\}}$ is a complete graph, so there is $i \in \{0, 1, \cdots, k-1\}$ such that $\{v_0, v_1, v_2\} \subseteq V(\mathcal{C}_i)$. Let $v_3 \in V(\mathcal{C}_j)$ with $j \in \{0, 1, \cdots, k-1\} \setminus \{i\}$, then $v_3 \text{---}_U \{v_0, v_1, v_2\}$. According to the nature of $\{v_1, v_3\}$ in G, we have the following cases:

• Case 1. $\{v_1, v_3\}$ is an oriented pair in G. W.l.o.g. we can assume $v_1 \longleftarrow_G v_3$, so $v_1 \longrightarrow_{G'} v_3$.
We have $U_{\restriction\{v_0,v_1,v_3\}} = v_3 \text{---} \{v_0 \ldots v_1\}$ and $\{v_0, v_1\}$ is an oriented pair in G, so from 2) of Lemma 2.3, $\{v_0, v_1\}$ is an interval of $G_{\restriction\{v_0,v_1,v_3\}}$, thus $v_0 \longleftarrow_G v_3$, so $v_0 \longrightarrow_{G'} v_3$. We have $U_{\restriction\{v_1,v_2,v_3\}} = v_3 \text{---} \{v_1 \ldots v_2\}$ and $\{v_1, v_2\}$ is a neutral pair in G, so from 3) of Lemma 2.3, $v_2 \longrightarrow_G v_3$, so $v_2 \longleftarrow_{G'} v_3$. Let $v_4 \in V(U) - V(\mathcal{C})$, then $v_4 \ldots_U \{v_0, v_1, v_2, v_3\}$. We have $U_{\restriction\{v_0,v_3,v_4\}} = \{v_0 \text{---} v_3\} \ldots v_4$ (resp. $U_{\restriction\{v_1,v_3,v_4\}} = \{v_1 \text{---} v_3\} \ldots v_4$, $U_{\restriction\{v_2,v_3,v_4\}} = \{v_2 \text{---} v_3\} \ldots v_4$) and $\{v_0, v_3\}$ (resp. $\{v_1, v_3\}$, $\{v_2, v_3\}$) is an oriented pair in G, so from 2) of Lemma 2.3, $\{v_0, v_3\}$ (resp. $\{v_1, v_3\}$, $\{v_2, v_3\}$) is an interval of $G_{\restriction\{v_0,v_3,v_4\}}$ (resp. $G_{\restriction\{v_1,v_3,v_4\}}$, $G_{\restriction\{v_2,v_3,v_4\}}$), thus $\{v_0, v_1, v_2, v_3\}$ is an interval of $G_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$. We set $H := G_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$ and $H' := G'_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$. According to the nature of $\{v_3, v_4\}$, we have the following subcases.

Case 1.1. $\{v_3, v_4\}$ is an oriented pair in G. W.l.o.g. we can assume that $v_3 \longrightarrow_G v_4$, so $v_3 \longrightarrow_{G'} v_4$. As $\{v_0, v_1, v_2, v_3\}$ is an interval of $G_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$, $\{v_0, v_1, v_2, v_3\} \longrightarrow_G v_4$ and

$\{v_0, v_1, v_2, v_3\} \longrightarrow_{G'} v_4$. We have $d^+_H(v_4) = 4$ and for all $x \in \{v_0, v_1, v_2, v_3, v_4\}$, $d^+_{H'}(x) \neq 4$ then $H' \not\simeq \overline{H}$. On the hother hand, $H' \not\simeq H$, indeed if σ is an isomorphism from H into H', since the only vertices not adjacent to a neutral pair are v_3 and v_4 and $d^+_H(v_4) = d^+_{H'}(v_4) = 0$, $d^+_H(v_3) \neq 0$, then $\sigma(v_4) = v_4$ and $\sigma(v_3) = v_3$. We get a contradiction with $d^+_H(v_3) = 3$ and $d^+_{H'}(v_3) = 2$.

Case 1.2. $\{v_3, v_4\}$ is a neutral pair in G. W.l.o.g. we can assume that $v_3 \underline{\quad}_G v_4$, so $v_3 \underline{\quad}_{G'} v_4$. As $\{v_0, v_1, v_2, v_3\}$ is an interval of $G_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3,v_4\}}$, $\{v_0, v_1, v_2, v_3\} \underline{\quad}_G v_4$ and $\{v_0, v_1, v_2, v_3\} \underline{\quad}_{G'} v_4$. We have H and H' have the same type $(5, 1)$, so $H' \not\simeq \overline{H}$. On the hother hand, $H' \not\simeq H$, indeed if σ is an isomorphism from H into H', since v_3 is the only vertex adjacent to exactly one neutral pair, then $\sigma(v_3) = v_3$. We get a contradiction with $d^+_H(v_3) = 2$ and $d^+_{H'}(v_3) = 1$.

• Case 2. $\{v_1, v_3\}$ is a neutral pair in G. W.l.o.g. we can assume that $v_1 \underline{\quad}_G v_3$, so $v_1 \ldots_{G'} v_3$.

We have $U_{\restriction\{v_0,v_1,v_3\}} = v_3 \underline{\quad} \{v_0 \ldots v_1\}$ and $\{v_0, v_1\}$ is an oriented pair in G, so from 2) of Lemma 2.3, we have $\{v_0, v_1\}$ is an interval of $G_{\restriction\{v_0,v_1,v_3\}}$, thus $v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$.

We have $v_0 \underline{\quad}_G v_3 \underline{\quad}_G v_1$ and $v_0 \ldots_{G'} v_3 \ldots_{G'} v_1$, which contradicts Lemma 2.4. □

Lemma 2.13. *Let G and G' be two digraphs on the same vertex set V such that G and G' are (≤ 5)-hypomorphic up to complementation and let $U := G \dot{+} G'$. Assume that U is not connected and let $\mathcal{C}$ be a connected component of U whose complement is not connected. Let $v_0, v_1, v_2 \in V(\mathcal{C})$ and $\mathcal{C} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ ($k \geq 2$). If $G_{\restriction\{v_0,v_1,v_2\}} = v_0 \longrightarrow v_1 \underline{\quad} v_2$ is a flag, then there are three distinct subclasses $\mathcal{C}_{i_0}, \mathcal{C}_{i_1}, \mathcal{C}_{i_2}$ such that, $v_0 \in \mathcal{C}_{i_0}$, $v_1 \in \mathcal{C}_{i_1}$ and $v_2 \in \mathcal{C}_{i_2}$, and $|\mathcal{C}_{i_0}| = |\mathcal{C}_{i_1}| = 1$.*

Proof. From Lemma 2.12, $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G}_{\restriction\{v_0,v_1,v_2\}} = v_1 \longrightarrow v_0 \underline{\quad} v_2$.

We have $\{v_2, v_0\}$ (resp. $\{v_2, v_0\}$) is a neutral edge in G reversed in G', so by Corollary 2.10, there is a subclass $\mathcal{C}_2$ containing v_2 and does not containing v_0, v_1.

Firstly, let $\mathcal{C}_0$ be a subclass such that $v_0 \in \mathcal{C}_0$. We prove that $|\mathcal{C}_0| = 1$. We assume by contradiction that $|\mathcal{C}_0| \geq 2$, let $v \in \mathcal{C}_0$. As $\overline{\mathcal{C}_0}$ is a connected component of $\overline{U}$, there are n vertices $v_0 = x_0, x_1, \cdots, x_{n-1} = v$ of $V(\mathcal{C}_0)$, such that $x_k \ldots_U x_{k+1}$ for all $k \in \{0, 1, \cdots, n-2\}$, so $x_1 \ldots_U v_0$. According to the nature of $\{x_1, v_0\}$, we have the following cases.

• Case 1. $\{x_1, v_0\}$ is oriented in G.

We have $\overline{U}_{\restriction\{v_0,v_2,x_1\}} = \{v_0 \underline{\quad} x_1\} \ldots v_2$ and $\{v_0, x_1\}$ is a an oriented pair in G, so from 2) of Lemma 2.3, $\{v_0, x_1\}$ is an interval of $G_{\restriction\{v_0,v_2,x_1\}}$, thus $x_1 \ldots_G v_2$ so $x_1 \underline{\quad}_{G'} v_2$. We have $x_1 \ldots_G v_2 \ldots_G v_0$ and $x_1 \underline{\quad}_{G'} v_2 \underline{\quad}_{G'} v_0$, which contradicts Lemma 2.4.

• Case 2. $\{x_1, v_0\}$ is a neutral pair in G.

We have $\overline{U}_{\restriction\{v_0,v_2,x_1\}} = \{v_0 \underline{\quad} x_1\} \ldots v_2$ and $\{v_0, x_1\}$ is a neutral pair in G, so from 3) of Lemma 2.3, $x_1 \underline{\quad}_G v_2$, so $x_1 \ldots_{G'} v_2$, thus we have $x_1 \underline{\quad}_G v_2 \underline{\quad}_G v_1$ and $x_1 \ldots_{G'} v_2 \ldots_{G'} v_1$, which contradicts Lemma 2.4.

Thus $|\mathcal{C}_0| = 1$. Therefore $v_1 \notin \mathcal{C}_0$, so there is a subclass $\mathcal{C}_1$ such that $v_1 \in \mathcal{C}_1$.

Secondly, to prove that $|\mathcal{C}_1| = 1$, it suffices to exchange the roles of $G_{\restriction\{v_0,v_2,x_1\}}$ and $G'_{\restriction\{v_0,v_2,x_1\}}$, then we come back to the previous case. □

3. Proof of Theorem 1.2

In this section we will prove Theorem 1.2 which gives the form of the pair of restrictions of G and G' on a connected component of $G \dot{+} G'$. The proof is obtained as follows:

1) is given by Proposition 3.1.
2) is given by Proposition 3.3.
3) is given by Proposition 3.7.

Proposition 3.1. *Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs, (≤ 5)-hypomorphic up to complementation. Let $U := G \dot{+} G'$. We assume that U is not connected. Let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. If $G_{\restriction V(\mathcal{C})}$ embeds a flag, then $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$ are isomorphic, and more precisely, the following assertions hold:*

1) If $|V(\mathcal{C})| = 3$ then $G'_{\restriction V(\mathcal{C})} = \overline{G}_{\restriction V(\mathcal{C})}$.

2) If $|V(\mathcal{C})| \geq 4$ then $|V(\mathcal{C})| = 4$ and the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is one of the following pairs: $\{\alpha_4, \overline{\alpha_4}\}$, $\{\beta_4, \overline{\beta_4}\}$, $\{\gamma_4^+, \gamma_4^-\}$, $\{\lambda_4^+, \lambda_4^-\}$, $\{(\alpha_4)^*, (\overline{\alpha_4})^*\}$, $\{(\beta_4)^*, (\overline{\beta_4})^*\}$, $\{(\gamma_4^+)^*, (\gamma_4^-)^*\}$, $\{(\lambda_4^+)^*, (\lambda_4^-)^*\}$.

Proof. As $G_{\restriction V(\mathcal{C})}$ embeds a flag, we can assume w.l.o.g. that there are $v_0, v_1, v_2 \in V(\mathcal{C})$ such that $v_0 \longrightarrow v_1 \underline{\quad} v_2$ in G. From Remark 2.1, $\mathcal{C} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ are the connected components of $\overline{\mathcal{C}}$ ($k \geq 2$). From Lemma 2.13, there are three distinct subclasses, w.l.o.g. $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$, of $\mathcal{C}$ such that $v_0 \in V(\mathcal{C}_0)$, $v_1 \in V(\mathcal{C}_1)$ and $v_2 \in V(\mathcal{C}_2)$. From Lemma 2.13, $|\mathcal{C}_0| = |\mathcal{C}_1| = 1$.

1) $|V(\mathcal{C})| = 3$. Then $G_{\restriction V(\mathcal{C})} = v_0 \longrightarrow v_1 \underline{\quad} v_2$. From Lemma 2.12, $G'_{\restriction V(\mathcal{C})} = \overline{G}_{\restriction V(\mathcal{C})} = v_1 \longrightarrow v_0 \underline{\quad} v_2$.

2) $|V(\mathcal{C})| \geq 4$.

Fact 1: Given a vertex v_3 in $V(C) \setminus \{v_0, v_1, v_2\}$. If $v_3 \underline{\quad}_U \{v_0, v_1, v_2\}$, then $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is one of the following pairs: $\{\alpha_4, \overline{\alpha_4}\}$, $\{\beta_4, \overline{\beta_4}\}$, $\{(\alpha_4)^*, (\overline{\alpha_4})^*\}$, $\{(\beta_4)^*, (\overline{\beta_4})^*\}$, and we have ($v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$).
Proof. Let $v_3 \in \mathcal{C}_3$. We have $v_3 \underline{\quad}_U \{v_0, v_1, v_2\}$.
If $\{v_2, v_3\}$ is void (resp. full) in G then $\{v_2, v_3\}$ is full (resp. void) in G'. This contradicts Lemma 2.4 applied to $\{v_0, v_2, v_3\}$ (resp. $\{v_1, v_2, v_3\}$). So $\{v_2, v_3\}$ is oriented in G.
• Case 1. $v_3 \longrightarrow_G v_2$. So $v_3 \longleftarrow_{G'} v_2$.
Lemma 2.4 applied to $\{v_0, v_2, v_3\}$ shows that $\{v_0, v_3\}$ is not void in G.
By Lemma 2.6, $G_{\restriction\{v_0,v_2,v_3\}}$ is not a peak, so $v_3 \not\longrightarrow_G v_0$. If $v_0 \longrightarrow_G v_3$ then $G_{\restriction\{v_0,v_2,v_3\}}$ is a 3-consecutivity and $G'_{\restriction\{v_0,v_2,v_3\}} \neq G^*_{\restriction\{v_0,v_2,v_3\}}$, which contradicts Lemma 2.11. So $\{v_0, v_3\}$ is full in G, thus void in G'.
Lemma 2.4 applied to $\{v_1, v_2, v_3\}$ shows that $\{v_1, v_3\}$ is not full in G.
By Lemma 2.6, $G_{\restriction\{v_1,v_2,v_3\}}$ is not a peak, so $v_3 \not\longrightarrow_G v_1$. If $v_1 \longrightarrow_G v_3$ then $G_{\restriction\{v_1,v_2,v_3\}}$ is a 3-consecutivity and $G'_{\restriction\{v_1,v_2,v_3\}} \neq G^*_{\restriction\{v_1,v_2,v_3\}}$, which contradicts Lemma 2.11. So $\{v_1, v_3\}$ is void in G, thus full in G'.
Set $C' := \{v_0, v_1, v_2, v_3\}$. Then $G_{\restriction V(C')} = \alpha_4$, $G'_{\restriction V(C')} = \overline{G}_{\restriction V(C')} = \overline{\alpha_4}$ and $G_{\restriction V(C')} \not\simeq G'_{\restriction V(C')}$. So we have ($v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$).
• Case 2. $v_3 \longleftarrow_G v_2$. So $v_3 \longrightarrow_{G'} v_2$.
Lemma 2.4 applied to $\{v_0, v_2, v_3\}$ shows that $\{v_0, v_3\}$ is not void in G.
By Lemma 2.6, $G_{\restriction\{v_0,v_2,v_3\}}$ is not a peak, so $v_0 \not\longrightarrow_G v_3$. If $v_3 \longrightarrow_G v_0$ then $G_{\restriction\{v_0,v_2,v_3\}}$ is a

3-consecutivity and $G'_{\restriction\{v_0,v_2,v_3\}} \neq G^*_{\restriction\{v_0,v_2,v_3\}}$, which contradicts Lemma 2.11. So $\{v_0, v_3\}$ is full in G, thus void in G'.
Lemma 2.4 applied to $\{v_1, v_2, v_3\}$ shows that $\{v_1, v_3\}$ is not full in G.
By Lemma 2.6, $G_{\restriction\{v_1,v_2,v_3\}}$ is not a peak, so $v_1 \not\longrightarrow_G v_3$. If $v_3 \longrightarrow_G v_1$ then $G_{\restriction\{v_1,v_2,v_3\}}$ is a 3-consecutivity and $G'_{\restriction\{v_1,v_2,v_3\}} \neq G^*_{\restriction\{v_1,v_2,v_3\}}$, which contradicts Lemma 2.11. So $\{v_1, v_3\}$ is void in G, thus full in G'.
Set $C' := \{v_0, v_1, v_2, v_3\}$. Then $G_{\restriction V(C')} = \beta_4$, $G'_{\restriction V(C')} = \overline{G}_{\restriction V(C')} = \overline{\beta_4}$ and $G_{\restriction V(C')} \not\simeq G'_{\restriction V(C')}$. So we have ($v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$). □

Fact 2: Given a vertex v_3 in $V(C) \setminus \{v_0, v_1, v_2\}$. If $v_3 \ldots_U v_2$, then $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is one of the following pairs: $\{\gamma_4^+, \gamma_4^-\}$, $\{\lambda_4^+, \lambda_4^-\}$, $\{(\gamma_4^+)^*, (\gamma_4^-)^*\}$, $\{(\lambda_4^+)^*, (\lambda_4^-)^*\}$, and we have ($v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$).
Proof. We have $v_3 \ldots_U v_2$ and $v_3 \underline{\quad}_U \{v_0, v_1\}$. According to the nature of $\{v_2, v_3\}$, we have the following subcases.
• Case 1. $v_2 \longrightarrow_G v_3$. So $v_2 \longrightarrow_{G'} v_3$.
We have $U_{\restriction\{v_0,v_2,v_3\}} = v_0 \underline{\quad} \{v_2 \ldots v_3\}$ and $\{v_2, v_3\}$ is a an oriented pair in G, so from 2) of Lemma 2.3, $\{v_2, v_3\}$ is an interval of $G_{\restriction\{v_0,v_2,v_3\}}$, thus $v_0 \ldots_G v_3$ so $v_0 \underline{\quad}_{G'} v_3$. We have $v_2 \ldots_G v_0 \ldots_G v_3$ and $v_2 \underline{\quad}_{G'} v_0 \underline{\quad}_{G'} v_3$, which contradicts Lemma 2.4.
• Case 2. $v_2 \longleftarrow_G v_3$. So $v_2 \longleftarrow_{G'} v_3$. Since $U = \overline{G'} \dot{+} \overline{G}$, by exchanging (G, G') by $(\overline{G'}, \overline{G})$, we come back to case 1.
• Case 3. $v_2 \underline{\quad}_G v_3$. So $v_2 \underline{\quad}_{G'} v_3$. The 3-hypomorphy up to complementation applied to $\{v_1, v_2, v_3\}$ (resp. $\{v_0, v_2, v_3\}$), gives $v_1 \ldots_G v_3$, so $v_1 \underline{\quad}_{G'} v_3$ (resp. $v_0 \ldots_{G'} v_3$, so $v_0 \underline{\quad}_G v_3$). Set $C' := \{v_0, v_1, v_2, v_3\}$. We have $G_{\restriction V(C')} = \gamma_4^+$, $G'_{\restriction V(C')} = \gamma_4^-$ and $G'_{\restriction V(C')} \simeq G_{\restriction V(C')}$. So we have ($v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$).
• Case 4. $v_2 \ldots_G v_3$. So $v_2 \ldots_{G'} v_3$.
The 3-hypomorphy up to complementation applied to $\{v_1, v_2, v_3\}$ (resp. $\{v_0, v_2, v_3\}$) gives $v_1 \underline{\quad}_{G'} v_3$, so $v_1 \ldots_G v_3$ (resp. $v_0 \underline{\quad}_G v_3$, so $v_0 \ldots_{G'} v_3$). Set $C' := \{v_0, v_1, v_2, v_3\}$. We have $G_{\restriction V(C')} = \lambda_4^+$, $G'_{\restriction V(C')} = \lambda_4^-$ and $G'_{\restriction V(C')} \simeq G_{\restriction V(C')}$. So we have ($v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$). □

As an immediate consequence of Fact 1 and Fact 2, we have the following fact.

Fact 3: For each v_3 in $V(C) \setminus \{v_0, v_1, v_2\}$, $v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$.

Fact 4: $|V(C)| = 4$.
Proof. To the contrary, we assume that $|V(C)| \geq 5$, so there are $v_3 \neq v_4$ in $V(C) \setminus \{v_0, v_1, v_2\}$. From Fact 3, ($v_0 \underline{\quad}_G v_3$ and $v_0 \ldots_{G'} v_3$), and ($v_0 \underline{\quad}_G v_4$ and $v_0 \ldots_{G'} v_4$) which contradicts Lemma 2.4. □

By Fact 4, $|V(C)| = 4$. Let $v_3 = V(C) \setminus \{v_0, v_1, v_2\}$. Since $|\mathcal{C}_0| = |\mathcal{C}_1| = 1$, we have $v_3 \underline{\quad}_U \{v_0, v_1\}$. We conclude using Fact 1 and Fact 2. □

We consider the following symmetric digraphs introduced in [5].
Let $n \geq 2$. Let X_n be an n-element set, $v_0, \cdots, v_{n-1}$ be an enumeration of X_n, $X_n^0 := \{v_i \in X_n : i \equiv 0 \ (mod\ 2)\}$ and $X_n^1 := X_n \setminus X_n^0$. Set $R_n := [X_n^0]^2 \cup [X_n^1]^2$, $S_n := \{\{v_{2i}, v_{2i+1}\} : 2i + 1 < n\}$, $S'_n := \{\{v_{2i+1}, v_{2i+2}\} : 2i + 2 < n - 1\}$. Let M_n and M'_n be the graphs with vertex set X_n and edge sets $E(M_n) := R_n \cup S_n$ and $E(M'_n) := R_n \cup S'_n$ respectively. Let $M''_n := (X_n, R_n \cup S'_n \cup$

$\{\{v_0, v_{n-1}\}\})$ for n even, $n \geq 4$. Finally, let $D_4 := (X_4, \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_2, v_3\}\})$ and $D'_4 := (X_4, \{\{v_0, v_2\}, \{v_0, v_3\}, \{v_1, v_2\}\})$. For example, $M_2 = v_0 \underline{\quad} v_1$, $M'_2 = v_0 \ldots v_1$. Note that M_2, M'_2, M_3, M'_3, M_4, M''_4, D_4, D'_4 were cited previously after Figure 7 and appear in the main result (Theorem 1.2).

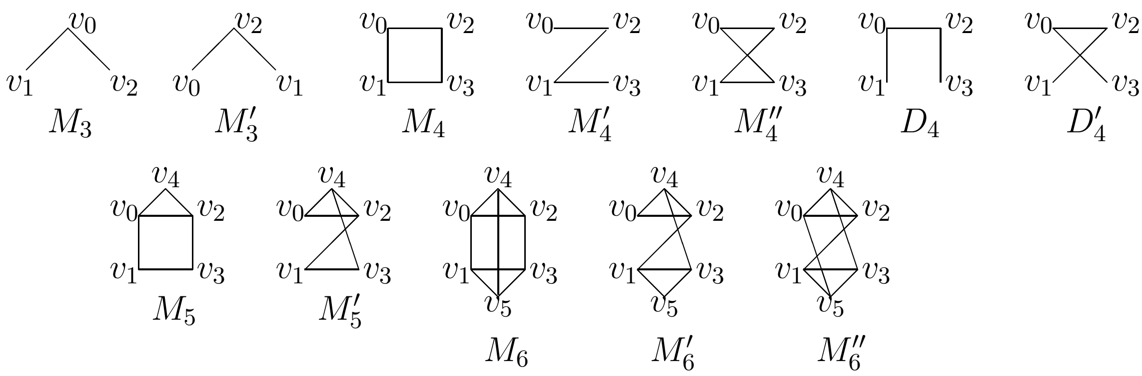

Figure 10. M_n, M'_n, M''_n, D_4, D'_4.

In [5], the following result was established.

Theorem 3.2. *(Theorem 3.15 of [5]) Let G and G' be two 3-hypomorphic up to complementation graphs with vertex set V, $U := G \dot{+} G'$ and U not connected. If $\mathcal{C}$ is a connected component of U of cardinality $n \geq 2$, then the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is one of the following:*

1) $\{M_n, M'_n\}$, $\{\overline{M_n}, \overline{M'_n}\}$, if $\mathcal{C}$ is a path.

2) $\{M_n, M''_n\}$, $\{\overline{M_n}, \overline{M''_n}\}$ $\{D_4, D'_4\}$, $\{\overline{D_4}, \overline{D'_4}\}$, if $\mathcal{C}$ is a cycle.

Proposition 3.3. *Let $G = (V, E)$ and $G' = (V, E')$ be two (≤ 5)-hypomorphic up to complementation digraphs. Let $U := G \dot{+} G'$. We assume that U is not connected. Let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. Let $v_0, v_1 \in V(\mathcal{C})$. If $\{v_0, v_1\}$ is a neutral pair in G reversed in G' and no flag is embeddable in $G_{\restriction V(\mathcal{C})}$, then $|V(\mathcal{C})| \leq 4$, $G_{\restriction V(\mathcal{C})}$ is a symmetric digraph and the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is one of the following:*

1) $\{M_2, M'_2\}$, $\{\overline{M_2}, \overline{M'_2}\}$, if $|V(\mathcal{C})| = 2$. So $G'_{\restriction V(\mathcal{C})} \simeq \overline{G}_{\restriction V(\mathcal{C})}$.

2) $\{M_3, M'_3\}$, $\{\overline{M_3}, \overline{M'_3}\}$, if $|V(\mathcal{C})| = 3$. So $G'_{\restriction V(\mathcal{C})} \simeq G_{\restriction V(\mathcal{C})}$.

3) $\{D_4, D'_4\}$, $\{\overline{D_4}, \overline{D'_4}\}$, $\{M_4, M''_4\}$, $\{\overline{M_4}, \overline{M''_4}\}$ if $|V(\mathcal{C})| = 4$. So $G'_{\restriction V(\mathcal{C})} \simeq G_{\restriction V(\mathcal{C})}$.

Proof. From Remark 2.1, $\mathcal{C} = S(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1})$, where S is a complete graph and $\overline{\mathcal{C}_0}, \overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_{k-1}}$ be the connected components of $\overline{\mathcal{C}}$ $(k \geq 2)$. Since $\{v_0, v_1\}$ is a neutral pair in G reversed in G', then From Corollary 2.10, there are two distinct subclasses $\mathcal{C}_0, \mathcal{C}_1$ of $\mathcal{C}$ such that $v_0 \in V(\mathcal{C}_0)$ and $v_1 \in V(\mathcal{C}_1)$. W.l.o.g. we assume that $v_0 \underline{\quad}_G v_1$, So $v_0 \ldots_{G'} v_1$.

Claim 3.4. *For all $v_2 \in V(\mathcal{C}) \setminus \{v_0, v_1\}$, $\{v_0, v_2\}$ and $\{v_1, v_2\}$ are neutral edges.*

Proof. We assume by contradiction w.l.o.g. that $\{v_0, v_2\}$ is oriented with $v_0 \longrightarrow_G v_2$. By Lemma 2.6, $G_{\restriction\{v_0,v_1,v_2\}}$ is not a peak, so $v_2 \not\longrightarrow_G v_1$. If $v_1 \longrightarrow_G v_2$ then $G_{\restriction\{v_0,v_1,v_2\}}$ is a 3-consecutivity and

$G'_{\restriction\{v_0,v_1,v_2\}} \neq G^*_{\restriction\{v_0,v_1,v_2\}}$, which contradicts Lemma 2.11. So $\{v_1,v_2\}$ is neutral in G and G'.
Since $G_{\restriction\{v_0,v_1,v_2\}}$ (resp. $G'_{\restriction\{v_0,v_1,v_2\}}$) is not a flag then $\{v_1,v_2\}$ is full in G (resp. void in G'). So we have ($v_1 \underline{\quad}_G \{v_0,v_2\}$ and $v_1 \cdots_{G'} \{v_0,v_2\}$), which contradicts Lemma 2.4. □

Claim 3.5. *$G_{\restriction V(\mathcal{C})}$ is a symmetric digraph.*

Proof. We assume by contradiction that there are $v_2 \neq v_3 \in V(\mathcal{C})\setminus\{v_0,v_1\}$ such that $\{v_2,v_3\}$ is oriented. By Claim 3.4, $\{v_0,v_2\}$, $\{v_0,v_3\}$, $\{v_1,v_2\}$, $\{v_1,v_3\}$ are neutral edges in G and G'. Since $G_{\restriction\{v_0,v_2,v_3\}}$ (resp. $G'_{\restriction\{v_0,v_2,v_3\}}$) is not a flag then $\{v_0,v_2\}$ and $\{v_0,v_3\}$ are neutral edges with the same nature in G (resp. in G'), which contradicts Lemma 2.4. □

Claim 3.6. *$|V(\mathcal{C})| \leq 4$ and $\mathcal{C}$ is a path P_2 or a path P_3 or a cycle C_4.*

Proof. By contradiction, if $|V(\mathcal{C})| \geq 5$, from Theorem 3.2, the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is one of the following: $\{M_n, M'_n\}$, $\{\overline{M}_n, \overline{M'}_n\}$, $\{M_n, M''_n\}$, $\{\overline{M}_n, \overline{M''}_n\}$. In all of these cases, $G'_{\restriction\{v_0,v_1,v_2,v_3\}} \not\simeq G_{\restriction\{v_0,v_1,v_2,v_3\}}$ and $G'_{\restriction\{v_0,v_1,v_2,v_3\}} \not\simeq \overline{G}_{\restriction\{v_0,v_1,v_2,v_3\}}$, which contradicts the 4-hypomorphy up to complementation, thus $|V(\mathcal{C})| \leq 4$. □

Now we prove Proposition 3.3.
1) If $|V(\mathcal{C})| = 2$, according to Theorem 3.2, $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\} = \{M_2, M'_2\}$ or $\{\overline{M_2}, \overline{M'}_2\}$. So $G'_{\restriction V(\mathcal{C})} \simeq \overline{G}_{\restriction V(\mathcal{C})}$.
2) If $|V(\mathcal{C})| = 3$, then Claim 3.5 and Theorem 3.2 give $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\} = \{M_3, M'_3\}$ or $\{\overline{M_3}, \overline{M'}_3\}$. Thus $G'_{\restriction V(\mathcal{C})} \simeq G_{\restriction V(\mathcal{C})}$.
3) If $|V(\mathcal{C})| = 4$, from the 4-hypomorphy up to complementation, $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is either $\{D_4, D'_4\}$ or $\{\overline{D_4}, \overline{D'_4}\}$ or $\{M_4, M''_4\}$ or $\{\overline{M_4}, \overline{M''_4}\}$. Thus $G'_{\restriction V(\mathcal{C})} \simeq G_{\restriction V(\mathcal{C})}$. □

Proposition 3.7. *Let $G = (V,E)$ and $G' = (V,E')$ be two digraphs, (≤ 5)-hypomorphic up to complementation. Let $U := G \dot{+} G'$. We assume that U is not connected. Let $\mathcal{C}$ be a connected component of U such that $\overline{\mathcal{C}}$ is not connected. If $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$ are 2-hypomorphic, then $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$ are (≤ 4)-hypomorphic and $\mathcal{C}$ is an interval of G and G'.*

Proof. The fact that $\mathcal{C}$ is an interval of G follows from 1) of Lemma 2.9.

Claim 3.8. *Any two neutral edges in $G_{\restriction V(\mathcal{C})}$ have no common vertex.*

Proof. By contradiction, assume that there are three vertices v_0, v_1, v_2 in $V(\mathcal{C})$ such that $\{v_0,v_1\}$ and $\{v_0,v_2\}$ are neutral edges in $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$. Since $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$ are 2-hypomorphic, then by Lemma 2.4, $\{v_0,v_1\}$ and $\{v_0,v_2\}$ are not of the same nature. From Lemma 2.12, if $G_{\restriction\{v_0,v_1,v_2\}}$ is a flag, then $G'_{\restriction\{v_0,v_1,v_2\}} = \overline{G}_{\restriction\{v_0,v_1,v_2\}}$ which contradicts the fact that no neutral edge in $G_{\restriction V(\mathcal{C})}$ is reversed in $G'_{\restriction V(\mathcal{C})}$. So $G_{\restriction\{v_0,v_1,v_2\}}$ is not a flag. Then $G_{\restriction\{v_0,v_1,v_2\}}$ is a symmetric digraph and $U_{\restriction\{v_0,v_1,v_2\}}$ is an empty graph, which contadicts 1) of Corollary 2.5. □

Claim 3.9. *Let v_0, v_1 and v_2 be distinct vertices in $\mathcal{C}$. If $G_{\restriction\{v_0,v_1,v_2\}}$ is not a tournament then $G_{\restriction\{v_0,v_1,v_2\}}$ is a 3-consecutivity and $G'_{\restriction\{v_0,v_1,v_2\}} = G^*_{\restriction\{v_0,v_1,v_2\}}$.*

Proof. As $G_{\restriction\{v_0,v_1,v_2\}}$ is not a tournament, we can assume that $\{v_0,v_1\}$ is a neutral edge. From Claim 3.8, $\{v_0,v_2\}$ and $\{v_1,v_2\}$ are oriented. From Lemma 2.6, $G_{\restriction\{v_0,v_1,v_2\}}$ is not a peak. Then $G_{\restriction\{v_0,v_1,v_2\}}$ is a 3-consecutivity. From Lemma 2.11, $G'_{\restriction\{v_0,v_1,v_2\}} = G^*_{\restriction\{v_0,v_1,v_2\}}$. □

From Claim 3.9, $G_{\restriction V(\mathcal{C})}$ and $G'_{\restriction V(\mathcal{C})}$ are (≤ 3)-hypomorphic.
Let $v_3 \in V(\mathcal{C})$, we will prove that $G'_{\restriction\{v_0,v_1,v_2,v_3\}} \simeq G_{\restriction\{v_0,v_1,v_2,v_3\}}$.
According to ℓ, the cardinal of the largest tournament in $G_{\restriction\{v_0,v_1,v_2,v_3\}}$, we have the following cases:

• Case 1. $\ell = 4$. From Lemma 2.7, $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ is not a diamond, then $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ is a 4-chain or $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ is obtained by dilating a vertex of the 3-directed cycle by an oriented pair. From Lemma 2.8, every 3-directed cycle in $G_{\restriction V(\mathcal{C})}$ is reversed in $G'_{\restriction V(\mathcal{C})}$. Then $G'_{\restriction\{v_0,v_1,v_2,v_3\}} \simeq G_{\restriction\{v_0,v_1,v_2,v_3\}}$.

• Case 2. $\ell = 3$. W.l.o.g. we can assume $G_{\restriction\{v_0,v_1,v_2\}} = \overrightarrow{C_3}$ and thus $G'_{\restriction\{v_0,v_1,v_2\}} = G^*_{\restriction\{v_0,v_1,v_2\}}$, or $G_{\restriction\{v_0,v_1,v_2\}}$ and $G'_{\restriction\{v_0,v_1,v_2\}}$ are two 3-chains.

Case 2.1. $G_{\restriction\{v_0,v_1,v_2\}} = \overrightarrow{C_3}$ and $G'_{\restriction\{v_0,v_1,v_2\}} = G^*_{\restriction\{v_0,v_1,v_2\}}$.
We have $v_0 \longrightarrow_G v_1 \longrightarrow_G v_2 \longrightarrow_G v_0$. As $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ is not a tournament, then, w.l.o.g. we can assume that $\{v_1, v_3\}$ is a neutral pair in G not reversed in G'.
From Claim 3.9, $G_{\restriction\{v_1,v_2,v_3\}}$ (resp. $G_{\restriction\{v_0,v_1,v_3\}}$) is a 3-consecutivity. So $v_2 \longrightarrow_G v_3$ and $v_2 \longleftarrow_{G'} v_3$ (resp. $v_0 \longleftarrow_G v_3$ and $v_0 \longrightarrow_{G'} v_3$). So, there is an isomorphism σ from $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ onto $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$ defined by $\sigma(v_0) = v_2$, $\sigma(v_2) = v_0$, $\sigma(v_1) = v_1$ and $\sigma(v_3) = v_3$.

Case 2.2. $G_{\restriction\{v_0,v_1,v_2\}}$ and $G'_{\restriction\{v_0,v_1,v_2\}}$ are two 3-chains. W.l.o.g. we assume that $G_{\restriction\{v_0,v_1,v_2\}} = v_0 < v_1 < v_2$. As $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ is not a tournament, we have the following subcases:

Case 2.2.1. $\{v_0, v_3\}$ is a neutral pair in G not reversed in G'.
From Claim 3.9, $G_{\restriction\{v_0,v_1,v_3\}}$ (resp. $G_{\restriction\{v_0,v_2,v_3\}}$) is a 3-consecutivity. So $v_1 \longrightarrow_G v_3$, $v_1 \longleftarrow_{G'} v_3$ and $v_0 \longleftarrow_{G'} v_1$ (resp. $v_2 \longrightarrow_G v_3$, $v_2 \longleftarrow_{G'} v_3$ and $v_0 \longleftarrow_{G'} v_2$).
So, there is an isomorphism σ from $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ into $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$ defined by $\sigma(v_0) = v_3$, $\sigma(v_3) = v_0$, and $\sigma(\{v_1, v_2\}) = \{v_1, v_2\}$.

Case 2.2.2. $\{v_1, v_3\}$ is a neutral pair in G not reversed in G'.
From Claim 3.9, $G_{\restriction\{v_0,v_1,v_3\}}$ (resp. $G_{\restriction\{v_1,v_2,v_3\}}$) is a 3-consecutivity. So $v_3 \longrightarrow_G v_0$, $v_3 \longleftarrow_{G'} v_0$ and $v_0 \longleftarrow_{G'} v_1$ (resp. $v_2 \longrightarrow_G v_3$, $v_2 \longleftarrow_{G'} v_3$ and $v_1 \longleftarrow_{G'} v_2$). The 3-hypomorphy up to complementation applied to $\{v_0, v_1, v_2\}$, gives $v_0 \longleftarrow_{G'} v_2$. So, there is an isomorphism σ from $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ into $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$ defined by $\sigma(v_0) = v_2$, $\sigma(v_2) = v_0$, $\sigma(v_1) = v_1$ and $\sigma(v_3) = v_3$.

Case 2.2.3. $\{v_2, v_3\}$ is a neutral pair in G not reversed in G'. As $U := G\dot{+}G' = \overline{G}\dot{+}\overline{G'}$, exchanging G by $\overline{G}$ and G' by $\overline{G'}$, we come back to Case 2.2.1.

• Case 3. $\ell = 2$. From Claim 3.9, $G_{\restriction\{v_0,v_1,v_2\}}$ is a 3-consecutivity and $G'_{\restriction\{v_0,v_1,v_2\}} = G^*_{\restriction\{v_0,v_1,v_2\}}$. W.l.o.g. we can assume that $v_0 \underline{\quad}_G v_1$ $v_1 \longrightarrow_G v_2$ and $v_2 \longrightarrow_G v_0$. As $\{v_0, v_1\}$ is a neutral pair then, by Claim 3.8, $\{v_0, v_3\}$ is an oriented pair in G. From Claim 3.9, $G_{\restriction\{v_0,v_2,v_3\}}$ is a 3-consecutivity and $G'_{\restriction\{v_0,v_2,v_3\}} = G^*_{\restriction\{v_0,v_2,v_3\}}$. So $v_0 \longrightarrow_G v_3$ and $\{v_2, v_3\}$ is a neutral pair in G. From Claim 3.9, $G_{\restriction\{v_1,v_2,v_3\}}$ is a 3-consecutivity and $G'_{\restriction\{v_1,v_2,v_3\}} = G^*_{\restriction\{v_1,v_2,v_3\}}$. So $v_3 \longrightarrow_G v_1$. Then $G'_{\restriction\{v_0,v_1,v_2,v_3\}} = G^*_{\restriction\{v_0,v_1,v_2,v_3\}}$. So, there is an isomorphism σ from $G_{\restriction\{v_0,v_1,v_2,v_3\}}$ into $G'_{\restriction\{v_0,v_1,v_2,v_3\}}$ defined by $\sigma(v_0) = v_1$, $\sigma(v_1) = v_0$, $\sigma(v_2) = v_2$ and $\sigma(v_3) = v_3$.
Now, the form of the pair $\{G_{\restriction V(\mathcal{C})}, G'_{\restriction V(\mathcal{C})}\}$ is given by the theorem of G.Lopez and C.Rauzy (Theorem 3.11 below) and, by the same theorem, $G'_{\restriction V(\mathcal{C})} \simeq G^*{}_{\restriction V(\mathcal{C})}$. □

3.1. *Theorem of G.Lopez and C.Rauzy*

The tournament T_h is defined on $2h+1$ vertices $0, 1, \cdots, 2h$ such that for each i, $(i, i+k)$ is an edge for $k \leq h$ ($i+k$ is considered modulo $2h+1$). A tournament R is a *dilatation of* T_h (denoted $R \in \mathcal{D}(T_h)$) if R is obtained from T_h by replacing, for all $k \leq 2h$, the vertex k by a chain p_k of finite cardinality with the following condition: for every x in p_k and for every y in p_j with $j \neq k$, $R(x, y) = T_h(k, j)$.

Lemma 3.10. *(Lemma 3a.2 [8]) If a tournament R is without diamond then it is a dilatation of some T_h by finite chains, that is $R = T_h(P_0, P_1, \ldots, P_{h-1})$ with P_i is a chain for all $i \in \{0, 1, \ldots, h-1\}$.*

Let $\mathcal{E}$ be the family of digraphs which are not tournaments, and embeds neither peaks, nor diamonds, nor adjacent neutral pairs. Then it follows the following usefull remark.

The morphology of the family $\mathcal{E}$ is described by G. Lopez and C. Rauzy [8] as follows. They begin by the description of the family S_n for each integer $n \geq 1$. An element of S_1 is a digraph on 2 vertices with a neutral pair. Let $\mathbb{Z}/2n\mathbb{Z}$ be the set $\{1, 2, \cdots, 2n\}$ modulo $2n$. For $n \geq 2$, a digraph is *an element of the family* S_n if there is a one-to-one enumeration of the vertices $(t_k : k \in \mathbb{Z}/2n\mathbb{Z})$, so that $\{t_i, t_j\}$ is a neutral pair if and only if $j = i+n$, and $t_i \longrightarrow t_j$ if there is $k \in \{1, \ldots, n-1\}$ such that $j = i+k$. The particular family $\mathcal{E}(S_n)$ of extensions of the digraphs family S_n is defined as follows: Let $\{t_k : k \in \mathbb{Z}/2n\mathbb{Z}\}$ be a set. With every $k \in \mathbb{Z}/2n\mathbb{Z}$ is associated a set s_k disjoint from $\{t_k : k \in \mathbb{Z}/2n\mathbb{Z}\}$ and may be empty, such that the s_k's are mutually disjoint.
Then $\mathcal{E} = \cup_{n\geq 1}\mathcal{E}(S_n)$ where $\mathcal{E}(S_n)$ is defined as follows. For $n \geq 1$, an element of $\mathcal{E}(S_n)$ is a digraph γ_n defined on $\{t_k : k \in \mathbb{Z}/2n\mathbb{Z}\} \cup \left(\bigcup_{k\in\mathbb{Z}/2n\mathbb{Z}} s_k\right)$ provided that:

(i) γ_n does not embed diamonds.

(ii) The subdigraph $\gamma_{n\restriction\{t_k:k\in\mathbb{Z}/2n\mathbb{Z}\}}$ is an element of the family S_n.

(iii) For every $i \in \mathbb{Z}/2n\mathbb{Z}$, the subdigraph $\gamma_{n\restriction s_i}$ is a chain and $t_i \longrightarrow_{\gamma_n} s_i \longrightarrow_{\gamma_n} t_{i+1}$. The set s_i is called *a sector* of γ_n.

(iv) For every $i \in \mathbb{Z}/2n\mathbb{Z}$, $\gamma_{n\restriction s_i\cup s_{i+n}}$ is a tournament. The set $s_i \cup s_{i+n}$ is called *a bisector* of γ_n.

(v) For every $i \in \mathbb{Z}/2n\mathbb{Z}$, $s_i \longrightarrow_{\gamma_n} t_{i+n}$ and

$$\left(\bigcup_{\substack{j=i+k+n\\k\in\{1,\ldots,n-1\}}} \{t_j\} \cup s_j\right) \longrightarrow_{\gamma_n} s_i \longrightarrow_{\gamma_n} \left(\bigcup_{\substack{j=i+k\\k\in\{1,\ldots,n-1\}}} \{t_j\} \cup s_j\right).$$

Theorem 3.11. *(G.Lopez, C.Rauzy [8]) Let $G = (V, E)$ and $G' = (V, E')$ be two digraphs, (≤ 4)-hypomorphic, and $\mathcal{C} \in D_{G,G'}$.*

1) If $G_{\restriction\mathcal{C}}$ is a tournament, then $G_{\restriction\mathcal{C}}$ is a diamond-free tournament and $G'_{\restriction V(\mathcal{C})} \simeq G^*{}_{\restriction V(\mathcal{C})}$.

2) If $G_{\restriction\mathcal{C}}$ has no 3-directed cycle, then $G_{\restriction\mathcal{C}}$ is either a chain or a near-chain or a $\overrightarrow{P_n}$ or a $\overrightarrow{P_n^f}$ or a $\overrightarrow{C_n}$ or a $\overrightarrow{C_n^f}$, and $G'_{\restriction V(\mathcal{C})} \simeq G_{\restriction V(\mathcal{C})} \simeq G^*{}_{\restriction V(\mathcal{C})}$.

3) If $G_{\restriction\mathcal{C}}$ has a 3-directed cycle and $G_{\restriction\mathcal{C}}$ is not a tournament, then there is an integer $n \geq 1$ such that $G_{\restriction\mathcal{C}} \in \mathcal{E}(S_n)$, and $G'_{\restriction V(\mathcal{C})} \simeq G^*{}_{\restriction V(\mathcal{C})}$.

4) Let $v_0, v_1, v_2 \in V(\mathcal{C})$. If $G_{\restriction\{v_0,v_1,v_2\}}$ is 3-consecutivity (resp. 3-cycle), then $G'_{\restriction\{v_0,v_1,v_2\}} = G^*_{\restriction\{v_0,v_1,v_2\}}$.

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References

[1] A. Ben Amira, B. Chaari, J. Dammak and H. Si Kaddour, *The* (≤ 5)*-hypomorphy of digraphs up to complementation*, Arab Journal of Mathematical Sciences **25** (2019), no. 1, 1–16.

[2] J. A. Bondy, A graph reconstructor's manual. *Surveys in combinatorics*, 1991 (Guildford, 1991), 221–252, London Math. Soc. Lecture Note Ser., 166, Cambridge Univ. Press, Cambridge, 1991.

[3] M. Bouaziz, Y. Boudabbous and N. El Amri, *Hereditary hemimorphy of* $\{-k\}$*-hemimorphic tournaments for* $k \geq 5$. J. Korean Math. Soc. **48** (2011), no. 3, 599–626.

[4] J. Dammak, G. Lopez, M. Pouzet and H. Si Kaddour, *Hypomorphy of graphs up to complementation*. JCTB, Series B **99** (2009), no. 1, 84–96.

[5] J. Dammak, G. Lopez, M. Pouzet and H. Si Kaddour, *Boolean sum of graphs and reconstruction up to complementation*. Advances in Pure and Applied Mathematics **4** (2013), 315–349.

[6] G. Lopez, *Deux résultats concernant la détermination d'une relation par les types d'isomorphie de ses restrictions*, C. R. Acad. Sci. Paris, t.**274**, Série A, (1972), 1525–1528.

[7] G. Lopez, *Sur la détermination d'une relation par les types d'isomorphie de ses restrictions*, C. R. Acad. Sci. Paris, t.**275**, Série A (1972), 951–953.

[8] G. Lopez and C. Rauzy, *Reconstruction of binary relations from their restrictions of cardinality* $2, 3, 4$ *and* $(n-1)$. I, Z. Math. Logik Grundlag. Math., **38**(1)(1992), 27–37.

[9] K. B. Reid and C. Thomassen, *Strongly self-complementary and hereditarily isomorphic tournaments*. Monatsh. Math. **81** (1976), no. 4, 291–304.

