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A note on the Branson-Paneitz curvature problem

Une note sur le problème de courbure de Branson-Paneitz

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ABSTRACT. In this note we revise the perturbation result of [7] on the prescribed Branson-Paneitz curvature problem on the n-dimensional unit sphere, $n \geq 6$. We remove condition $(\mathbf{A_1})$ of ([7], Theorem 1.3) and we prove an entirely new perturbation theorem.
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1. Introduction

Let S^n, $n \geq 5$, be the unit sphere of $\mathbb{R}^{n+1}$ and let $g_0 = \sum_{i=1}^{n+1} dx_i^2$ its standard metric. Given a function $K : S^n \to \mathbb{R}$, the Branson-Paneitz curvature problem on S^n consists in finding a new metric g conformal to g_0 such that the corresponding Branson-Paneitz curvature Q_g equals K. Here Q_g is defined by

$$Q_g = -\frac{1}{2(n-1)}\Delta_g R_g^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g - \frac{2}{(n-2)^2}|Ric_g|^2,$$

where R_g and Ric_g denote the scalar curvature and the Ricci curvature of (S^n, g) respectively, see [9, 17, 22]. If we write $g = u^{\frac{4}{n-4}}g_0$, where $u : S^n \to \mathbb{R}$ is a positive smooth function, then the above problem is equivalent to the solvability of the critical semilinear equation

$$\begin{cases} P_{g_0}u = \Delta_{g_0}^2 u - a_n\Delta_{g_0}u + b_n u = \frac{n-4}{2}Ku^{\frac{n+4}{n-4}}, \\ \\ u > 0, \text{ on } S^n, \end{cases} \tag{1.1}$$

where $a_n = \frac{1}{2}(n^2-2n-4)$ and $b_n = \frac{n-4}{16}n(n^2-4)$.

Equation (1.1) is called critical in the sense that the associated variational functional fails to satisfy the Palais-Smale condition. This is a consequence of the compactness defect of the Sobolev embedding $H_2^2(S^n) \hookrightarrow L^{\frac{2n}{n-4}}(S^n)$. Moreover besides the necessary condition that $\max_{S^n} K(x)$ be positive, there are obstructions of Kazdan-Warner type found in [13].

Considerable studies have been devoted to the problem of existence of solutions of (1.1) and related problems. In [10, 14, 21] the authors studied the Yamabe-type problem; that is when $K \equiv 1$ (see also the recent work of Gursky and Malchiodi [19] which treat the case of closed semi-positive riemannian manifolds). For non-constant function K, various important studies on the problem trying to understand under what conditions on K, (1.1) is solvable. See [1, 3, 7, 11, 12, 15, 16, 18] and references therein.

In [7] Bensouf and Chtioui studied problem (1.1) in the perturbative setting, that is when K is close to a positive constant function. They proved the following result.

(nd) : Assume that K is a Morse function on S^n such that

$$|\Delta_{g_0} K(x)| + |\nabla_{g_0} K(x)| \neq 0, \forall x \in S^n.$$

Denote,

$$\mathcal{K} = \{x \in S^n, \nabla_{g_0} K(x) = 0\} \text{ and } \mathcal{K}^+ = \{x \in \mathcal{K}, -\Delta_{g_0} K(x) > 0\}.$$

For any $x \in \mathcal{K}$, we denote $ind(K, x)$ the Morse index of K at x.

Let k be a non-negative integer. We say that $k \in (A_1)$ if for any $y \in \mathcal{K}^+$, $n - ind(K, x) \neq k + 1$.

Theorem 1.1. *[7] Let K be a given function on S^n, $n \geq 6$ satisfying (nd)-condition.*

If

$$\max_{k \in (\mathbf{A_1})} \Big| 1 - \sum_{\substack{y \in \mathcal{K}^+, \\ n - ind(K,y) \leq k}} (-1)^{n - ind(K,y))} \Big| \neq 0, \tag{1.2}$$

then (1.1) *has a solution provided K is close to 1.*

Our objective in this note is to improve the above result by removing condition $(\mathbf{A_1})$ on the integer k in the counting index formula (1.2). This leads to an interesting result of new type. More precisely we shall prove the following perturbation result. Let y_0 be an absolute maximum of K on S^n. Evidently $y_0 \in \mathcal{K}^+$ under (nd)-condition.

Theorem 1.2. *Let K be a given function on S^n, $n \geq 7$ satisfying (nd)-condition. If*

$$\mathcal{K}^+ \setminus \{y_0\} \neq \emptyset,$$

then (1.1) *has a solution provided K is close to 1.*

The above non degeneracy condition (nd) can be relaxed as follows. Assume that $(f)_\beta$: K is a C^1 function on S^n such that around any $y \in \mathcal{K}$ there exists a real $\beta(y) = \beta$ such that in the geodesic normal coordinates system centered at y the following expansion holds,

$$K(x) = K(y) + \sum_{k=1}^{n} b_k \big|(x-y)_k\big|^\beta + o\big(\|x - y\|^\beta\big).$$

Here $b_k(y) = b_k \in \mathbb{R} \setminus \{0\}$, $\forall k = 1, ..., n$, $\sum_{k=1}^{n} b_k(y) \neq 0$ and

$$\frac{1}{\beta^*(y)} + \frac{1}{\beta^*(y')} > \frac{2}{n-4},$$

$\forall y \neq y' \in \mathcal{K}$, where $\beta^*(z) = \min(\beta(z), n)$, $\forall z \in \mathcal{K}$.

Denoting,

$$\widetilde{\mathcal{K}^+} = \Big\{y \in \mathcal{K}, -\sum_{k=1}^{n} b_k > 0\Big\}.$$

We then have

Theorem 1.3. *Let K be a given function on S^n, $n \geq 6$, satisfying $(f)_\beta$-condition. There exists a positive constant δ_0 (which depends only on K on $S^n \setminus \bigcup_{y \in \mathcal{K}} B(y, \rho_0)$, $\rho_0 > 0$ and small) such that if $1 < \beta < n + \delta_0$ and*

$$\widetilde{\mathcal{K}^+} \setminus \{y_0\} \neq \emptyset,$$

then equation (1.1) *has a solution provided that K is close to 1.*

It is easy to see that for $n \geq 7$, (nd)-condition implies $(f)_\beta$-condition, with $\beta = \beta(y) = 2$ for any $y \in \mathcal{K}$ and $\widetilde{\mathcal{K}^+} = \mathcal{K}^+$.

Our argument to prove Theorem 1.2 and Theorem 1.3 follows that of [4], [8] and [20] where the scalar curvature problem on closed riemannian manifolds was studied using algebraic topological tools.

In the next section we state some preliminaries related to the variational structure of (1.1). In section 3, we study the description of the critical points at infinity of the problem under $(f)_\beta$-condition, $1 < \beta < n + \delta_0$. In section 4, we prove Theorems 1.2 and 1.3.

2. Preliminaries

Problem (1.1) has a variational structure. The associated variational function is

$$J(u) = \frac{\int_{S^n} P_{g_0} u \, u \, dv g_0}{\left(\int_{S^n} K u^{\frac{2n}{n-4}} dv g_0 \right)^{\frac{n-4}{n}}}, \; u \in H_2^2(S^n) \setminus \{0\}.$$

Let

$$\Sigma = \left\{ u \in H_2^2(S^n), \; \|u\|^2 = \int_{S^n} P_{g_0} u \, u \, dv g_0 = 1 \right\} \text{ and } \Sigma^+ = \left\{ u \in \Sigma, \; u \geq 0 \right\}.$$

Using the above natation, problem (1.1) is equivalent to finding the critical points of J in Σ^+. Due to the compactness defect of the embedding $H_2^2(S^n) \hookrightarrow L^{\frac{2n}{n-4}}(S^n)$, J fails to satisfy the Palais-Smale condition. This leads to the failure of the standard critical point theory. The sequences failing the Palais-Smale condition are characterized as follows.

Let $a \in S^n$ and $\lambda > 0$. We define

$$\delta_{(a \; \lambda)}(x) = c_n \frac{\lambda^{\frac{n-4}{2}}}{\left(1 + \frac{\lambda^2 - 1}{2}\big(1 - \cos d(a, x)\big)\right)^{\frac{n-4}{2}}},$$

where c_n is a fixed positive constant such that $\delta_{(a \; \lambda)}$ satisfies

$$P_{g_0} u = u^{\frac{n+4}{n-4}}, \; u > 0 \text{ on } S^n.$$

For $p \in \mathbb{N}$, and $\varepsilon > 0$, we define

$$V(p, \varepsilon) = \begin{cases} u \in \Sigma, \; \exists \, a_1, ..., a_p \in S^n, \; \lambda_1, ..., \lambda_p > \varepsilon^{-1}, \text{ and } \alpha_1, ..., \alpha_p > 0, \text{ s. t} \\ \| \; u - \sum_{i=1}^p \alpha_i \delta_{(a_i, \, \lambda_i)} \; \| < \varepsilon, \left| J(u)^{\frac{n}{n-4}} \alpha_i^{\frac{4}{n-4}} K(a_i) - 1 \right| < \varepsilon, \\ \forall 1 \leq i \leq p \text{and } \varepsilon_{ij} < \varepsilon \, \forall 1 \leq i \neq j \leq p. \end{cases}$$

Here $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \frac{\lambda_i\lambda_j}{2}(1 - \cos \mathrm{d}(a_i, a_j))\right)^{-\frac{n-4}{2}}$.

Following [5] we have,

Proposition 2.1. *Assume that J has no critical point in Σ^+. Let $(u_n)_n$ be a sequence in Σ^+ such that $J(u_n)$ is bounded and $\partial J(u_n) \to 0$. Then there exist a positive integer p, a positive sequence (ε_k), tending to zero and an extracted subsequence of (u_k) denoted again (u_k) such that $u_k \in V(p, \varepsilon_k)$, $\forall k$.*

For $u \in V(p, \varepsilon)$, we introduce the minimization problem

$$\min_{\alpha_i>0,\, \lambda_i>\varepsilon^{-1},\, a_i\in S^n} \left\| u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \right\|. \tag{2.1}$$

Following [6], we have

Proposition 2.2. *Let $p \geq 1$ and $\varepsilon > 0$ small enough. For any $u \in V(p, \varepsilon)$, problem* (2.1) *has a unique solution $(\bar{\alpha}, \bar{\lambda}, \bar{a})$ up to a permutation.*

Denoting $v = u - \sum_{i=1}^{p} \bar{\alpha}_i \delta_{(\bar{a}_i, \bar{\lambda}_i)}$. Then v satisfies the following orthogonality condition:

$$(V_0):\ \Big\langle v, \varphi \Big\rangle = 0,\ \forall\, \varphi \in \left\{ \delta_{(\bar{a}_i,\bar{\lambda}_i)}, \frac{\partial \delta_{(\bar{a}_i,\bar{\lambda}_i)}}{\partial \bar{\lambda}_i}, \frac{\partial \delta_{(\bar{a}_i,\bar{\lambda}_i)}}{\partial \bar{a}_i}, i = 1, ..., p \right\}.$$

Here $\langle ., . \rangle$ denotes the inner product in $H_2^2(S^n)$ associated to the norm $\|.\|$.

The following Morse Lemma gets rid of the v-contribution. It is proved in ([12], Lemma 3.1).

Proposition 2.3. *There exists a C^1-map which to any (α, a, λ) such that $\sum_{i=1}^{p} \alpha_i\, \delta_{(a_i,\lambda_i)}$ belongs to $V(p, \varepsilon)$ associates $\overline{v} = \overline{v}(\alpha, a, \lambda) \in H_2^2(S^n)$, $\|\overline{v}\| < \varepsilon$ and $\overline{v}$ is the unique solution of the problem*

$$\min_{v\, satisfies\, (V_0)} J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v \right).$$

Moreover,

$$\begin{aligned} \|\overline{v}\| \ &\leq\ c \sum_{i=1}^{p} \left[\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{\frac{n+4}{2n}}}{\lambda_i^{\frac{n+4}{2}}} \right] \\ &\quad +\ c \begin{cases} \displaystyle\sum_{k\neq r} \varepsilon_{k\,r}^{\frac{n+4}{2(n-4)}} \Big(\log \varepsilon_{kr}^{-1} \Big)^{\frac{n+4}{2n}}, & n \geq 12 \\ \displaystyle\sum_{k\neq r} \varepsilon_{k\,r} \Big(\log \varepsilon_{kr}^{-1} \Big)^{\frac{n-4}{n}}, & n < 12. \end{cases} \end{aligned}$$

We now state the definition of a critical point at infinity.

Definition 2.1. *[5] A critical point at infinity of J is a limit of a non-precompact flow line $u(s)$ of the gradient $(-\partial J)$. According to Propositions 2.1 and 2.2, $u(s)$ have to be of the form*

$$u(s) = \sum_{i=1}^{p} \alpha_i(s)\, \delta_{(a_i(s), \lambda_i(s))} + v(s), \quad \forall s >> 1.$$

Denoting $\overline{\alpha}_i := \lim_{s\longrightarrow+\infty} \alpha_i(s)$ *and* $\overline{y}_i := \lim_{s\longrightarrow+\infty} a_i(s)$, *we denote by*

$$\sum_{i=1}^{p} \overline{\alpha}_i \delta_{(\overline{y}_i,\, \infty)} \text{ or } (\overline{y}_1, ..., \overline{y}_p)_\infty$$

such a critical point at infinity.

3. Critical points at infinity

In this section we characterise the critical point at infinity of J under $(f)_\beta$-condition, $1 < \beta < n + \delta_0$, where δ_0 is a fixed constant given in Corollary 3.1. Such a characterization is obtained through the construction of a specific pseudo gradient W in $V(p, \varepsilon)$, $p \geq 1$, satisfying the Palais-Smale condition out side the critical points at infinity. More precisely, we prove the following result.

Theorem 3.1. *Let K be a positive function satisfying $(f)_\beta$-condition. There exists $\delta_0 > 0$ such that if $\beta \in (1, n + \delta_0)$, then the critical points at infinity of J in $V(p, \varepsilon)$, $p \geq 1$, are*

$$(y_1, ..., y_p)_\infty = \sum_{i=1}^{p} \frac{1}{K(y_i)^{\frac{n-4}{2}}} \delta_{(y_i,\, \infty)},$$

where $y_1, ..., y_p \in \widetilde{\mathcal{K}^+}$ with $y_i \neq y_j$, $\forall 1 \leq i \neq j \leq p$.

The index of J at $(y_1, ..., y_p)_\infty$ equals to

$$i(y_1, ..., y_p)_\infty = p - 1 + \sum_{i=1}^{p} n - \widetilde{i}(y_j),$$

where $\widetilde{i}(y_j) = \sharp\Big\{b_k(y_j),\ k = 1, ..., n, \text{ such that } b_k(y_j) < 0\Big\}$.

To prove Theorem 3.1, we need first to expand the gradient of J in $V(p, \varepsilon)$, $p \geq 1$. This allows us to understand the variation of J and hint the construction of the required vector field W.

Lemma 3.1. *Assume that K is a positive function satisfying $(f)_\beta$-condition with, $\beta \in (1, \infty)$. Let $u = \sum_{i=1}^{p} \alpha_i\, \delta_{(a_i,\, \lambda_i)} \in V(p, \varepsilon)$. For any i, $1 \leq i \leq p$ such that $a_i \in B(y_i, \rho_0)$, $y_i \in \mathcal{K}$ and ρ_0 is a small positive constant chosen so that expansion $(f)_\beta$ holds in $B(y_i, 2\rho_0)$, we have*

$$\Big\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \Big\rangle \;=\; J(u)\frac{\alpha_i^2(n-4)}{2K(a_i)} c(y_i) \sum_{k=1}^{n} b_k(y_i) \begin{cases} \dfrac{1}{\lambda_i^{\beta(y_i)}}, & \text{if } \beta(y_i) < n \\ \dfrac{log\lambda_i}{\lambda_i^{\beta(y_i)}}, & \text{if } \beta(y_i) = n \\ \dfrac{1}{\lambda_i^{n}}, & \text{if } \beta(y_i) > n \end{cases}$$

$$-\; c_1 J(u) \sum_{j\neq i} \alpha_i \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(\frac{1}{\lambda_i^n}\Big) + O\Big(\mid a_i - y_i \mid^{\beta(y_i)}\Big)$$

$$
\begin{aligned}
&+ \sum_{j\neq i} o\Big(\varepsilon_{ij}\Big) + \Big[o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big),\ \textit{if}\ \beta(y_i)\neq n\Big] \\
&+ \Big[o\Big(\frac{log\lambda_i}{\lambda_i^{\beta(y_i)}}\Big),\ \textit{if}\ \beta(y_i) = n\Big].
\end{aligned}
$$

Here,

$$
c(y_i) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \frac{|z_1|^{\beta(y_i)}(|x|^2-1)}{(1+|x|^2)^{n+1}}dx, \textit{if}\ \beta(y_i) < n \\ 1, \qquad\qquad\qquad\qquad \textit{if}\ \beta(y_i) = n \\ \dfrac{\rho_0^{\beta(y_i)-n} w_{n-1}}{\beta(y_i)-n}, \qquad \textit{if}\ \beta(y_i) > n \end{cases} \textit{, and}\ c_1 = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+4}{2}}}.
$$

Proof. We apply estimates (3.3), (3.4) and (3.5) of [12]. We have

$$
\begin{aligned}
\Big\langle \partial J(u), \alpha_i\lambda_i \frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}\Big\rangle &= 2J(u)\Big[-c_1\sum_{j\neq i}\alpha_i\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \\
&- \alpha_i^{\frac{2n}{n-4}}J(u)^{\frac{n}{n-4}}\int_{S^n} K(x)\delta_{(a_i,\lambda_i)}^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}dv_{g_0}\Big].
\end{aligned} \tag{3.1}
$$

In order to expand the above integral $I := \displaystyle\int_{S^n} K(x)\delta_{(a_i,\lambda_i)}^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}dv_{g_0}$, we perform a stereographic projection with respect to a point q of S^n. For simplicity, we identify a point $x\in S^n$ with its projection in $\mathbb{R}^n$ and we identify the function K and its composition with the stereographic projection.

By an elementary composition, we have

$$
\delta_{(a_i,\lambda_i)}^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i} = \frac{n-4}{2}\lambda_i^n\frac{1-\lambda_i^2|x-a_i|^2}{\big(1+\lambda_i^2|x-a_i|^2\big)^{n+1}}.
$$

Therefore,

$$
I = \int_{\mathbb{R}^n}\big(K(x)-K(y_i)\big)\delta_{(a_i,\lambda_i)}^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}dx,
$$

since, $\displaystyle\int_{\mathbb{R}^n}\delta_{(a_i,\lambda_i)}^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}dx = 0.$

Outside some neighborhood of a_i, we have

$$
\begin{aligned}
I_1 &:= \int_{B(a_i,\rho_0)^c}\big(K(x)-K(y_i)\big)\delta_{(a_i,\lambda_i)}^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}dx \\
&\leq \frac{n-4}{2}\sup_{B(a_i,\rho_0)^c}\big|K(x)-K(y_i)\big|\int_{B(a_i,\rho_0)^c}\frac{\big|1-\lambda_i^2|x-a_i|^2\big|}{\big(1+\lambda_i^2|x-a_i|^2\big)^{n+1}}\lambda_i^n dx, \\
&\leq (n-4)\sup_{S^n}K(x)\,w_{n-1}\int_{\lambda_i\rho_0}^{\infty}\frac{|1-r^2|}{(1+r^2)^{n+1}}r^{n-1}dr,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad (n-4)\sup_{S^n} K(x)\, w_{n-1}\frac{\rho_0^{-n}}{\lambda_i^n},\\
&\leq \quad O\Big(\frac{1}{\lambda_i^n}\Big). \qquad (3.2)
\end{aligned}
$$

In side $B(a_i,\rho_0)$, we have

$$
\begin{aligned}
I_2 \quad &:= \int\limits_{B(a_i,\rho_0)} \big(K(x)-K(y_i)\big)\delta_{(a_i,\lambda_i)}^{\frac{n+4}{n-4}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}dx\\
&= \frac{n-4}{2}\sum_{k=1}^n b_k \int\limits_{B(a_i,\rho_0)} \big|(x-y_i)_k\big|^{\beta(y_i)}\frac{1-\lambda_i^2|x-a_i|^2}{\big(1+\lambda_i^2|x-a_i|^2\big)^{n+1}}\lambda_i^n dx\\
&+ \quad o\Big(\int\limits_{B(a_i,\rho_0)} \big|(x-y_i)\big|^{\beta(y_i)}\frac{\big|1-\lambda_i^2|x-a_i|^2\big|}{\big(1+\lambda_i^2|x-a_i|^2\big)^{n+1}}\lambda_i^n dx\Big).
\end{aligned}
$$

Setting $z=\lambda_i(x-a_i)$, we obtain that

$$
\begin{aligned}
I_2 \quad &= \frac{n-4}{2}\sum_{k=1}^n b_k \int\limits_{B(0,\lambda_i\rho_0)} \frac{\big|(z+\lambda_i(a_i-y_i)_k\big|^{\beta(y_i)}}{\lambda_i^{\beta(y_i)}}\frac{1-|z|^2}{\big(1+|z|^2\big)^{n+1}}dz\\
&+ \quad o\Big(\int\limits_{B(0,\lambda_i\rho_0)} \frac{\big|(z+\lambda_i(a_i-y_i)_k\big|^{\beta(y_i)}}{\lambda_i^{\beta(y_i)}}\frac{|1-|z|^2|}{\big(1+|z|^2\big)^{n+1}}dz\Big).
\end{aligned}
$$

Observe that,

$$
\begin{aligned}
&\int\limits_{B(0,\lambda_i\rho_0)} \frac{\big|z+\lambda_i(a_i-y_i)_k\big|^{\beta(y_i)}}{\lambda_i^{\beta(y_i)}}\frac{1-|z|^2}{\big(1+|z|^2\big)^{n+1}}dz\\
&= \frac{1}{\lambda_i^{\beta(y_i)}}\int\limits_{B(0,\lambda_i\rho_0)} \frac{\big|z_k\big|^{\beta(y_i)}\Big(1-|z|^2\Big)}{\big(1+|z|^2\big)^{n+1}}dz\\
&+ \quad O\Big(\frac{\big|a_i-y_i\big|}{\lambda_i^{\beta(y_i)-1}}\int\limits_{B(0,\lambda_i\rho_0)} \frac{|z|^{\beta(y_i)-1}|1-|z|^2|}{\big(1+|z|^2\big)^{n+1}}dz\Big)+O\Big(\big|a_i-y_i\big|^{\beta(y_i)}\Big)\\
&= -\frac{1}{\lambda_i^{\beta(y_i)}}\begin{cases} c(1+o(1)), & \text{if } \beta(y_i)<n\\ log\lambda_i(1+o(1)), & \text{if } \beta(y_i)=n\\ \rho_0^{\beta(y_i)-n}\dfrac{w_{n-1}}{\beta(y_i)-n}\dfrac{1}{\lambda_i^{n-\beta(y_i)}}(1+o(1)), & \text{if } \beta(y_i)>n \end{cases}\\
&+ \quad O\Big(|a_i-y_i|^{\beta(y_i)}\Big),
\end{aligned}
$$

where $c = \displaystyle\int_{\mathbb{R}^n} \frac{\left|z_k\right|^{\beta(y_i)}(|z|^2-1)}{\left(1+|z|^2\right)^{n+1}} dz$. Therefore,

$$
\begin{aligned}
I_2 &= -\frac{n-4}{2} c(y_i) \sum_{k=1}^{n} b_k \begin{cases} \dfrac{1}{\lambda_i^{\beta(y_i)}}, & \text{if } \beta(y_i) < n \\ \dfrac{\log \lambda_i}{\lambda_i^{\beta(y_i)}}, & \text{if } \beta(y_i) = n \\ \dfrac{1}{\lambda_i^{n}}, & \text{if } \beta(y_i) > n \end{cases} \\
&+ O\Big(|a_i - y_i|^{\beta(y_i)}\Big) + \left[o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big), \text{ if } \beta(y_i) \neq n\right] \\
&+ \left[o\Big(\frac{\log \lambda_i}{\lambda_i^{\beta(y_i)}}\Big), \text{ if } \beta(y_i) = n\right].
\end{aligned} \tag{3.3}
$$

The proof of Lemma 3.1 follows from (3.1), (3.2), (3.3) and the fait that $\alpha_i^{\frac{8}{n-4}} K(a_i) J(u)^{\frac{n}{n-4}} = 1 + o(1)$. □

Observe that the sign of the leader term in the expansion of lemma 3.1 is unknown in the case of $\beta(y_i) > n$. It is of the form $\dfrac{c(y_i)}{\lambda_i^n} + O\left(\frac{1}{\lambda_i^n}\right)$, where $O\left(\frac{1}{\lambda_i^n}\right)$ is independent of the flatness order $\beta(y_i)$. Nevertheless, if we restrict our attention to the case of $\beta(y_i) > n$ and $\beta(y_i)$ is close to n, by the expression of $c(y_i)$ we can get a sign of this leader term and therefore we can prove Theorem 3.1. Other ways, the analysis of the problem will be blurry and the characterization of the critical points at infinity remains open.

Corollary 3.1. *Assume that K is a positive function satisfying $(f)_\beta$-condition. There exists $\delta_0 > 0$ which depend only on the function K on $S^n \setminus \bigcup_{y \in \mathcal{K}} B(y, \rho_0)$, such that if $1 < \beta < n + \delta_0$, then for any $u = \sum_{i=1}^{p} \alpha_i \, \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$ and for any index i, $1 \leq i \leq p$ such that $a_i \in B(y_i, \rho_0)$ we have*

$$
\begin{aligned}
\Big\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \Big\rangle &= J(u) \frac{\alpha_i^2 (n-4)}{n K(a_i)} c(y_i) \sum_{k=1}^{n} b_k(y_i) \begin{cases} \dfrac{1}{\lambda_i^{\beta(y_i)}}, & \textit{if } \beta(y_i) < n \\ \dfrac{log \lambda_i}{\lambda_i^{\beta(y_i)}}, & \textit{if } \beta(y_i) = n \\ \dfrac{1}{\lambda_i^{n}}, & \textit{if } \beta(y_i) > n \end{cases} \\
&- c_1 J(u) \sum_{j \neq i} \alpha_i \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(\mid a_i - y_i \mid^{\beta(y_i)} \Big) \\
&+ O\Big(\frac{|a_i - y_i|^{\beta(y_i)-1}}{\lambda_i}\Big) + \sum_{j \neq i} o\Big(\varepsilon_{ij}\Big) + \Big[o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big), \\
&\quad \textit{if } \beta(y_i) \neq n\Big] + \Big[o\Big(\frac{log \lambda_i}{\lambda_i^{\beta(y_i)}}\Big), \textit{ if } \beta(y_i) = n\Big].
\end{aligned}
$$

Observe that the term $O(\frac{1}{\lambda_i^n})$ of the expansion of Lemma 3.1 is removed in the expansion of Corollary 3.1.

For any critical point y of K, we define a small neighborhood of y in S^n as follows. Let γ be a small positive constant. Define

$$N(y) = \Big\{a \in S^n,\ a \in B(y, \rho_0) \text{ and } \lambda^{\beta^*(y)}|a-y|^{\beta(y)} < \gamma\Big\},$$

where $\beta^*(y) = \min(n, \beta(y))$.

For any $p \geq 1$, we define

$$V^\infty(p,\varepsilon) = \Big\{u = \sum_{i=1}^{p} \alpha_i\, \delta_{(a_i,\, \lambda_i)} \in V(p,\varepsilon),\ \text{s.t, } a_i \in N(y_i),\ y_i \in \mathcal{K},$$

$$\forall i = 1,\ldots,p \text{ and } y_i \neq y_j,\ \forall 1 \leq i \neq j \leq p\Big\}.$$

In $V^\infty(p,\varepsilon)$, the expansion of Corollary 3.1 can be improved as follows.

Corollary 3.2. *Let K be a positive function satisfying $(f)_\beta$-condition, $1 < \beta < n + \delta_0$. For any $u = \sum_{i=1}^{p} \alpha_i\, \delta_{(a_i,\, \lambda_i)} \in V^\infty(p,\varepsilon)$ and for any index i, $1 \leq i \leq p$, we have*

$$\begin{aligned}\Big\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}\Big\rangle &= J(u)\frac{\alpha_i^2(n-4)}{nK(a_i)} c(y_i) \sum_{k=1}^{n} b_k(y_i) \begin{cases} \dfrac{1}{\lambda_i^{\beta(y_i)}}, \textit{if } \beta(y_i) < n \\ \dfrac{log\lambda_i}{\lambda_i^{\beta(y_i)}}, \textit{if } \beta(y_i) = n \\ \dfrac{1}{\lambda_i^{n}}, \quad \textit{if } \beta(y_i) > n \end{cases} \\ &- c_1 J(u) \sum_{j\neq i} \alpha_i \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \Big[o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big),\ \textit{if } \beta(y_i) \neq n\Big] \\ &+ \Big[o\Big(\frac{log\lambda_i}{\lambda_i^{\beta(y_i)}}\Big),\ \textit{if } \beta(y_i) = n\Big] + \sum_{j\neq i} o\Big(\varepsilon_{ij}\Big).\end{aligned}$$

In ([11], Theorem 3.1) and ([12], step 4 and 5) it is proved that $V(p,\varepsilon)\backslash V^\infty(p,\varepsilon)$, $p \geq 1$ does not contain any critical points at infinity. The proof can be extended to the case of $1 < \beta < n + \delta_0$.

More precisely,

Proposition 3.1. *[11, 12] Assume that K is positive and satisfying $(f)_\beta$-condition, $\beta \in (1, n+\delta_0)$. There exists a bounded pseudo-gradient W_1 in $V(p,\varepsilon)\backslash V^\infty(p,\varepsilon)$ such that for any $u = \sum_{i=1}^{p} \alpha_i\, \delta_{(a_i,\, \lambda_i)} \in V(p,\varepsilon)\backslash V^\infty(p,\varepsilon)$, we have*

i) $\Big\langle \partial J(u), W(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\widetilde{\beta}}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \sum_{j\neq i} \varepsilon_{ij}\Big),$

ii) $\Big\langle \partial J(u+\overline{v}), W(u) + \dfrac{\partial \overline{v}}{\partial(\alpha, a, \lambda)} W(u) \Big\rangle \leq -c\Big(\sum_{i=1}^{p} \dfrac{1}{\lambda_i^{\widetilde{\beta}}} + \sum_{i=1}^{p} \dfrac{|\nabla K(a_i)|}{\lambda_i} \sum_{j\neq i} \varepsilon_{ij}\Big),$

iii) for any $i = 1, \dots, p$, $\lambda_i(s)$ remains a bounded function as long as the associated flow line $u(s) = \sum_{i=1}^{p} \alpha_i(s)\, \delta(a_i(s), \lambda_i(s))$ stays in $V(p, \varepsilon) \backslash V^{\infty}(p, \varepsilon)$.

To prove Theorem 3.1, we then only focus our attention to study the concentration phenomenon in $V^{\infty}(p, \varepsilon),\ p \geq 1$. We prove the following result.

Proposition 3.2. *Assume that K is a positive function satisfying $(f)_\beta$-condition, $1 < \beta < n + \delta_0$. There exists a bounded pseudo-gradient W_2 in $V^{\infty}(p, \varepsilon)$ such that for any $u \in V^{\infty}(p, \varepsilon)$, inequalities (i) and (ii) of Proposition 3.1 hold. Moreover, the only case where the $\max_{1\leq i\leq p} \lambda_i(s)$ is not bounded is when $a_i(s)$ tends to a critical point $y_i \in \widetilde{\mathcal{K}^+}$, $\forall i = 1, \dots, p$ with $y_i \neq y_j$, $\forall 1 \leq i \neq j \leq p$.*

Proof. Let $u = \sum_{i=1}^{p} \alpha_i\, \delta_{(a_i,\, \lambda_i)} \in V^{\infty}(p, \varepsilon)$. The construction of $W_2(u)$ will be related to the following two cases.

- Case 1. $y_i \in \widetilde{\mathcal{K}^+}, \forall 1 \leq i \leq p$.

In this region we increase all the parameters $\lambda_i,\ i = 1, \dots, p$ with respect to the differential equation $\dot{\lambda}_i = \lambda_i$. The corresponding vector field is

$$Z_i(u) = \alpha_i \lambda_i \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial \lambda_i},\ i = 1, \dots, p.$$

Using the asymptotic expansion of Corollary 3.2, we get

$$\begin{aligned}
\Big\langle \partial J(u), Z_i(u) \Big\rangle &= \frac{n-4}{n} J(u) \frac{\alpha_i^2 c(y_i)}{K(a_i)} \sum_{k=1}^{n} b_k(y_i) \begin{cases} \dfrac{1}{\lambda_i^{\beta^*(y_i)}}, & \text{if } \beta(y_i) \neq n \\ \dfrac{log\lambda_i}{\lambda_i^{\beta(y_i)}}, & \text{if } \beta(y_i) = n \end{cases} \\
&+ \sum_{j\neq i} O(\varepsilon_{ij}) + \Big[o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big), \text{if } \beta(y_i) \neq n\Big] + \Big[o\Big(\frac{log\lambda_i}{\lambda_i^{\beta(y_i)}}\Big), \\
&\quad \text{if } \beta(y_i) = n\Big] \\
&\leq -c \begin{cases} \dfrac{1}{\lambda_i^{\beta^*(y_i)}}, & \text{if } \beta(y_i) \neq n \\ \dfrac{log\lambda_i}{\lambda_i^{\beta(y_i)}}, & \text{if } \beta(y_i) = n \end{cases} + \sum_{j\neq i} O(\varepsilon_{ij}),
\end{aligned}$$

since $\sum_{k=1}^{n} b_k(y_i) < 0$. Define

$$W_2^1(u) = \sum_{i=1}^{n} Z_i(u).$$

By the above expansion, we have

$$\Big\langle \partial J(u), W_2(u)\Big\rangle \leq -c\sum_{i=1}^{p}\begin{cases}\dfrac{1}{\lambda_i^{\beta^*(y_i)}}, & \text{if } \beta^*(y_i)\neq n\\ \dfrac{log\lambda_i}{\lambda_i^{\beta(y_i)}}, & \text{if } \beta(y_i)=n\end{cases} + \sum_{j\neq i} O(\varepsilon_{ij}).$$

$$\leq -c\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{j\neq i} O(\varepsilon_{ij}). \quad (3.4)$$

We claim the following :

Claim 1. For any $u=\sum_{i=1}^{p}\alpha_i\,\delta_{(a_i,\,\lambda_i)}\in V^{\infty}(p,\varepsilon)$ and for any $1\leq i\neq j\leq p$, we have

$$\varepsilon_{ij}=o\big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\big)+o\big(\frac{1}{\lambda_j^{\beta^*(y_j)}}\big),\ \text{as } \varepsilon \text{ is small.}$$

Indeed, in $V^{\infty}(p,\varepsilon)$ we have

$$\varepsilon_{ij}\sim\frac{1}{(\lambda_i\lambda_j)^{\frac{n-4}{2}}},\ \forall i\neq j.$$

Let $\alpha>0$ small enough. If $\lambda_j^{\frac{n-4}{2}}\geq\frac{1}{\varepsilon^{\alpha}}\lambda_i^{\beta^*(y_i)-\frac{n-4}{2}}$, then

$$\varepsilon_{ij}\leq\varepsilon^{\alpha}\frac{1}{\lambda_i^{\beta^*(y_i)}}.$$

Therefore $\varepsilon_{ij}=o\big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\big)$ as ε tends to zero.

If $\lambda_j^{\frac{n-4}{2}}\leq\frac{1}{\varepsilon^{\alpha}}\lambda_i^{\beta^*(y_i)-\frac{n-4}{2}}$, then $\beta^*(y_i)-\frac{n-4}{2}$ has to be positive, (if not $\lambda_j^{\frac{n-4}{2}}\leq\frac{1}{\varepsilon^{\alpha}}<\frac{1}{\varepsilon^{\frac{n-4}{2}}}$, taking $\alpha<\dfrac{n-4}{2}$ and therefore, $\lambda_j<\frac{1}{\varepsilon}$). It follows that

$$\frac{n-4}{\lambda_j^{2\beta^*(y_i)-(n-4)}}\leq\varepsilon^{-\frac{2\alpha}{2\beta^*(y_i)-(n-4)}}\lambda_i,$$

and thus

$$\frac{1}{\lambda_i^{\frac{n-4}{2}}}\leq\varepsilon^{-\frac{(n-4)\alpha}{2\beta^*(y_i)-(n-4)}}\lambda_j^{-\frac{n-4}{2}\frac{n-4}{2\beta^*(y_i)-(n-4)}}.$$

Consequently

$$\begin{aligned}\varepsilon_{ij} &\leq \varepsilon^{-\frac{(n-4)\alpha}{2\beta^*(y_i)-(n-4)}}\lambda_j^{-\frac{n-4}{2}\left(1+\frac{n-4}{2\beta^*(y_i)-(n-4)}\right)},\\ &\leq \varepsilon^{-\frac{(n-4)\alpha}{2\beta^*(y_i)-(n-4)}}\lambda_j^{-\beta^*(y_i)-\frac{n-4}{2\beta^*(y_i)-(n-4)}\left(\beta^*(y_i)+\beta^*(y_j)-\frac{2\beta^*(y_i)\beta^*(y_j)}{n-4}\right)},\end{aligned}$$

since $\dfrac{1}{\beta^*(y_i)}+\dfrac{1}{\beta^*(y_j)}>\dfrac{2}{n-4}$. Using the fact that $\lambda_j>\varepsilon^{-1}$, we get,

$$\varepsilon_{ij}\leq\varepsilon^{\frac{(n-4)}{2\beta^*(y_i)-(n-4)}\left(\beta^*(y_i)+\beta^*(y_j)-\frac{2\beta^*(y_i)\beta^*(y_j)}{n-4}-\alpha\right)}\frac{1}{\lambda_j^{\beta^*(y_j)}}.$$

Taking $\alpha < \beta^*(y_i) + \beta^*(y_j) - \frac{2\beta^*(y_i)\beta^*(y_j)}{n-4}$, Claim 1 follows.

Inequality (3.4) and Claim 1 yield

$$\begin{aligned}\Big\langle \partial J(u), W_2^1(u)\Big\rangle &\leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{i\neq j}\varepsilon_{ij}\Big)\\ &\leq -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\Big),\end{aligned}$$

since under $(f)_\beta$-condition, $\dfrac{|\nabla K(a_i)|}{\lambda_i} \leq \dfrac{c}{\lambda_i^{\beta^*(y_i)}}$.

• <u>Case 2.</u> There exists an index i, $1 \leq i \leq p$, such that $y_i \notin \widetilde{\mathcal{K}^+}$.
In this case we denote I the set of all the index i, $1 \leq i \leq p$, such that $y_i \notin \widetilde{\mathcal{K}^+}$. For any $i \in I$, we set

$$\dot{\lambda_i} = -\lambda_i \text{ and } -Z_i(u) = -\alpha_i\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}.$$

Using the expansion of Corollary 3.2, we get

$$\begin{aligned}\Big\langle \partial J(u), Z_i(u)\Big\rangle &= \frac{n-4}{n}J(u)\frac{\alpha_i^2 c(y_i)}{K(a_i)}\ \sum_{k=1}^{n} b_k(y_i)\begin{cases}\dfrac{1}{\lambda_i^{\beta^*(y_i)}}, \text{ if } \beta(y_i)\neq n\\ \dfrac{log\lambda_i}{\lambda_i^{\beta(y_i)}}, \ \text{ if } \beta(y_i) = n\end{cases}\\ &+ \sum_{j\neq i}O(\varepsilon_{ij}) + \Big[o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big), \text{ if } \beta(y_i)\neq n\Big] + \Big[o\Big(\frac{log\lambda_i}{\lambda_i^{\beta(y_i)}}\Big),\\ &\quad \text{if } \beta(y_i) = n\Big]\\ &\leq -c\begin{cases}\dfrac{1}{\lambda_i^{\beta^*(y_i)}}, \text{ if } \beta(y_i)\neq n\\ \dfrac{log\lambda_i}{\lambda_i^{\beta(y_i)}}, \ \text{ if } \beta(y_i) = n\end{cases} + \sum_{j\neq i}O(\varepsilon_{ij}),\end{aligned}$$

since $\sum_{k=1}^{n} b_k(y_i) > 0, \forall i \in I$. Define

$$V_I(u) = -\sum_{i\in I} Z_i(u).$$

The above expansion and Claim 1 yield,

$$\begin{aligned}\Big\langle \partial J(u), V_I(u)\Big\rangle &\leq -c\sum_{i\in I}\begin{cases}\dfrac{1}{\lambda_i^{\beta^*(y_i)}}, \text{ if } \beta(y_i)\neq n\\ \dfrac{log\lambda_i}{\lambda_i^{\beta(y_i)}}, \ \text{ if } \beta(y_i) = n\end{cases} + \sum_{i=1}^{p} o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big),\\ &\leq -c\sum_{i\in I}\frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{i=1}^{p} o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big).\end{aligned}$$

Define

$$\tilde{I} = \Big\{i, 1 \leq i \leq p, \text{ s.t.}, \lambda_i^{\beta^*(y_i)} \geq \frac{1}{2}\min_{k\in I}\lambda_k^{\beta^*(y_k)}\Big\}.$$

Of course $I \subset \tilde{I}$ and the preceding inequality can be improved as follows

$$\Big\langle \partial J(u), V_I(u) \Big\rangle \leq -c \sum_{i \in \tilde{I}} \frac{1}{\lambda_i^{\beta^*(y_i)}} + \sum_{i \notin \tilde{I}} o\Big(\frac{1}{\lambda_i^{\beta^*(y_i)}}\Big).$$

We now define,

$$W_2^2(u) = V_I(u) + \sum_{i \notin \tilde{I}} \alpha_i \lambda_i \frac{\partial \delta(a_i, \lambda_i)}{\partial \lambda_i}.$$

Using the fact that $\sum_{k=1}^{n} b_k(y_i) < 0,\ \forall i \notin \tilde{I}$ and the above expansion, we obtain that

$$\Big\langle \partial J(u), W_2^2(u) \Big\rangle \leq -c \sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta^*(y_i)}}.$$

Also by Claim 1 and $(f)_\beta$-condition, the following holds

$$\Big\langle \partial J(u), W_2^2(u) \Big\rangle \leq -c \Big(\sum_{i=1}^{p} (\frac{1}{\lambda_i^{\beta^*(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}) + \sum_{i \neq j} \varepsilon_{ij} \Big).$$

Let $W_2(u)$, $u \in V^\infty(p, \varepsilon)$ be a convex combination of $W_2^1(u)$ and $W_2^2(u)$. It is straightforward to see that W_2 satisfies the required properties of Proposition 3.2. □

Proof of Theorem 3.1 The required pseudo-gradient $W \in V(p, \varepsilon)$, $p \geq 1$, is defined by a convex combination of W_1 and W_2, where W_1 and W_2 are the pseudo-gradient defined in Proposition 3.1 and Proposition 3.2 respectively. By construction the only critical points at infinity of $J \in V(p, \varepsilon)$ are

$$(y_1, \dots, y_p)_\infty = \sum_{j=1}^{p} \frac{1}{K(y_j)^{\frac{n-4}{2}}} \delta_{(y_i, \infty)},$$

where $y_i \in \widetilde{\mathcal{K}^+},\ \forall i = 1, \dots, p$ and $y_i \neq y_j,\ \forall 1 < i \neq j \leq p$.

Arguing as in ([2], Lemma 4.2), near each critical point at infinity $(y_1, \dots, y_p)_\infty$, J can be expanded as follows

$$\begin{aligned} J\Big(\sum_{i=1}^{p} \alpha_i \delta_(a_i, \lambda_i) + \bar{v}\Big) &= \Big(\sum_{i=1}^{p} \frac{S_n}{K(y_i)^{\frac{n-4}{2}}}\Big)^{\frac{2}{n}} \Big(1 - \sum_{i=1}^{p} \sum_{k=1}^{n} b_k(y_i) \big|(a_i - y_i)_k\big|^{\beta(y_i)} \\ &- |h|^2 + \sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta^*(y_i)}}\Big), \end{aligned} \tag{3.5}$$

where $h \in \mathbb{R}^{p-1}$ and S_n is the best constant of Sobolev. Using the fact that $b_k(y_i) \neq 0, \forall k = 1, ..., n$, the index of J at $(y_1, \dots, y_p)_\infty$ equals to $\sum_{j=1}^{p} \Big(n - i(y_j)\Big) + p - 1$. This conclude the proof of Theorem 3.1. □

4. Proof of the existence results

Proof of Theorem 1.3 We argue by a contradiction. Assume that (1.1) has no solution. It follows from Proposition 2.1 and Theorem 3.1 that the critical points at infinity of J are

$$(y_1,\dots,y_p)_\infty=\sum_{i=1}^{p}\frac{1}{K(y_i)^{\frac{n-4}{2}}}\delta_{(y_i,\infty)},\; p\geq 1,$$

with $y_i\in\widetilde{\mathcal{K}^+},\ \forall i=1,\dots,p$ and $y_i\neq y_j,\ \forall 1<i\neq j\leq p$. Using expansion (3.5), the level of J at $(y_1,\dots,y_p)_\infty$ equals to

$$C_\infty(y_1,\dots,y_p)_\infty=\Big(\sum_{i=1}^{p}\frac{S_n}{K(y_i)^{\frac{n-4}{2}}}\Big)^{\frac{2}{n}}$$

Denote,

$$J^1(u)=\frac{1}{\Big(\int_{S^n}u^{\frac{2n}{n-4}}dvg_0\Big)},\; u\in\Sigma^+.$$

J^1 represents the Euler-Lagrange functional of problem (1.1) when the functional $K=1$ on S^n. In view of uniqueness result of Lin [21], the solution of (1.1), when $K=1$ on S^n define a contractible $(n+1)$-dimensional manifold

$$Z=\Big\{\delta_{(a,\lambda)},\, a\in S^n,\, \lambda>0\Big\}.$$

Moreover, J^1 has no critical point at infinity in Σ^+ and for any $a\in S^n$ and $\lambda>0$,

$$J^1\Big(\delta_{(a,\lambda)}\Big)=\inf_{u\in\Sigma^+}J^1(u)=S_n^{\frac{2}{n}}.$$

Let $\eta_0>0$ be a fixed small constant chosen so that

$$S_n^{\frac{2}{n}}+\eta_0<\big(2S_n\big)^{\frac{2}{n}}-\eta_0\text{and}\big(2S_n\big)^{\frac{2}{n}}+\eta_0<\big(3S_n\big)^{\frac{2}{n}}-\eta_0$$

. Denote $\parallel K-1\parallel_{L^\infty(S^n)}$ the maximum of $|K(x)-1|,\ x\in S^n$. By the description of the critical values at infinity of J, we have

$$C_\infty(y)_\infty<S_n^{\frac{2}{n}}+\frac{\eta_0}{4},\ \forall y\in\widetilde{\mathcal{K}^+},$$

$$C_\infty(y_i,y_j)_\infty\in\Big(\big(2S_n\big)^{\frac{2}{n}}-\frac{\eta_0}{4},\ \big(2S_n\big)^{\frac{2}{n}}+\frac{\eta_0}{4}\Big),\ \forall y_i\neq y_j,\in\widetilde{\mathcal{K}^+},$$

$$C_\infty(y_1,\dots,y_p)_\infty>\big(3S_n\big)^{\frac{2}{n}}-\frac{\eta_0}{4},\ \forall p\geq 3,$$

provided $\parallel K-1\parallel_{L^\infty(S^n)}$ small enough.

The following estimate is proved in [7]. For a function $J:\Sigma\to\mathbb{R}$ and $c\in\mathbb{R}$, we denote

$$J_c=\Big\{u\in\Sigma^+,\ J(u)\leq c\Big\}.$$

Lemma 4.1. *[7]*

$$J(u)=J^1(u)\bigg(1+O\Big(\parallel K-1\parallel_{L^\infty(S^n)}\Big)\bigg).$$

It follows that for $\| K-1 \|_{L^\infty(S^n)}$ small enough, we have

$$J_{S_n^{\frac{2}{n}}+\frac{\eta_0}{4}} \subset J^1_{S_n^{\frac{2}{n}}+\frac{\eta_0}{2}} \subset J_{S_n^{\frac{2}{n}}+\eta_0} \tag{4.1}$$

and

$$J_{\left(2S_n\right)^{\frac{2}{n}}+\frac{\eta_0}{4}} \subset J^1_{\left(2S_n\right)^{\frac{2}{n}}+\frac{\eta_0}{2}} \subset J_{\left(2S_n\right)^{\frac{2}{n}}+\eta_0}. \tag{4.2}$$

From (4.1), we use the topological arguments of ([7], pages 477-479). We derive that

$$\chi\Big(J_{S_n^{\frac{2}{n}}+\eta_0}\Big) = 1 = \sum_{y_j\in\widetilde{\mathcal{K}^+}} (-1)^{n-\widetilde{i}(y_j)}, \tag{4.3}$$

where χ denotes the Euler-Poincar$\acute{e}$ characteristic. Moreover by (4.2) we obtain that

$$\chi\Big(J_{\left(2S_n\right)^{\frac{2}{n}}+\eta_0}\Big) = 1 = \sum_{y_j\in\widetilde{\mathcal{K}^+}} (-1)^{n-\widetilde{i}(y_j)} + \sum_{y_i\neq y_j\in\widetilde{\mathcal{K}^+}} (-1)^{1+n-\widetilde{i}(y_i)+n-\widetilde{i}(y_j)}, \tag{4.4}$$

(4.3), (4.4) yield,

$$\sum_{y_i\neq y_j\in\widetilde{\mathcal{K}^+}} (-1)^{n-\widetilde{i}(y_i)+n-\widetilde{i}(y_j)} = 0.$$

We now use exactly the computation of ([20], p.16). We get $\sharp\,\widetilde{\mathcal{K}^+} = 1$. This implies that $\widetilde{\mathcal{K}}\backslash\{y_0\} = \emptyset$. This contradicts the assumption of Theorem 3.1. □

Proof of Theorem 1.2 It is a consequence of Theorem 1.3. □

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