From Ingham to Nazarov's inequality: a survey on some trigonometric inequalities

D'Ingham à l'inégalité de Nazarov : un survey sur quelques inégalités trigonométriques

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ABSTRACT. The aim of this paper is to give an overview of some inequalities about L^p-norms ($p = 1$ or $p = 2$) of harmonic (periodic) and non-harmonic trigonometric polynomials. Among the material covered, we mention Ingham's Inequality about L^2 norms of non-harmonic trigonometric polynmials, the proof of the Littlewood conjecture by Mc Gehee, Pigno and Smith on the lower bound of the L^1 norm of harmonic trigonometric polynomials as well as its counterpart in the non-harmonic case due to Nazarov. For the later one, we give a quantitative estimate that completes our recent result with an estimate of L^1-norms over small intervals.
We also give some stronger lower bounds when the frequencies satisfy some more restrictive conditions (lacunary Fourier series, "multi-step arithmetic sequences").
Most proofs are close to existing ones and some open questions are mentionned at the end.
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1. Introduction

The aim of this paper is to give an overview of the main estimates of L^p-norms of harmonic and non-harmonic trigonometric polynomials, when $p = 2$ and $p = 1$. In many fields of mathematics, ranging from number theory (*see e.g.* the previous survey [1]) to signal processing and PDEs, one is lead to inverstigate such norms. Our main motivation comes from the use of Ingham's inequality (lower and upper estimates of L^2-norms of non-harmonic trigonometric polynomials) in control theory of PDEs. We refer the interrested reader to the book by Komornik and Loreti [17].

Let us now be more precise. We will here restrict attention to $p = 2$ or $p = 1$ and investigate L^p-norm estimates of (harmonic) trigonometric polynomials

$$\int_{-1/2}^{1/2} \left|\sum_{k=-N}^{N} c_k e^{2i\pi n_k t}\right|^p \mathrm{d}t \quad (c_k)_{k\in\mathbb{Z}} \subset \mathbb{C},\ (n_k)_{k\in\mathbb{Z}} \subset \mathbb{Z}$$

or non-harmonic trigonometric polynomials

$$\frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=-N}^{N} c_k e^{2i\pi\lambda_k t}\right|^p \mathrm{d}t \quad T > 0,\ (c_k)_{k\in\mathbb{Z}} \subset \mathbb{Z} \subset \mathbb{C},\ (\lambda_k)_{k\in\mathbb{Z}} \subset \mathbb{Z} \subset \mathbb{R}$$

as well as their limits when $T \to +\infty$

$$\limsup_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k=-N}^{N} c_k e^{2i\pi\lambda_k t}\right|^p \mathrm{d}t \quad T > 0,\ c_k \subset \mathbb{C},\ \lambda_k \subset \mathbb{R}$$

(that is Besikovich norms). Note for future use that, when all the λ_k's are integers, the Besikovich norm is the same as the usual $L^p([-1/2,1/2])$-norm.

The by far simplest case is $p=2$ and harmonic trigonometric polynomials as Parseval's relation states that

$$\int_{-1/2}^{1/2}\left|\sum_{k=-N}^{N} c_k e^{2i\pi kt}\right|^2 \mathrm{d}t=\sum_{k=-N}^{N}|c_k|^2.$$

The situation becomes much deeper for non-harmonic trigonometric polynomials but is well understood thanks to a deep result by Ingham:

Theorem A (Ingham)**.** *Let $\gamma>0$ and $T>\dfrac{1}{\gamma}$, then there are two constants $A=A(T,\gamma)$ and $B=B(T,\gamma)$ such that*

– for every sequence $(\lambda_k)_{k\in\mathbb{Z}}$ with $\lambda_{k+1}-\lambda_k\geq\gamma$,

– for every complex sequence $(c_k)_{k=-N,\ldots,N}$
for every integer $N\geq 1$,

$$A\sum_{k=-N}^{N}|c_k|^2\leq\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N} c_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t\leq B\sum_{k=-N}^{N}|c_k|^2. \tag{1.1}$$

When $T=\dfrac{1}{\gamma}$, there is a sequence $(\lambda_k)_{k\in\mathbb{Z}}$ with $\lambda_{k+1}-\lambda_k\geq\gamma$ such that the lower bound in (1.1) *does not hold for every N and every $(c_k)_{k=-N,\ldots,N}\subset\mathbb{C}$ unless $A=0$.*

Note that the upper bound in (1.1) holds for every $T>0$, the restriction $T>1$ is only needed for the lower bound, which is the most interresting one for control theory. In the next section, we will present Ingham's proof and also compute the constants A,B. We will also present an argument due to Haraux that allows to obtain the lower bound with T arbitrarily small when $|\lambda_{k+1}-\lambda_k|\to+\infty$ when $k\to\pm\infty$. This result also follows from an earlier theorem of Kahane [16].

Once the L^2-theory settled, we will move to the L^1-theory which is much richer. To start with, the behavior of the L^1-norm of a trigonometric polynomial may depend on the frequencies. There are two extreme cases:

– the frequencies form an arithmetic sequence, *e.g.* the Dirichlet kernel. One might expect the L^1-norm to be small in view of the standard estimate (*see e.g.* [29, Section 8.3]): when $N\to+\infty$

$$\int_{-1/2}^{1/2}\left|\sum_{k=-N}^{N} e^{2i\pi kt}\right| \mathrm{d}t=\frac{4}{\pi^2}\ln N+O(1).$$

– The frequencies form a geometric sequence, or more generally a lacunary sequence ($n_{k+1}\geq qn_k$ with $q>1$) then the L^1-norms are much large: for N large enough

$$\int_{-1/2}^{1/2}\left|\sum_{k=0}^{N} e^{2i\pi n_k t}\right| \mathrm{d}t\geq C\sqrt{N}.$$

We will prove this last estimate in Section 3. In view, of those estimates Littlewood conjectured that the Dirichlet kernel has worse possible behavior, namely that

$$L_N := \inf_{n_0<n_1<\cdots<n_N} \int_{-1/2}^{1/2} \left|\sum_{k=0}^{N} e^{2i\pi n_k t}\right| \mathrm{d}t \geq C \log N$$

for some constant $C \leq \dfrac{4}{\pi^2}$. The first non-trivial estimate was obtained by Cohen [5] who proved that

$$L_N \geq C(\ln N/\ln\ln N)^{1/8}$$

for $N \geq 4$. Subsequent improvements are due to Davenport [6], Fournier [8] and crucial contributions by Pichorides [23, 24, 25, 26] leading to $L_N \geq C\ln N/(\ln\ln N)^2$. Finally, Littlewood's conjecture was proved independently by Konyagin [18] and Mc Gehee, Pigno, Smith [19] in 1981. In both papers, Littlewood's conjecture is actually obtained as a corollary of a stronger result (and they are not consequences of one another). The second one is the one we will focus on here and is given by the following result:

Theorem B (Mc Gehee, Pigno & Smith [19]). *For $n_1 < n_2 < \cdots < n_N$ integers and $a_1, \ldots, a_N$ complex numbers,*

$$C_{MPS} \sum_{k=1}^{N} \frac{|a_k|}{k} \leq \int_{-1/2}^{1/2} \left|\sum_{k=1}^{N} a_k e^{2i\pi n_k t}\right| \mathrm{d}t$$

where C_{MPS} is a universal constant ($C_{MPS} = 1/30$ would do).

We will present its proof below. We would like to insist that this is the worse possible behavior. To start, as we already mentionned, in the lacunary case, the lower bound is $\sqrt{N}$. We will show a recent result of Hanson [10] who considered a family of sets (so-called "strongly multi-dimensional sets") of which the simplest example is a multi-step arithmetic sequence. By that, we mean a sequence $n_{jM+k} = jD + kd$, $j \in \mathbb{Z}$, $k = 0, \ldots, M-1$ and $Md \ll D$. In other words, we take an arithmetic sequence with large step D and, after each element of the sequence, we add a small piece of an arithmetic sequence with small step. It is then shown that

$$C \ln N \ln M \leq \int_{-1/2}^{1/2} \left|\sum_{j=0}^{N}\sum_{k=0}^{M} e^{2i\pi(jD+kd)t}\right| \mathrm{d}t.$$

We also present an example of Newman that constructed $a_0, \ldots, a_N$ all of modulus 1 such that

$$\sqrt{N} - C \leq \int_{-1/2}^{1/2} \left|\sum_{k=0}^{N} a_k e^{2i\pi kt}\right| \mathrm{d}t.$$

This shows that it is possible to be far away from the lower bound (and actually near to the best possible upper bound).

The last part of this survey is devoted to L^1-norms of non-harmonic trigonometric polynomials. The best result to day is the following:

Theorem C. *Let $\lambda_0 < \lambda_2 < \cdots < \lambda_N$ be N distinct real numbers and $a_0, \ldots, a_N$ be complex numbers. Then*

i. we have

$$\frac{1}{26}\sum_{k=1}^{N}\frac{|a_k|}{k+1} \leq \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N}a_k e^{2i\pi\lambda_k t}\right|\,\mathrm{d}t.$$

ii. If further $a_0, \ldots, a_N$ all have modulus larger than 1, $|a_k| \geq 1$ then

$$\frac{4}{\pi^3}\ln N \leq \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N}a_k e^{2i\pi\lambda_k t}\right|\,\mathrm{d}t.$$

iii. Assume further that for $k = 0, \ldots, N-1$, $\lambda_{k+1} - \lambda_k \geq 1$, then, for every $T > 1$, there exists a constant $C(T)$ such that, for every $a_0, \ldots, a_N \in \mathbb{C}$,

$$C(T)\sum_{k=1}^{N}\frac{|a_k|}{k+1} \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N}a_k e^{2i\pi\lambda_k t}\right|\,\mathrm{d}t.$$

Moreover,

(a) for $T \geq 72$ we can take $C(T) = \dfrac{1}{122}$;

(b) for $1 < T \leq 2$, $C(T) = O\big((T-1)^{-15/2}\big)$.

Let us comment on this theorem. To start with, the theorem is essentially a quantitative version of a result of F. Nazarov [21] and is due to the authors together with K. Kellay [15]. Only the case $1 < T \leq 2$ has not been presented before. To be more precise, when the λ_k's are integers, this is of course the result of Mc Gehee, Pigno and Smith while Point 2 comes from a slight modification of their argument by Stegeman and Yabuta. Note that the constants are the same as those in Theorem B (and are actually a bit better). While it is obvious that this inequality implies Theorem B (with the same constant), there is an elegent argument by Hudson and Leckband [12] that allows to show that the converse is also true. We will present this argument below. Point 3 is due to Nazarov with non-explicit constant. A direct proof of this theorem is given in [15] with the explicit constants mentionned above. Only the case of small T has not been presented so far and is thus the main novelty of the present paper. It is obtained by a very mild modification of Nazarov's proof. Note also that once we have established Point 3 for some T_0, it is valid for all $T \geq T_0$ so that we actually recover Nazarov's orginal result (with much worse constants than the ones stated above).

The remaining of the paper is organised as follows: Section 2 is devoted to the L^2 case and we prove the inequalities in Ingham's Theorem in Section 2.1 and the necessity of the condition $T > 1$ in Section 2.2. Further, Haraux's argument is presented in Section 2.3.

In Section 3, we investigate the L^1 norms when the frequencies are integers. We devote Section 3.1 to lacunary trigonometric polynomials. Section 3.3 is devoted to Mc Gehee, Pigno, Smith's proof of Littlewood's conjecture. In Section 3.3 we present Hanson's result on strongly multidimensional sequences and we conclude in Section 3.4 with the result of Newman.

Section 4 is devoted to the case of non-harmonic trigonometric polynomials. We start in Section 4.1 with the argument of Hudson and Leckband and then extend the case of lacunary trigonometric polynomials to the non-harmonic setting and the Besikovich L^1-norms. We then present Ingham's L^1-Inequality. We conclude this section with the proof of the quantitative version of Nazarov's theorem for small T.

In the last section, we present a few open questions.

2. L^2 estimates

2.1. *Ingham's inequalities*

The aim of this section is to show the following: let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be a 1-separated sequence, $|\lambda_k - \lambda_\ell| \geq 1$ if $k \neq \ell$. Let $\mathcal{P}(\Lambda)$ be the set of (non-harmonic) trigonometric polynomials

$$\mathcal{P}(\Lambda) = \left\{ P(t) := \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \ : (a_k)_{k\in\mathbb{Z}} \subset \mathbb{C} \text{ with finite support} \right\}.$$

Note that, if $P \in \mathcal{P}(\Lambda)$ is given, then the a_k's are determined by

$$a_k = \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} P(t) e^{-2i\pi\lambda_k t}\,\mathrm{d}t.$$

We can then define two natural norms on $\mathcal{P}(\Lambda)$, namely

$$\|P\|_{L^2([-T/2,T/2])} := \left(\frac{1}{T} \int_{-T/2}^{T/2} |P(t)|^2\,\mathrm{d}t \right)^{\frac{1}{2}}$$

and

$$\|P\|_{\ell^2} := \left(\sum_{k\in\mathbb{Z}} |a_k|^2 \right)^{\frac{1}{2}}.$$

Our aim in this section is to show that, when $T > 1$, these two norms are equivalent. This is done by proving two inequalities. The first one is the direct inequality:

Proposition 2.1 (Ingham's direct inequality)**.** *Let $\gamma > 0$. Let $(a_k)_{k\in\mathbb{Z}}$ be a finitely supported sequence of complex numbers and $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers with $\lambda_{k+1} - \lambda_k \geq \gamma$. For every $T > 0$,*

$$\frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t} \right|^2 dt \leq 2\frac{\gamma T + 1}{\gamma T} \sum_{k\in\mathbb{Z}} |a_k|^2. \tag{2.1}$$

Proof. Changing variable $t = s/\gamma$ we may assume that $\gamma = 1$. We consider the function H on $\mathbb{R}$ defined by

$$h(x) = \begin{cases} \cos \pi x & \text{when } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

As h is real and even, its Fourier transform is given by

$$\widehat{h}(t) = 2 \int_0^{1/2} \cos \pi x \cos 2\pi x t \, \mathrm{d}x = \frac{2}{\pi} \frac{\cos \pi t}{1 - 4t^2}$$

with the understanding that $\widehat{h}(1/2) = \frac{1}{2}$. From this, one shows that $\widehat{h}(x) \geq \frac{1}{2}$ for $|x| \leq \frac{1}{2}$. Finally, let

$$g(x) = h * h(x) = \begin{cases} \dfrac{\sin \pi|x| - \pi(|x| - 1) \cos \pi x}{2\pi} & \text{when } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

One easily shows that g is even, non-negative, supported in $[-1, 1]$ and that $g(0) = \frac{1}{2}$. Further its Fourier transform is $\widehat{g}(t) = \widehat{h}(t)^2$. In particular, $\widehat{g}(t) \geq 0$ and $\widehat{g}(t) \geq \frac{1}{4}$ for $|t| \leq \frac{1}{2}$.

But then, if $(a_k)_{k \in \mathbb{Z}}$ is finitely supported and $P(t) = \sum_{k \in \mathbb{Z}} a_k e^{2i\pi\lambda_k t}$,

$$\begin{aligned} \int_{-1/2}^{1/2} |P(t)|^2 \, \mathrm{d}t &\leq 4 \int_{-1/2}^{1/2} \widehat{g}(t) |P(t)|^2 \, \mathrm{d}t \leq 4 \int_{\mathbb{R}} \widehat{g}(t) \left| \sum_{k \in \mathbb{Z}} a_k e^{2i\pi\lambda_k t} \right|^2 \mathrm{d}t \\ &= 4 \sum_{j,k \in \mathbb{Z}} a_j \overline{a_k} \int_{\mathbb{R}} \widehat{g}(t) e^{2i\pi(\lambda_j - \lambda_k)t} \, \mathrm{d}t \\ &= 4 \sum_{j,k \in \mathbb{Z}} a_j \overline{a_k} g(\lambda_j - \lambda_k). \end{aligned}$$

Note that the sums are actually finite. Further, if $j \neq k$ then $|\lambda_j - \lambda_k| \geq 1$ and, as g is supported in $[-1, 1]$, we then have $g(\lambda_j - \lambda_k) = 0$. This implies that

$$\int_{-1/2}^{1/2} |P(t)|^2 \, \mathrm{d}t \leq 4g(0) \sum_{j \in \mathbb{Z}} |a_j|^2$$

so that the inequality is proven for $T = 1$ since $4g(0) = 2$.

For $T < 1$ we simply write

$$\frac{1}{T} \int_{-T/2}^{T/2} |P(t)|^2 \, \mathrm{d}t \leq \frac{1}{T} \int_{-1/2}^{1/2} |P(t)|^2 \, \mathrm{d}t \leq \frac{2}{T} \sum_{j \in \mathbb{Z}} |a_j|^2.$$

To conclude, notice first that, if $I=[a-1/2,a+1/2]$ and $P(t)=\sum_{j\in\mathbb{Z}} a_j e^{2i\pi\lambda_j t}$ then

$$\begin{aligned}\int_I |P(t)|^2\,\mathrm{d}t &= \int_{-1/2}^{1/2} |P(a+t)|^2\,\mathrm{d}t = \int_{-1/2}^{1/2} \left|\sum_{j\in\mathbb{Z}} a_j e^{2i\pi\lambda_j a} e^{2i\pi\lambda_j t}\right|^2 \mathrm{d}t\\ &\leq 2\sum_{j\in\mathbb{Z}} |a_j e^{2i\pi\lambda_j a}|^2 = 2\sum_{j\in\mathbb{Z}} |a_j|^2\end{aligned}$$

from the case $T=1$.

Now let $T>1$ and cover the interval $[-T/2,T/2]$ by $K=\lceil T\rceil \leq T+1$ intervals $I_1,\ldots,I_K$ of length 1. Then

$$\frac{1}{T}\int_{-T/2}^{T/2} |P(t)|^2\,\mathrm{d}t \leq \frac{1}{T}\sum_{j=1}^{K}\int_{I_j} |P(t)|^2\,\mathrm{d}t \leq 2\frac{T+1}{T}\sum_{j\in\mathbb{Z}} |a_j|^2.$$

This completes the proof. □

We now show that a converse inequality also holds, but this time with the extra condition that $T>1$:

Proposition 2.2 (Ingham's converse inequality). *Let $\gamma>0$. Let $(a_k)_{k\in\mathbb{Z}}$ be a finitely supported sequence of complex numbers and $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers with $\lambda_{k+1}-\lambda_k\geq\gamma$. For every $T>\frac{1}{\gamma}$,*

$$C(T,\gamma)\sum_{k\in\mathbb{Z}} |a_k|^2 \leq \frac{1}{T}\int_{-T/2}^{T/2} \left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 dt \tag{2.2}$$

with

$$C(T,\gamma)=\begin{cases} \dfrac{\pi^2}{8}\dfrac{(\gamma T)^2-1}{(\gamma T)^3} & \text{for } \dfrac{1}{\gamma}<T\leq\dfrac{2}{\gamma}\\ \dfrac{\pi^2}{64} & \text{for } T\geq\dfrac{2}{\gamma}\end{cases}. \tag{2.3}$$

Proof. Changing variable $t=s/\gamma$ we find that $C(T,\gamma)=C(\gamma T,1)$ so that we may assume that $\gamma=1$. We will prove this inequality in three steps. We first establish this inequality for $1<T\leq 2$.

As in the previous proof, let h be again defined by $h(x)=\mathbb{1}_{[-1/2,1/2]}(x)\cos\pi x$. Notice that, as h is non-negative, even, continous with support $[-1/2,1/2]$, then $h*h$ is non-negative, even, continous with support $[-1,1]$.

Next $h\in H^1(\mathbb{R})$ with $h'=-\pi\mathbb{1}_{[-1/2,1/2]}\sin\pi x$ and

$$\widehat{h'}(t)=4it\frac{\cos\pi t}{1-4t^2}$$

thus

$$\widehat{h' * h'}(t) = -(2\pi t)^2 \widehat{H}^2(t)$$

We now consider $k_T = \pi^2 T^2 h * h + h' * h'$ so that k_T is continuous, real valued, even and supported in $[-1, 1]$.

$$\widehat{k_T}(t) = \pi^2 \left(T^2 - 4t^2\right) \widehat{h}^2(t)$$

is even (so k_T is the Fourier transform of $\widehat{k_T}$) and in L^1. Further $\widehat{k_T}$ is non-negative on $[-T/2, T/2]$ and negative on $\mathbb{R} \setminus [-T/2, T/2]$.

This implies that

$$\begin{aligned}\int_{-T/2}^{T/2} \widehat{k_T}(t) \left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t &\geq \int_{\mathbb{R}} \widehat{k_T}(t) \left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \\ &= \sum_{k,\ell\in\mathbb{Z}} a_k \overline{a_\ell} \int_{\mathbb{R}} \widehat{k_T}(t) e^{2i\pi(\lambda_k - \lambda_\ell)t} \,\mathrm{d}t \\ &= \sum_{k,\ell\in\mathbb{Z}} a_k \overline{a_\ell} G_T(\lambda_k - \lambda_\ell) = \sum_{k\in\mathbb{Z}} |a_k|^2 k_T(0).\end{aligned}$$

In the last line, we use that $|\lambda_k - \lambda_\ell| \geq 1$ when $k \neq \ell$ thus $k_T(\lambda_k - \lambda_\ell) = 0$.

Now, for $\xi \in [-T/2, T/2]$,

$$\widehat{k_T}(\xi) = \pi^2(T^2 - 4\xi^2)\widehat{h}^2(\xi) \leq \pi^2(T^2 - 4\xi^2)\widehat{h}^2(0) \leq 4T^2$$

while

$$k_T(0) = \pi^2 \int_{-1/2}^{1/2} T^2 \cos^2 \pi t - \sin^2 \pi t \,\mathrm{d}t = \frac{\pi^2}{2}(T^2 - 1)$$

which leads to

$$\int_{-T/2}^{T/2} \left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \geq \frac{\pi^2}{8} \frac{T^2 - 1}{T^2} \sum_{k\in\mathbb{Z}} |a_k|^2. \tag{2.4}$$

For $2 \leq T \leq 6$, we simply write

$$\int_{-T/2}^{T/2} \left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \geq \int_{-1}^{1} \left|\sum_{k\in\mathbb{Z}} a_k e^{2i\pi\lambda_k t}\right|^2 \mathrm{d}t \geq \frac{3\pi^2}{32} \sum_{k\in\mathbb{Z}} |a_k|^2$$

where the second inequality is (2.4) with $T = 2$, establishing (2.2) with $C = \dfrac{3\pi^2}{32T} \geq \dfrac{\pi^2}{64}$.

Now let $T\geq 6$ and $M_T=[T/2]$ so that $M_T\geq\dfrac{T}{2}-1\geq\dfrac{T}{3}$. For $j=0,\ldots,M_T-1$, let $t_j=-T/2+j+1$ so that the intervals $[t_j-1,t_j+1[$ are disjoint and $\bigcup_{j=0}^{N-1}[t_j-1,t_j+1[\subset[-T/2,T/2]$ thus

$$\begin{aligned}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}}b_ke^{2i\pi\lambda_kt}\right|^2\mathrm{d}t &\geq \sum_{j=0}^{M_T-1}\int_{t_j-1}^{t_j+1}\left|\sum_{k\in\mathbb{Z}}b_ke^{2i\pi\lambda_kt}\right|^2\mathrm{d}t\\ &= \sum_{j=0}^{M_T-1}\int_{-1}^{1}\left|\sum_{k\in\mathbb{Z}}b_ke^{2i\pi\lambda_kt_j}e^{2i\pi\lambda_kt}\right|^2\mathrm{d}t.\end{aligned}$$

Now, apply (2.4) with $a_k=b_ke^{2i\pi\lambda_kt_j}$ and $T=2$ to get

$$\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}}b_ke^{2i\pi\lambda_kt}\right|^2\mathrm{d}t\geq\frac{3\pi^2}{32}\frac{M_T}{T}\sum_{k\in\mathbb{Z}}|a_k|^2\geq\frac{\pi^2}{32}\sum_{k\in\mathbb{Z}}|a_k|^2,$$

establishing (2.2) with $C=\dfrac{\pi^2}{32}$. □

Finally, let us notice that, with a change of variable, and a simple limiting argument to remove the condition on the support of (a_k), we have just proved the following

Theorem 2.3 (Ingham)**.** *Let $\gamma>0$ and $T>\dfrac{1}{\gamma}$ and let $C(T,\gamma)$ be given by* (2.3)*. Then*

– for every sequence of real numbers $\Lambda=\{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lambda_{k+1}-\lambda_k\geq\gamma$;

– for every sequence $(a_k)_{k\in\mathbb{Z}}\in\ell^2(\mathbb{Z},\mathbb{C})$,

$$C(T,\gamma)\sum_{k\in\mathbb{Z}}|a_k|^2\leq\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k\in\mathbb{Z}}a_ke^{2i\pi\lambda_kt}\right|^2dt\leq 2\frac{\gamma T+1}{\gamma T}\sum_{k\in\mathbb{Z}}|a_k|^2. \tag{2.5}$$

2.2. *The condition* $T>\frac{1}{\gamma}$

We now show that the condition $T>\dfrac{1}{\gamma}$ can not be fully removed for (2.2) to hold for every Λ and every $P\in\mathcal{P}(\Lambda)$.

Proposition 2.4 (Ingham)**.** *Let $\gamma>0$. There exists a real sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ with $\lambda_{k+1}-\lambda_k\geq\gamma$ and a family of sequences $\big(a_k(\alpha)\big)_{k\in\mathbb{Z},0<\alpha<1/2}$ such that, if*

$$C\sum_{k=-N}^{N}|a_k|^2\leq\int_{-1/2\gamma}^{1/2\gamma}\left|\sum_{k=-N}^{N}a_ke^{2i\pi\lambda_kt}\right|^2dt \tag{2.6}$$

holds for every $N\geq 1$ and every $c_{-N},\ldots,c_N\in\mathbb{C}$, then $C=0$.

In other words, the condition $T > \dfrac{1}{\gamma}$ is necessary in Ingham's inequality to obtain an meaningful lower bound.

Proof. After scaling we again assume that $\gamma = 1$. Let $0 < \alpha < 1/2$ and define, for $|z| < 1$,

$$g_\alpha(z) = (1+z)^{-\alpha} = \frac{\exp\left(-i\alpha \arctan \frac{\Im(z)}{1+\Re(z)}\right)}{|1+z|^\alpha} \quad , \quad |z| < 1.$$

Of course, we may also write g_α as a power series

$$g_\alpha(z) = \sum_{n=0}^{+\infty} \frac{(-\alpha)_n}{n!} z^n$$

where $(\alpha)_0 = 1$, $(-\alpha)_n = -\alpha(-\alpha-1)\cdots(-\alpha-n+1)$.

Next define

$$\begin{aligned} f_\alpha(r,t) &= 2\Re\big(e^{i\pi(\alpha+1)t} g_\alpha(re^{2i\pi t})\big) \\ &= e^{i\pi(\alpha+1)t} g_\alpha(re^{2i\pi t}) + e^{-i\pi(\alpha+1)t} g_\alpha(re^{-2i\pi t}) \\ &= \sum_{n=0}^{+\infty} e^{2i\pi\left(n+\frac{\alpha+1}{2}\right)t} + \sum_{n=0}^{+\infty} \frac{(\alpha)_n}{n!} r^n e^{-2i\pi\left(n+\frac{\alpha+1}{2}\right)t}. \end{aligned}$$

Now set $\Lambda = \{\lambda_j\}_{j\in\mathbb{Z}}$ with $\lambda_j = j + \dfrac{\alpha+1}{2}$ when $j \geq 0$ and $\lambda_j = j + 1 - \dfrac{\alpha+1}{2}$ for $j \leq -1$, then $\lambda_{j+1} - \lambda_j \geq 1$ (and even $= 1$ excepted for $|\lambda_0 - \lambda_{-1}| = 1 + \alpha$). In particular, if we set

$$P_{m,r}(t) = \sum_{n=0}^{m} \frac{(-\alpha)_n}{n!} r^n e^{2i\pi\left(n+\frac{\alpha+1}{2}\right)t} + \sum_{n=0}^{m} \frac{(-\alpha)_n}{n!} r^n e^{-2i\pi\left(n+\frac{\alpha+1}{2}\right)t} := \sum_{k\in\mathbb{Z}} a_{m,r}(k) e^{2i\pi\lambda_k t} \in \mathcal{P}(\Lambda)$$

and $P_{m,r} \to f_\alpha$ when $m \to +\infty$, uniformely over $t \in [-1/2, 1/2]$.

Further, Parseval's relation reads

$$\sum_{n=0}^{+\infty} \left|\frac{(-\alpha)_n}{n!} r^n\right|^2 = \int_{-1/2}^{1/2} \left|\sum_{n=0}^{+\infty} \frac{(-\alpha)_n}{n!} r^n e^{2i\pi nt}\right|^2 \mathrm{d}t = \int_{-1/2}^{1/2} |g_\alpha(re^{2i\pi t})|^2 \,\mathrm{d}t$$

thus

$$\lim_{m\to+\infty} \sum_{k\in\mathbb{Z}} |a_{m,r}(k)|^2 = \lim_{m\to+\infty} 2 \sum_{n=0}^{+\infty} \left|\frac{(\alpha)_n}{n!} r^n\right|^2 = 2 \int_{-1/2}^{1/2} |g_\alpha(re^{2i\pi t})|^2 \,\mathrm{d}t.$$

It follows that, if we had

$$\int_{-1/2}^{1/2} |P_{m,r}(t)|^2 \,\mathrm{d}t \geq C \sum_{k\in\mathbb{Z}} |a_{m,r}(k)|^2 \tag{2.7}$$

then, letting $m \to +\infty$, we would also have

$$\int_{-1/2}^{1/2} |f_\alpha(r,t)|^2\,\mathrm{d}t \geq 2C \int_{-1/2}^{1/2} |g_\alpha(re^{2i\pi t})|^2\,\mathrm{d}t \tag{2.8}$$

for every $0 < r < 1$ and every $0 < \alpha < \dfrac{1}{2}$.

But, if we fix $t \in]-1/2, 1/2[$ then, when $r \to 1$,

$$g_\alpha(re^{\pm 2i\pi t}) = \frac{1}{\left(1 + re^{\pm 2i\pi t}\right)^\alpha} \to \frac{1}{\left(1 + e^{\pm 2i\pi t}\right)^\alpha} = \frac{e^{\mp i\alpha\pi t}}{2^\alpha \cos^\alpha \pi t}$$

(this is where we use that $T \leq 1$) while

$$|g_\alpha(re^{\pm 2i\pi t})|^2 = \frac{1}{((1-r)^2 + 4r\cos^2\pi t)^\alpha} \leq \frac{1}{4\cos^{2\alpha}\pi t}$$

for $\frac{1}{2} < r < 1$. Similar bounds follow for $f_\alpha(r,t)$:

$$f_\alpha(r,t) = e^{i\pi(\alpha+1)t} g_\alpha(re^{2i\pi t}) + e^{-i\pi(\alpha+1)t} g_\alpha(re^{-2i\pi t}) \to \frac{e^{i\pi t} + e^{-i\pi t}}{2^\alpha \cos^\alpha \pi t} = \frac{1}{2^{\alpha-1}\cos^{\alpha-1}\pi t}$$

while

$$|f_\alpha(r,t)|^2 \leq \frac{1}{\cos^{2\alpha}\pi t}.$$

When $2\alpha < 1$ the majorants are integrable so that we can let $r \to 1$ in (2.8). This leads to

$$2^{2-2\alpha} \int_{-1/2}^{1/2} \frac{\mathrm{d}t}{\cos^{2\alpha-2}\pi t} \geq 2^{1-2\alpha} C \int_{-1/2}^{1/2} \frac{\mathrm{d}t}{\cos^{2\alpha}\pi t}. \tag{2.9}$$

Letting $\alpha \to \dfrac{1}{2}$, the left hand side stays bounded while the right hand side goes to $+\infty$ unless $C = 0$. □

2.3. *Sequences with large gaps*

The results in this section are due to Haraux [11], though they are presented in a slightly different way. We here follow closely the presentation of Haraux's results in [17].

Lemma 2.5 (Haraux). *Let $\Lambda \subset \mathbb{R}$ be a sequence such that $\gamma(\Lambda) = \inf_{k \neq \ell \in \mathbb{Z}} |\lambda_k - \lambda_\ell| > 1$ and let $T > \gamma(\Lambda)^{-1}$.*

Assume that there exists $0 < C \leq 1 \leq B$ such that, for every $(a_\lambda)_{\lambda\in\Lambda}$ finitely supported sequence of complex numbers,

$$C \sum_{\lambda\in\Lambda} |a_\lambda|^2 \leq \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{\lambda\in\Lambda} a_\lambda e^{2i\pi\lambda t} \right|^2 \mathrm{d}t \leq B \sum_{\lambda\in\Lambda} |a_\lambda|^2.$$

Let $\mu \in \mathbb{R} \setminus \Lambda$ and, for sake of simplicity, assume that $\gamma(\Lambda \cup \{\mu\}) \geq 1$. Let $0 < \delta < \min\left(T, \dfrac{1}{4}\right)$, then there is a D with $0 < D < C$ such that, for every $(a_\lambda)_{\lambda \in \Lambda \cup \{\mu\}}$

$$D \sum_{\lambda \in \Lambda} |a_\lambda|^2 \leq \frac{1}{T+\delta} \int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}} \left| \sum_{\lambda \in \Lambda \cup \{\mu\}} a_\lambda e^{2i\pi\lambda t} \right|^2 \mathrm{d}t$$

Moreover, D can be taken of the form

$$D = \frac{C}{10B} \delta^4.$$

Note that, as δ is arbitrarily small, as $T > \gamma(\Lambda)^{-1}$ is arbitrarily near to $\gamma(\Lambda \setminus \{\lambda_0\})^{-1}$ so is $T + \delta$.

Proof. Set

$$f(t) = \sum_{\lambda \in \Lambda \cup \{\mu\}} a_\lambda e^{2i\pi\lambda t}.$$

For $0 < \delta < \dfrac{1}{4}$, we define

$$\begin{aligned} g_\delta(t) &= f(t) - \frac{1}{\delta} \int_{-\delta/2}^{\delta/2} e^{-2i\pi\mu s} f(t+s)\,\mathrm{d}s \\ &= \sum_{j \in \Lambda} a_\lambda \left(1 - \frac{\sin \pi(\lambda - \mu)\delta}{\pi(\lambda - \mu)\delta}\right) e^{2i\pi\lambda t} := \sum_{\lambda \in \Lambda} b_\lambda(\delta) e^{2i\pi\lambda t}. \end{aligned}$$

The first observation is that, as $|\lambda - \mu| \geq 1$,

$$\left|1 - \frac{\sin \pi(\lambda - \mu)\delta}{\pi(\lambda - \mu)\delta}\right|^2 \geq \eta_\delta := \inf_{s \geq \pi\delta} \left|1 - \frac{\sin s}{s}\right|^2.$$

Futher, as for $\lambda \neq \lambda' \in \Lambda$, then $|\lambda - \lambda'| \geq \gamma(\Lambda)$, the hypothesis of the lemma states that, as $T > \dfrac{1}{\gamma(\Lambda)}$,

$$\frac{1}{T} \int_{-T/2}^{T/2} |g_\delta(t)|^2 \,\mathrm{d}t \geq C \sum_{\lambda \in \Lambda} |b_\lambda(\delta)|^2 \geq C\eta_\delta \sum_{\lambda \in \Lambda} |a_\lambda|^2. \tag{2.10}$$

On the other hand, from Cauchy-Schwarz,

$$\begin{aligned} |g_\delta(t)|^2 &\leq 2|f(t)|^2 + 2\left|\frac{1}{\delta} \int_{-\delta/2}^{\delta/2} e^{-2i\pi\lambda_0 s} f(t+s)\,\mathrm{d}s\right|^2 \\ &\leq 2|f(t)|^2 + 2\frac{1}{\delta} \int_{-\delta/2}^{\delta/2} |f(t+s)|^2 \,\mathrm{d}s \end{aligned}$$

$$= \quad 2|f(t)|^2 + \frac{2}{\delta}\int_{t-\delta/2}^{t+\delta/2} |f(s)|^2 \,\mathrm{d}s.$$

It follows that

$$\begin{aligned}
\frac{1}{T}\int_{-T/2}^{T/2} |g_\delta(t)|^2 \,\mathrm{d}t &\leq \frac{2}{T}\int_{-T/2}^{T/2} |f(t)|^2 \,\mathrm{d}t + \frac{2}{\delta}\frac{1}{T}\int_{-T/2}^{T/2}\int_{t-\delta/2}^{t+\delta/2} |f(s)|^2, \mathrm{d}s \,\mathrm{d}t \\
&= \frac{2}{T}\int_{-T/2}^{T/2} |f(t)|^2 \,\mathrm{d}t + \frac{2}{T}\int_{-T/2-\delta/2}^{T/2+\delta/2} |f(s)|^2 \frac{1}{\delta}\int_{\max(-T/2,s-\delta/2)}^{\min(T/2,s+\delta/2)} \mathrm{d}t \,\mathrm{d}s \\
&\leq \frac{2}{T}\int_{-T/2}^{T/2} |f(t)|^2 \,\mathrm{d}t + \frac{2}{T}\int_{-T/2-\delta/2}^{T/2+\delta/2} |f(s)|^2 \,\mathrm{d}s \\
&\leq 4\frac{1}{T}\int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}} |f(s)|^2 \,\mathrm{d}s \leq 8\frac{1}{T+\delta}\int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}} |f(s)|^2 \,\mathrm{d}s
\end{aligned}$$

since we assumed that $T+\delta \leq 2T$.

With (2.10), we have thus shown that

$$\sum_{\lambda\in\Lambda} |a_\lambda|^2 \leq \frac{8}{C\eta_\delta}\frac{1}{T+\delta}\int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}} |f(s)|^2 \,\mathrm{d}s. \tag{2.11}$$

We now have to add $|a_\mu|^2$ on the left hand side. As $a_\mu = f - \sum_{\lambda\in\Lambda} a_\lambda e^{2i\pi\lambda t}$,

$$\begin{aligned}
|a_\mu|^2 &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |a_\mu|^2 \,\mathrm{d}t \leq 2\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 \,\mathrm{d}t + 2\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left|\sum_{\lambda\in\Lambda} a_\lambda e^{2i\pi\lambda t}\right|^2 \mathrm{d}t \\
&\leq 4\frac{1}{T+\delta}\int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}} |f(t)|^2 \,\mathrm{d}t + 2B\sum_{\lambda\in\Lambda} |a_\lambda|^2
\end{aligned}$$

where we used again that $T+\delta \leq 2T$ and the hypothesis of the lemma. But then, from (2.11), we get

$$|a_\mu|^2 \leq \left(4 + \frac{16B}{C\eta_\delta}\right)\frac{1}{T+\delta}\int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}} |f(s)|^2 \,\mathrm{d}s.$$

Adding this to (2.11), we finally get that

$$\begin{aligned}\sum_{\lambda\in\Lambda\cup\{\mu\}}|a_\lambda|^2 &\le \left(4+\frac{8\big(2B+1\big)}{C\eta_\delta}\right)\frac{1}{T+\delta}\int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}}|f(s)|^2\,\mathrm{d}s\\ &\le \left(4+\frac{24}{\eta_\delta}\right)\frac{B}{C}\frac{1}{T+\delta}\int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}}|f(s)|^2\,\mathrm{d}s\end{aligned}$$

since $C\le B\le 1$.

It remains to give an estimate of η_δ. First, as $\pi\delta\le\frac{\pi}{4}$,

$$\eta_\delta := \inf_{s\ge\pi\delta}\left|1-\frac{\sin s}{s}\right|^2=\left|1-\frac{\sin\pi\delta}{\pi\delta}\right|^2$$

Here we use *e.g.* that the $1-\dfrac{\sin s}{s}$ is increasing on $[0,\pi]$, ≥ 1 on $[\pi,2\pi]$ and

$$\left|1-\frac{\sin s}{s}\right|\ge 1-\frac{1}{2\pi}\ge 1-\frac{2\sqrt{2}}{\pi}=1-\frac{\sin\pi/4}{\pi/4}$$

for $s\ge 2\pi$. Writing $1-\dfrac{\sin s}{s}=t^2\sum_{k=0}^{+\infty}(-1)^{k-1}\dfrac{t^{2k}}{(2k+3)!}$ and using Cauchy products of power series, it is not hard to see that

$$\left(1-\frac{\sin s}{s}\right)^2=t^4\sum_{k=0}^{+\infty}(-1)^k w_k t^{2k}$$

with $w_0=\dfrac{1}{36}$, $w_1=\dfrac{1}{360}$ and $0\le w_{k+1}\le w_k$ for every k. In particular, for $t\le 1$,

$$\left|1-\frac{\sin s}{s}\right|^2\ge\frac{t^4}{36}\left(1-\frac{t^2}{10}\right).$$

In particular, as $\delta\pi\le\frac{\pi}{4}$,

$$\eta_\delta\ge\frac{\pi^4}{36}\left(1-\frac{\pi^2}{160}\right)\delta^4\ge\frac{5}{2}\delta^4.$$

As $\delta\le\dfrac{1}{4}$, it follows that

$$4+\frac{24}{\eta_\delta}\le\left(\frac{1}{64}+\frac{48}{5}\right)\delta^{-4}\le 10\frac{B}{C}\delta^{-4}.$$

We thus obtain the inequality with $D=\dfrac{C}{10B}\delta^4$ as claimed. □

We can now use this lemma to improve Ingham's Inequality.

Corollary 2.6 (Haraux [11]). *Let Λ be a sequence such that $\gamma(\Lambda) = \min_{\lambda\neq\lambda'\in\Lambda} |\lambda-\lambda'| \geq 1$ and let $F=\{\lambda_1,\ldots,\lambda_n\}\subset\Lambda$. Let $S > \dfrac{1}{\gamma(\Lambda\setminus F)}$. Then there exists two constants $0<D<1<B$ such that, for every finitely supported sequence $(a_\lambda)_{\lambda\in\Lambda}$,*

$$D\sum_{\lambda\in\Lambda}|a_\lambda|^2 \leq \frac{1}{S}\int_{-S/2}^{S/2}\left|\sum_{\lambda\in\Lambda}a_\lambda e^{2i\pi\lambda t}\right|^2 \mathrm{d}t \leq B\sum_{\lambda\in\Lambda}|a_\lambda|^2. \tag{2.12}$$

Proof. Set $\Lambda_0=\Lambda$ and, for $j=1,\ldots,n$, $\Lambda_j=\Lambda\setminus\{\lambda_1,\ldots,\lambda_n\}$ and $\gamma_j=\min_{\lambda\neq\lambda'\in\Lambda_j}|\lambda-\lambda'|\geq 1$.

From Ingham's Theorem, we know that

– for every $2>T>\dfrac{1}{\gamma_n}$

$$\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{\lambda\in\Lambda_n}a_\lambda e^{2i\pi\lambda t}\right|^2\mathrm{d}t \geq C_n\sum_{\lambda\in\Lambda_n}|a_\lambda|^2$$

with $C_n=\dfrac{\pi^2}{2^6}(T\gamma_n-1)$;

– for every $2>T>0$, and $j=0,\ldots,n$,

$$\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{\lambda\in\Lambda_j}a_\lambda a e^{2i\pi\lambda t}\right|^2\mathrm{d}t \leq B_j\sum_{\lambda\in\Lambda_j}|a_\lambda|^2$$

with $B_j=\dfrac{6(T\gamma_j+1)}{T\gamma_j}=6\left(1+\dfrac{1}{\gamma_j T}\right)$.

In particular, the upper bound in (2.12) is already established.

Now let $\dfrac{1}{\gamma(\Lambda\setminus F)}<T<S$ and $\delta=\dfrac{S-T}{n}$. One might take $T=\dfrac{1}{2}\left(S+\dfrac{1}{\gamma(\Lambda\setminus F)}\right)$ so that $\delta=\dfrac{1}{2n}\left(S-\dfrac{1}{\gamma(\Lambda\setminus F)}\right)$.

From Hauraux's lemma, we get

$$\frac{1}{T+\delta}\int_{-\frac{T+\delta}{2}}^{\frac{T+\delta}{2}}\left|\sum_{\lambda\in\Lambda_{n-1}}a_\lambda e^{2i\pi\lambda t}\right|^2\mathrm{d}t \geq C_{n-1}\sum_{\lambda\in\Lambda_{n-1}}|a_\lambda|^2$$

with $C_{n-1}=\delta^4\dfrac{C_n}{10B_n}$. We can therefore apply Haraux's lemma again, till we reach Λ_0. At each step, the constant C_{n-j} is replace by $C_{n-j-1}=\delta^4\dfrac{C_{n-j}}{10B_{n-j}}$ while the integral ranges over $\left[-\frac{T+(j+1)\delta}{2},\frac{T+(j+1)\delta}{2}\right]$.

This finally leads to

$$\frac{1}{T+n\delta}\int_{-\frac{T+n\delta}{2}}^{\frac{T+n\delta}{2}}\left|\sum_{\lambda\in\Lambda_{n-1}}a_\lambda e^{2i\pi\lambda t}\right|^2\mathrm{d}t \geq C_0\sum_{\lambda\in\Lambda}|a_\lambda|^2$$

with

$$C_0 = \left(\frac{\delta^4}{10}\right)^n \frac{C_n}{B_1 B_2 \cdots B_n} = \frac{\pi^2}{2^6}\left(\frac{\delta^4}{60}\right)^n \frac{\gamma_n T - 1}{\prod_{j=1}^n \left(1 + \frac{1}{\gamma_j T}\right)}$$

As $T + n\delta = S$, this establishes (2.12). □

3. L^1 estimates with integer frequencies

3.1. *Lacunary trigonometric polynomials*

In this section, we consider a sequence of integers (n_k) such that $\dfrac{n_{k+1}}{n_k} \geq q > 1$. Such sequences are called q-lacunary in the sense of Hadamard (or simply lacunary).

First note that a q-lacunary sequence is a finite union of q'-lacunary sequences with $q' \geq 3$. Indeed, it $q \geq 3$ there is nothing to prove and for $1 < q < 3$, take N an integer such that $q^N \geq 3$ and write $n_k^{(\ell)} = n_{\ell+kN}$ then $\left(n_k^{(\ell)}\right)_k$ is q^N-Lacunary and $\{n_k\} = \bigcup_{\ell=0}^{N-1}\{n_k^{(\ell)}\}$. Next, for $q \geq 3$, q-lacuary sequences have a particular arithmetic property:

Lemma 3.1. *Let $q \geq 3$ and $(n_k)_{k\geq 0}$ a sequence such that $n_0 \geq 1$ and $n_{k+1} \geq q n_k$. Consider two finite sequences $\varepsilon_\ell, \eta_\ell \in \{-1, 0, 1\}$ for $\ell = 0, \ldots, m$ and assume that*

$$\sum_{\ell=0}^m \varepsilon_\ell n_\ell = \sum_{\ell=0}^m \eta_\ell n_\ell \tag{3.1}$$

then $\varepsilon_\ell = \eta_\ell$ for every ℓ.

In other words, an integer can be represented in at most one way as $\sum \pm n_\ell$. Such a sequence is called *quasi-independent*.

Proof. First observe that $n_j \leq \dfrac{1}{q^{m-j}} n_m = \dfrac{q^j}{q^m} n_m$ for $j = 0, \ldots, m$.

Assume that (3.1) holds and define $\nu_\ell = \varepsilon_\ell - \eta_\ell$ so that

$$\sum_{\ell=0}^m \nu_\ell n_\ell = 0.$$

Assume towards a contradiction that there is an ℓ such that $\nu_\ell \neq 0$. Without loss of generality, we may assume that the largest such ℓ is m and, up to exchanging ε_ℓ and η_ℓ, that $\nu_m \geq 1$.

Observe that $\nu_\ell \in \{-2, -1, 0, 1, 2\}$ so that we obtain the desired contradiction writing

$$\begin{aligned} 0 = \sum_{\ell=0}^m \nu_\ell n_\ell &= \nu_m n_m + \sum_{\ell=0}^{m-1} \nu_\ell n_\ell \geq n_m - 2\sum_{\ell=0}^{m-1} n_\ell \geq n_m - 2\sum_{\ell=0}^{m-1} \frac{q^\ell}{q^m} n_m \\ &= \left(1 - \frac{2}{q^m}\frac{q^m - 1}{q - 1}\right) n_m = \frac{q^{m+1} - 3q^m + 2}{(q-1)q^m} n_m > 0 \end{aligned}$$

since $q \geq 3$ and $n_m > 0$. □

Note that this result is valid when the n_k's are real, not only for integers.

The aim of this section is to prove that trigonometric polynomials with lacunary frequencies have large L^1-norms of which the following estimate is a particular case: there exists a constant $C > 0$ such that, for every N,

$$\int_{-1/2}^{1/2} \left| \sum_{k=1}^{N} e^{2i\pi n_k t} \right| \mathrm{d}t \geq C\sqrt{N}.$$

This follows from a more general theorem which estimates L^p-norms of lacunary Fourier series. The aim of this section is to present this result. To do so, we follow closely [29, Chapter V.8] which goes through Rademacher series first, so let us introduce those series.

To start, let us denote by $\mathcal{D}_k = \{[j2^{-k-1}, (j+1)2^{-k-1}[, \ j = 0, \ldots, 2^{k+1} - 1\}$ the dyadic intervals of generation k and $\mathcal{D} = \bigcup_{k\geq 0} \mathcal{D}_k$ the set of all dyadic intervals. Also, if $I, J \in \mathcal{D}$ then either $I \cap J = \emptyset$ or $I \subset J$ or $J \subset I$. The Rademacher functions of generation k are then functions that take alternative values $+1$ and -1 on successive intervals in $\mathcal{D}_k$, that is

$$r_k(t) = \sum_{j=0}^{2^{k+1}-1} (-1)^j \mathbb{1}_{]j2^{-k-1},(j+1)2^{-k-1}[} = \operatorname{sign}\bigl(\sin(2\pi 2^j t)\bigr).$$

The first observation is that, if $I \in \mathcal{D}_\ell$ and $k > \ell$ then r_k takes the value $+1$ on half of I and -1 on the other half so that $\int_I r_k = 0$. A first consequence is that r_k is orthogonal to r_ℓ in $L^2([0,1])$ since r_ℓ is constant on each $I \in \mathcal{D}_\ell$ so that $\int_I r_k r_\ell = 0$ and $\mathcal{D}_\ell$ is a covering of $[0,1]$. Moreover, as $|r_k| = 1$, the family $(r_k)_{k\geq 0}$ is an orthonormal sequence in $L^2([0,1])$.

In particular, we now fix a sequence $(c_j)_{j\geq 0}$ such that $\sum_{k=0}^{+\infty} |c_k|^2$ converges, we can define

$$f = \sum_{k=0}^{+\infty} c_k r_k$$

and this series converges in $L^2([0,1])$ thus $f \in L^2([0,1])$. We actually have a bit better:

Theorem 3.2. *If* $\sum_{k=0}^{+\infty} |c_k|^2 < +\infty$ *then there exists* $f \in L^2([0,1])$ *defined by*

$$f = \sum_{k=0}^{+\infty} c_k r_k$$

and this series converges both in $L^2([0,1])$ *and almost everywhere.*

Proof. The L^2 convergence has already been established. Further, let $F = \int f$ be the indefinite integral of f and let $E \subset [0,1]$ be the set of Lebesgue points of f so that $|E| = 1$ and on E, F' exists and is finite.

Now let, $S_n[f]$ be the n-th partial sum of this series

$$\mathcal{S}_n[f](x) = \sum_{k=0}^{n} c_k r_k(x).$$

As $\mathcal{S}_n[f] \to f$ in $L^2([0,1])$, for every $0 \le a < b \le 1$,

$$\left| \int_a^b \big(f(x) - \mathcal{S}_n[f](x)\big)\,\mathrm{d}x \right| \le \int_0^1 |f(x) - \mathcal{S}_n[f](x)|\,\mathrm{d}x \le \left(\int_0^1 |f(x) - \mathcal{S}[f](x)|^2\,\mathrm{d}x \right)^{1/2} \to 0.$$

We have just shown that, if I is an interval, then $\int_I \mathcal{S}_n[f] \to \int_I f$ thus also, if we fix $\ell \ge 1$,

$$\int_I (\mathcal{S}_n[f] - \mathcal{S}_{\ell-1}[f]) \to \int_I (f - \mathcal{S}_{\ell-1}[f]).$$

On the other hand, if $I \in \mathcal{D}_{\ell-1}$ and $k \ge \ell$, then $\int_I r_k = 0$ so that $\int_I \mathcal{S}_n[f] = \int_I \mathcal{S}_{\ell-1}[f]$. Letting $n \to +\infty$ we obtain that

$$\int_I f(x)\,\mathrm{d}x = \int_I \mathcal{S}_{\ell-1}[f](x)\,\mathrm{d}x \quad \text{for every} I \in \mathcal{D}_{\ell-1}.$$

Next, let $x_0 \in E$ not a dyadic rational ($x_0 \ne \dfrac{p}{2^q}$, $p, q \in \mathbb{N}$) and let $I_k =]j2^{-k}, (j+1)2^{-k}[$ be such that $x_0 \in E \cap I_k$. Then, as $\mathcal{S}_{k-1}[f]$ is constant over I_k

$$\mathcal{S}_{k-1}[f](x_0) = \frac{1}{|I_k|} \int_{I_k} \mathcal{S}_k[f](x)\,\mathrm{d}x = \frac{1}{|I_k|} \int_{I_k} f(x)\,\mathrm{d}x \to F'(x_0)$$

when $k \to +\infty$. □

The second result is that f is actually in every L^p space:

Theorem 3.3. *Let $(c_k) \in \ell^2$ and $f = \sum_{k=0}^{+\infty} c_k r_k$. Then, for $1 \le p < +\infty$, $f \in L^p([0,1])$. Moreover, there exists A_p, B_p, depending on p only, such that*

$$A_p \left(\sum_{k=0}^{+\infty} |c_k|^2 \right)^{\frac{1}{2}} \le \left(\int_0^1 |f(x)|^p\,\mathrm{d}x \right)^{\frac{1}{p}} \le B_p \left(\sum_{k=0}^{+\infty} |c_k|^2 \right)^{\frac{1}{2}}.$$

Proof. Let us first notice that the theorem holds for $p = 2$ since

$$\gamma := \left(\int_0^1 |f(x)|^2\,\mathrm{d}x \right)^{\frac{1}{2}} = \left(\sum_{k=0}^{+\infty} |c_k|^2 \right)^{\frac{1}{2}}$$

i.e. the inequalities are equalities with $A_2 = B_2 = 1$.

Next, let us notice that this implies the lower bound when $p > 2$ with $A_p = 1$ since then, with Hölder

$$\left(\int_0^1 |f(x)|^p\,\mathrm{d}x \right)^{\frac{1}{p}} \ge \left(\int_0^1 |f(x)|^2\,\mathrm{d}x \right)^{\frac{1}{2}} = \gamma.$$

It also implies the upper bound with $B_p = 1$ for $p < 2$ since now Hölder implies that

$$\left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\int_0^1 |f(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \gamma$$

Further, take $2(m-1) < p \leq 2m$ for some integer $m \geq 2$, and assume that that the upper bound

$$\left(\int_0^1 |f(x)|^{2m} \,\mathrm{d}x\right)^{\frac{1}{2m}} \leq B_{2m}\gamma \tag{3.2}$$

holds. Then Hölder implies that

$$\left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\int_0^1 |f(x)|^{2m} \,\mathrm{d}x\right)^{\frac{1}{2m}} \leq B_{2m}\gamma$$

that is, $B_p \leq B_{2m}$ for $2(m-1) < p \leq 2m$.

Next, let us show that the upper bound for $p = 4$ implies the lower bound for $p < 2$. Assume for the moment that we are able to prove that

$$\left(\int_0^1 |f(x)|^4 \,\mathrm{d}x\right)^{\frac{1}{4}} \leq B_4\gamma.$$

Let $1 \leq q < 2$ and write $2 = qt + 4(1-t)$, that is, take $t = \dfrac{2}{4-q}$. Then, from Hölder

$$\begin{aligned}\gamma^2 &= \int_0^1 |f(x)|^2 \,\mathrm{d}x = \int_0^1 |f(x)|^{qt}|f(x)|^{4(1-t)} \,\mathrm{d}x \leq \left(\int_0^1 |f(x)|^q \,\mathrm{d}x\right)^t \left(\int_0^1 |f(x)|^4 \,\mathrm{d}x\right)^{1-t} \\ &\leq (B_4\gamma)^{4(1-t)} \left(\int_0^1 |f(x)|^q \,\mathrm{d}x\right)^t = (B_4\gamma)^{2-qt} \left(\int_0^1 |f(x)|^q \,\mathrm{d}x\right)^t\end{aligned}$$

thus

$$\left(\int_0^1 |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \geq B_4^{1-\frac{4-q}{q}}\gamma.$$

So it remains to prove (3.2) for every $m \geq 2$. Notice also that it is enough to prove this inequality with real c_k's. The constant in the complex case is then multiplied by 2: write $f = f_r + if_i$ where $f_r = \sum \Re(c_k)r_k$ and $f_i = \sum \Im(c_k)r_k$. Then

$$\|f\|_{2m} \leq \|f_r\|_{2m} + \|f_i\|_{2m} \leq B_{2m}^{\mathbb{R}} \left[\left(\sum_{k=0}^{+\infty} |\Re(c_k)|^2\right)^{\frac{1}{2}} + \left(\sum_{k=0}^{+\infty} |\Im(c_k)|^2\right)^{\frac{1}{2}}\right] \leq 2B_{2m}^{\mathbb{R}}\gamma$$

since $|\Re(c_k)|, |\Im(c_k)| \leq |c_k|$.

To conclude, we write

$$\int_0^1 S_n[f](x)^{2m}\,\mathrm{d}x = \sum_{\ell_0+\cdots+\ell_n=2m} A_{\ell_0,\ldots,\ell_n} c_0^{\ell_0}\cdots c_n^{\ell_n} \int_0^1 r_0^{\ell_0}(x)\cdots r_n^{\ell_n}(x)\,\mathrm{d}x$$

where $\ell_j \geq 0$ for every j and

$$A_{\ell_0,\ldots,\ell_n} = \frac{(\ell_0+\cdots+\ell_n)!}{\ell_0!\cdots\ell_j!}.$$

Now observe that

$$\int_0^1 r_0^{\ell_0}(x)\cdots r_n^{\ell_n}(x)\,\mathrm{d}x = \begin{cases} 1 & \text{if all the } \ell_j\text{'s are even} \\ 0 & \text{otherwise} \end{cases}$$

and that

$$\left(\sum_{k=0}^n c_k^2\right)^m = \sum_{\ell_0+\cdots+\ell_n=m} A_{\ell_0,\ldots,\ell_n} (c_0^2)^{\ell_0}\cdots(c_n^2)^{\ell_n}.$$

Further, when $\ell_0+\cdots+\ell_n = m$,

$$\frac{A_{2\ell_0,\ldots,2\ell_n}}{A_{\ell_0,\ldots,\ell_n}} = \frac{(m+1)(m+2)\cdots 2m}{\prod_{j=0}^n(\ell_j+1)(\ell_j+2)\cdots 2\ell_j} \leq \frac{(m+1)(m+2)\cdots 2m}{2^m} \leq m^m$$

(with the convention that the denominator is $(\ell_j+1)(\ell_j+2)\cdots 2\ell_j = 1$ when $\ell_j = 0$). It follows that

$$\int_0^1 S_n[f](x)^{2m}\,\mathrm{d}x \leq m^m \left(\sum_{k=0}^n |c_k|^2\right)^m.$$

As $S_n[f] \to f$ a.e., we conclude that

$$\left(\int_0^1 |f(x)|^{2m}\,\mathrm{d}x\right)^{\frac{1}{2m}} \leq m^{1/2}\left(\sum_{k=0}^{+\infty}|c_k|^2\right)^{\frac{1}{2}}$$

that is $B_{2m} = 2m^{1/2}$. □

The estimate $B_{2m} = 2m^{1/2}$ allows to improve a bit the result:

Corollary 3.4. *Let* $(c_k) \in \ell^2$ *and* $f = \sum_{k=0}^{+\infty} c_k r_j$. *Then, for every* $\mu > 0$, $\exp(\mu|f|^2) \in L^1([0,1])$.

Proof. Let us fix $\mu > 0$. We first show that if $\gamma := \|c_j\|_2$ is small enough, then $\exp(\mu|f|^2) \in L^1([0,1])$. Indeed

$$\int_0^1 \exp(\mu|f(x)|^2)\,\mathrm{d}x = \sum_{m=0}^{+\infty}\frac{\mu^m}{m!}\int_0^1 |f(x)|^{2m}\,\mathrm{d}x \leq \sum_{m=0}^{+\infty}\frac{m^m}{m!}(4\mu\gamma^2)^m. \tag{3.3}$$

But $\frac{m^m}{m!} \leq \sum_{n=0}^{+\infty} \frac{m^n}{n!} = e^m$ so that

$$\int_0^1 \exp(\mu|f(x)|^2)\,\mathrm{d}x \leq \sum_{m=0}^{+\infty} (4e\mu\gamma^2)^m = \frac{1}{1-4e\mu\gamma^2} < +\infty$$

provided $\gamma^2 < \frac{1}{4e\mu}$.

Next, take any $f \in L^1(0,1)$, and apply the first part to $f - \mathcal{S}_n[f] = \sum_{j=n+1}^{+\infty} c_j r_j$. As $\gamma_n^2 := \sum_{j=n+1}^{+\infty} |c_j|^2 \to 0$, for n large enough $\gamma_n^2 < \frac{1}{8e\mu}$ thus $\exp(2\mu|f - \mathcal{S}_n[f]|^2) \in L^1([0,1])$.

Finally, as $|f|^2 \leq 2|f - \mathcal{S}_n[f]| + 2|\mathcal{S}_n[f]|^2$, we have

$$\exp(\mu|f|^2) \leq \exp(2\mu|f - \mathcal{S}_n[f]|^2)\exp(2\mu|\mathcal{S}_n[f]|^2) \in L^1$$

since $|\mathcal{S}_n[f]| \in L^\infty$ thus also $\exp(2\mu|\mathcal{S}_n[f]|^2) \in L^\infty$. □

Next, we consider series of the form

$$\sum_{j=0}^{+\infty} c_j e^{2i\pi jt} r_j(x).$$

The idea is that such series are of the form $\sum \pm c_j e^{2i\pi jt}$, that is, chosing $x \in (0,1)$ at random, we randomly change the sign of c_j. Our first result is the following:

Theorem 3.5. *Let* $(c_k) \in \ell^2$ *and* $f_x(t) = \sum_{k=0}^{+\infty} c_k r_k(x) e^{2i\pi kt}$. *Then, for almost every* $x \in (0,1)$, *the series converges almost everywhere in* $t \in (0,1)$ *and* $f_x \in L^p([0,1])$ *for every* $1 \leq p < +\infty$.

Proof. Let E be the set of $(x,t) \in [0,1]^2$ where the series defining f converges.

According to Theorem 3.2, for every $t \in [0,1]$, the set $E_t^2 = \{(x,t) \in E\}$ has measure $|E_x| = 1$. It follows that $|E| = 1$ but then, for almost every $x \in [0,1]$, $E_x^1 = \{(x,t) \in E\}$ has also measure $|E_x^1| = 1$.

Next, set $\gamma = \|c_k\|_2$ and fix $n \geq 1$. As in (3.3),

$$\frac{\mu^n}{n!}\int_0^1 |f_x(t)|^{2n}\,\mathrm{d}x = \sum_{m=0}^{+\infty} \frac{\mu^m}{m!}\int_0^1 |f_x(t)|^{2m}\,\mathrm{d}x = \int_0^1 \exp(\mu|f_x(t)|^2)\,\mathrm{d}x \leq \frac{1}{1-4e\mu\gamma^2} \tag{3.4}$$

provided $\mu < \frac{1}{4e\gamma^2}$. It follows that

$$\int_0^1\int_0^1 |f_x(t)|^{2n}\,\mathrm{d}t\,\mathrm{d}x = \int_0^1\int_0^1 |f_x(t)|^{2n}\,\mathrm{d}x\,\mathrm{d}t \leq \frac{n!}{(1-4e\mu\gamma^2)\mu^n} < +\infty.$$

But then, for every n, there is a set $F_n \subset [0,1]$ with $|F_n| = 0$ such that, if $x \in [0,1] \setminus F_n$, $\displaystyle\int_0^1 |f_x(t)|^{2n}\,\mathrm{d}t < +\infty$. Setting $F = \bigcup F_n$, $|F| = 0$ and, for every $x \in [0,1] \setminus F$, for every n, $f_x \in L^{2n}$. Using the inclusion $L^{2n}([0,1]) \subset L^p([0,1])$ when $p \leq 2n$, we obtain that, for almost every x, $f_x \in L^p([0,1])$ for every $p \geq 1$, as claimed. □

We can now prove the main result of this section:

Theorem 3.6. *Let $q > 1$ and $(n_j)_{j\geq 0}$ be a q-lacunary sequence of integers, $n_0 \geq 1$ and $n_{j+1} \geq qn_j$. Let $1 \leq p < +\infty$. There are two constants $A_{p,q}, B_{p,q}$ such that, if $(c_j)_{j\geq 0} \in \ell^2$, then $g(t) = \sum_{j\geq 0} c_j e^{2i\pi n_j t}$ is in $L^p([0,1])$ with*

$$A_{p,q}\left(\sum_{j=0}^{+\infty}|c_j|^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 \left|\sum_{j\geq 0} c_j e^{2i\pi n_j t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_{p,q}\left(\sum_{j=0}^{+\infty}|c_j|^2\right)^{\frac{1}{2}}. \tag{3.5}$$

Remark 3.7. Note that a simple change of variable also shows that, for every integer M,

$$A_{p,q}\left(\sum_{j=0}^{+\infty}|c_j|^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{M}\int_{-M/2}^{M/2} \left|\sum_{j\geq 0} c_j e^{2i\pi \frac{n_j}{M} t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_{p,q}\left(\sum_{j=0}^{+\infty}|c_j|^2\right)^{\frac{1}{2}}. \tag{3.6}$$

Also, we may assume that $q \to A_{p,q}, B_{p,q}$ are continuous.

Proof. The beginning of the proof is the same as for Theorem 3.3. Parseval's identify shows that (3.5) is satisfied when $p = 2$ with $A_{2,q} = B_{2,q} = 1$. The lower bound is then automatically satisfied for $p \geq 2$ with $A_{p,q} = 1$ while the upper bound is satisfied for $p \leq 2$ with $B_{2,q} = 1$. Finally, if we establish the upper bound for $p > 2$, using Hölder's inequality in the same way as in the proof of Theorem 3.3, the lower bound follows for $p < 2$ with $A_{2,q} = B_{4,q}^{1-\frac{4-p}{p}}$. Also, it is enough to prove the upper bound when $p = 2m$, $m \geq 2$ and then, if $2(m-1) < p \leq 2m$, $B_{p,q} \leq B_{2m,q}$. Another reduction is that, by homogeneity, it is enough to prove the theorem when $\sum_{j=0}^{+\infty}|c_j|^2 = 1$.

A further restriction is that it is enough to prove the theorem for $q \geq 3$. Indeed, for $1 < q < 3$, we introduce an integer N_q such that $q^{N_q} \geq 3$ and write $n_k^{(\ell)} = n_{kN_q+\ell}$ for $\ell = 0, \ldots, N_q - 1$. Then $n_{k+1}^{(\ell)} \geq q^{N_q} n_k^{(\ell)}$. If the theorem is established when $q \geq 3$ then, for each ℓ, the upper bound in (3.5) reads

$$\left(\int_0^1 \left|\sum_{k\geq 0} c_{kN_q+\ell} e^{2i\pi n_k^{(\ell)} t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_{p,q^{N_q}}\left(\sum_{k=0}^{+\infty}|c_{kN_q+\ell}|^2\right)^{\frac{1}{2}}.$$

But then, with the triangular inequality in L^p,

$$\left(\int_0^1 \left|\sum_{j\geq 0} c_j e^{2i\pi n_j t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} = \left(\int_0^1 \left|\sum_{\ell=0}^{N_q-1}\sum_{k\geq 0} c_{kN_q+\ell} e^{2i\pi n_k^{(\ell)} t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\leq \sum_{\ell=0}^{N_q-1}\left(\int_0^1\left|\sum_{k\geq 0} c_{kN_q+\ell}e^{2i\pi n_k^{(\ell)}t}\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\\
&\leq B_{p,q^{N_q}}\sum_{\ell=0}^{N_q-1}\left(\sum_{k=0}^{+\infty}|c_{kN_q+\ell}|^2\right)^{\frac{1}{2}}\\
&\leq N_q^{1/2}B_{p,q^{N_q}}\left(\sum_{\ell=0}^{N_q-1}\sum_{k=0}^{+\infty}|c_{kN_q+\ell}|^2\right)^{\frac{1}{2}}\\
&= N_q^{1/2}B_{p,q^{N_q}}\left(\sum_{j\geq 0}|c_j|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

where we have used Cauchy-Schwarz in $\mathbb{R}^{N_q}$ in the next to last line.

A last reduction comes from the observation that, for every k

$$
\int_0^1 \exp(\mu|g(t)|^2)\,\mathrm{d}t = \sum_{n=0}^{+\infty}\frac{\mu^n}{n!}\int_0^1 |g(t)|^{2n}\,\mathrm{d}t \geq \frac{\mu^m}{m!}\int_0^1 |g(t)|^{2m}\,\mathrm{d}t
$$

It is therefore enough to prove that there is a $\mu(q)$ and a $C>0$ such that, if $\mu<\mu(q)$

$$
\int_0^1 \exp(\mu|g(t)|^2)\,\mathrm{d}t \leq C \tag{3.7}
$$

which would then imply that

$$
\int_0^1 |g(t)|^{2k}\,\mathrm{d}t \leq C\frac{m!}{\mu^m}
$$

as desired.

In order to prove (3.7), let us introduce

$$
f_x(t) = \sum_{j\geq 0} c_j r_{n_j}(x)e^{2i\pi n_j t}.
$$

Integrating (3.4) with respect to t and using Fubini, we deduce that

$$
\int_0^1\int_0^1 \exp(\mu|f_x(t)|^2)\,\mathrm{d}t\,\mathrm{d}x \leq K := \frac{1}{1-4e\mu\gamma^2}.
$$

But then, there is an x_0 (that we can assume not to be a dyadic rational $x_0\neq 2^j/k$) such that

$$
\int_0^1 \exp(\mu|f_{x_0}(t)|^2)\,\mathrm{d}t \leq K.
$$

Next, we consider the Riesz product

$$
P_k(x) = \prod_{j=0}^{k}\bigl(1+r_{n_j}(x_0)\cos 2\pi n_j t\bigr) = \prod_{j=0}^{k}\left(1+r_{n_j}(x_0)\frac{e^{2\pi n_j t}+e^{-2\pi n_j t}}{2}\right) = \sum_{j\in\mathbb{Z}}\gamma_j e^{2i\pi j t}
$$

where the Fourier coefficients have the following property:

– $\gamma_0 = 1$;

– $\gamma_j = 0$ if j is an integer that is not of the form $\sum \pm n_\ell$, in particular when $|j| > \sum_{\ell=0}^{n} n_\ell$;

– if $j = \sum \varepsilon_\ell n_\ell$ with $\varepsilon_\ell \in \{-1, 0, 1\}$. As $q > 3$, this ε_ℓ's are unique. Then $\gamma_j = \prod_{\varepsilon_\ell \neq 0} \frac{r_{n_\ell}(x_0)}{2}$. In particular, $\gamma_{n_j} = \frac{r_{n_j}(x_0)}{2}$ for $j = 0, \ldots, k$ and $\gamma_{n_j} = 0$ for $j > k$.

As a consequence, the partial sums of the Fourier series of g are given by

$$S_{n_k}[g](t) := \sum_{j=0}^{k} c_j e^{2i\pi n_j t} = \sum_{j=0}^{k} c_j r_j(x_0)^2 e^{2i\pi n_j t} = 2\int_0^1 f_{x_0}(s) P_k(t-s)\,\mathrm{d}s.$$

Note that $P_k \geq 0$ and $\int_0^1 P_k(t)\,\mathrm{d}t = \gamma_0 = 1$ so that $\nu_k = P_k(t)\,\mathrm{d}t$ is a probability measure. As $\varphi(s) = \exp(\mu s^2)$ is increasing and convex, we apply Jensen's inequality (with the measure ν_k) to obtain

$$\varphi\left(\frac{1}{2} S_{n_k}[g](t)\right) \leq \varphi\left(\int_0^1 |f_{x_0}(s)| P_k(t-s)\,\mathrm{d}s\right) \leq \int_0^1 \varphi\big(|f_{x_0}(s)|\big) P_k(t-s)\,\mathrm{d}s.$$

Integrating over $[0,1]$ and using Fubini, we get

$$\int_0^1 \varphi\left(\frac{1}{2} S_{n_k}[g](t)\right)\,\mathrm{d}t \leq \int_0^1 \varphi\big(|f_{x_0}(s)|\big) \int_0^1 P_k(t-s)\,\mathrm{d}t\,\mathrm{d}s = \int_0^1 \varphi\big(|f_{x_0}(s)|\big)\,\mathrm{d}s \leq K.$$

Letting $k \to +\infty$, we obtain

$$\int_0^1 \exp\left(\frac{\mu}{2}|g(t)|^2\right)\,\mathrm{d}t \leq K$$

as claimed (up to $\mu/2$ instead of μ). □

3.2. *The proof of Littlewood's conjecture by Mc Gehee, Pigno and Smith*

We will now give the proof of the Littlewood conjecture, *i.e.* of Theorem B, following closely [4]. Let us recall the statement:

Theorem 3.8 (Mc Gehee, Pigno, Smith)**.** *There exists a constant $C_{MPS} < \frac{4}{\pi^2}$ such that, if $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence of integers and (a_k) is a complex sequence with finite support, then*

$$C_{MPS} \sum_{k=0}^{+\infty} \frac{|a_k|}{k+1} \leqslant \int_0^1 \left|\sum_{k=0}^{+\infty} a_k e^{2i\pi n_k t}\right| \mathrm{d}t.$$

The proof given below will give $C_{MPS} = \dfrac{1}{96}$ which is not the best possible. We avoid minor technicalities which would lead to $C_{MPS} = \dfrac{1}{30}$ (*see e.g.* [7, 28]). The proof in [15] requires the introduction of various parameters and a cumbersome optimization of those parameters in the final step and leads to $C_{MPS} = \dfrac{1}{26}$.

Strategy of the proof

It will be convenient to write both $\widehat{F}(s) = c_s(F)$ for the Fourier coefficients of $F \in L^2([0,1])$.

We fix a trigonometric polynomial

$$\phi(t) = \sum_{k=0}^{N} a_k e^{2i\pi n_k t} \quad \text{and} \quad S = \sum_{k=0}^{N} \frac{|a_k|}{k+1}. \tag{3.8}$$

We then write $|a_k| = a_k u_k$ with u_k complex numbers of modulus 1 and we introduce

$$T_0(t) = \sum_{k=0}^{N} \frac{u_k}{k+1} e^{-2i\pi n_k t}. \tag{3.9}$$

Then by Parseval's identity we have

$$S := \sum_{k=0}^{N} \frac{|a_k|}{k+1} = \sum_{k=0}^{N} \widehat{\phi}(n_k)\widehat{T_0}(-n_k) = \int_0^1 \phi(t) T_0(t)\,\mathrm{d}t. \tag{3.10}$$

so that

$$S \leqslant \|\phi\|_1 \|T_0\|_{L^\infty([0,1])}.$$

The issue is that, typically we have no control over the L^∞ norm of T_0 other than the trivial and explosive control by $\sum \frac{1}{k}$, so we will correct T_0 into another test function T_1 as follows;

1. The L^∞ norm of the corrected function T_1 is controlled by a constant C, $\|T_1\|_\infty \le C$.
2. $\widehat{T_1}(-n_k)$ only differs a little from $\widehat{T_0}(-n_k)$ for $0 \leqslant k \leqslant N$, say $|\widehat{T_1}(-n_k) - \widehat{T_0}(-n_k)| \le \dfrac{\widehat{T_0}(-n_k)}{2} = \dfrac{1}{2k}$, while we impose no condition on the behavior of $\widehat{T_1}(n)$ for $n \neq -n_k$.

We would then conclude as follows:

$$\left| S - \int_0^1 \phi(t) T_1(t)\,\mathrm{d}t \right| = \left| \int_0^1 \phi(t)(T_0 - T_1)(t)\,\mathrm{d}t \right| = \left| \sum_{k=0}^{N} \widehat{\phi}(\lambda_k)(\widehat{T_0}(-n_k) - \widehat{T_1}(-n_k)) \right|$$

with Parseval. With the triangular inequality and our expected estimate $|\widehat{T_1}(-n_k) - \widehat{T_0}(-n_k)| \le \dfrac{1}{2k}$, we then conclude that

$$\left| S - \int_0^1 \phi(t) T_1(t)\,\mathrm{d}t \right| \le \sum_{k=1}^{N} |\widehat{\phi}(\lambda_k)|\, |\widehat{T_0}(-n_k) - \widehat{T_1}(-n_k)| \le \sum_{k=1}^{N} \frac{|a_k|}{2k} = \frac{S}{2}.$$

But then

$$\frac{1}{2}S \leq \left| \int_0^1 \phi(t)T_1(t)\,\mathrm{d}t \right| \leq \|T_1\|_\infty \int_0^1 |\phi(t)|\,\mathrm{d}t \leq C \int_0^1 |\phi(t)|\,\mathrm{d}t$$

which is the expected result with $A = \dfrac{1}{2C}$.

The proof of the Theorem 3.8 (Theorem B) thus amounts to proving the following lemma:

Lemma 3.9. *There exists a universal constant C and a $T_1 \in L^\infty$ such that*

1. $\|T_1\|_\infty \leqslant C$
2. $|\widehat{T}_1(-n_k) - \widehat{T}_0(-n_k)| \leqslant \frac{1}{2}|\widehat{T}_0(-n_k)| = \frac{1}{2}\frac{1}{k+1}$ *for* $0 \leqslant k \leqslant N$

where T_0 is the function given by (3.9).

The remaining of this section is devoted to the proof of this lemma.

Proof of Lemma 3.9

First note that, up to eventually adding extra zeros to the sequence (a_k) and adding $\lambda_{N+j} = \lambda_N + j$ to the sequence (λ_k), we may assume that $N = 2^{m+1} - 1$.

We start by decomposing T_0 into a sum of dyadic blocs on which the amplitude $|\widehat{T}_0(-\lambda_k)| = \dfrac{1}{k}$ is more or less constant. More precisely, for $j = 0, \ldots m$ we set $\mathcal{D}_j = [2^j, 2^{j+1}[$ and

$$f_j = \sum_{k \in \mathcal{D}_j} \frac{u_k}{k} e^{-2i\pi n_k t}.$$

The function T_0 now appears as the partial sum of order m of the series $\sum f_j$, in other words

$$T_0 = \sum_{j=0}^{m} f_j.$$

Let us start with a simple lemma:

Lemma 3.10. *With the above notations, we have*

1. $\|f_j\|_{L^\infty([0,1])} \leqslant 1$.
2. $\|f_j\|_{L^2([0,1])} \leqslant 2^{-j/2}$.

Proof. By construction, $|I_j| = 2^j$ hence

$$\|f_j\|_\infty \leqslant \sum_{k \in \mathcal{D}_j} \frac{1}{k} \leqslant \frac{|\mathcal{D}_j|}{2^j} = 1.$$

On the other hand, Parseval implies that

$$\|f_j\|_2^2 = \sum_{k=2^j}^{2^{j+1}-1} \frac{1}{k^2} \leqslant \frac{2^j}{4^j} = 2^{-j}$$

as claimed. □

Now, write the Fourier series of each $|f_j| \in L^2([0,1])$

$$|f_j| = \sum_{s\in\mathbb{Z}} c_s(|f_j|)e^{2i\pi st}.$$

To each $|f_j|$, we associate $h_j \in L^2([-\pi,\pi])$ defined via its Fourier series as

$$h_j(t) = c_0(|f_j|) + 2\sum_{s=1}^{\infty} c_s(|f_j|)e^{2i\pi st}.$$

Lemma 3.11. *For $0 \le j \le n$, the following properties hold*

1. $\mathrm{Re}(h_j) = |f_j| \leqslant 1$;
2. $\|h_j\|_{L^2([0,1])} \le \sqrt{2}\|f_j\|_{L^2([0,1])}$.

Proof. First, as $|f_j|$ is real valued, $c_0(|f_j|)$ is also real while $\overline{c_s(|f_j|)} = c_{-s}(|f_j|)$ for every $s \ge 1$. Hence

$$\overline{h_j}(t) = c_0(|f_j|) + 2\sum_{s=1}^{\infty} c_{-s}(|f_j|)e^{-2i\pi st}$$

and thus

$$\mathrm{Re}(h_j) = \frac{h_j + \overline{h_j}}{2} = c_0(|f_j|) + \sum_{s\neq 0} c_s(|f_j|)e^{2i\pi st} = |f_j| \leqslant 1$$

by lemma 3.10.

By Parseval's identity and again $\overline{c_s(|f_j|)} = c_{-s}(|f_j|)$,

$$\begin{aligned}\|h_j\|_2^2 &= \big|c_0(|f_j|)\big|^2 + 4\sum_{s=1}^{\infty}\big|c_s(|f_j|)\big|^2 = \big|c_0(|f_j|)\big|^2 + 2\sum_{s=1}^{\infty}\big|c_s(|f_j|)\big|^2 + 2\sum_{s=1}^{\infty}\big|c_{-s}(|f_j|)\big|^2 \\ &\leqslant 2\sum_{s\in\mathbb{Z}} |c_s(|f_j|)|^2 = 2\|f_j\|_2^2\end{aligned}$$

as claimed. □

We now define a sequence $(F_j)_{j=0,\dots,n}$ inductively through

$$F_0 = f_0 \quad \text{and} \quad F_{j+1} = F_j e^{-\eta h_{j+1}} + f_{j+1}$$

where $0 < \eta \le 1$ is a real number to be adjusted later on. Further set

$$E_\eta := \sup_{0<x\le 1} \frac{x}{1-e^{-\eta x}} = \frac{1}{\eta}\sup_{0<x\le\eta} \frac{x}{1-e^{-x}} = \frac{1}{1-e^{-\eta}} \le \frac{2}{\eta}.$$

Lemma 3.12. *For* $0 \le j \le n$, $\|F_j\|_\infty \leqslant \dfrac{2}{\eta}$.

Proof. By definition of E_η, if $C \le E_\eta$ and $0 \le x \le 1$, then $Ce^{-\eta x} + x \le E_\eta e^{-\eta x} + x \le E_\eta$.

We can now prove by induction over j that $|F_j| \le E_\eta$. First, when $j = 0$, from Lemma 3.10 we get

$$\|F_0\|_\infty = \|f_0\|_\infty \le 1 \le E_\eta.$$

Assume now that $\|F_j\|_\infty \le E_\eta$, then

$$\begin{aligned} |F_{j+1}(t)| &= |F_j(t)e^{-\eta h_{j+1}(t)} + f_{j+1}(t)| \le |F_j(t)|e^{-\eta \Re\left(h_{j+1}(t)\right)} + |f_{j+1}(t)| \\ &= |F_j(t)|e^{-\eta|f_{j+1}(t)|} + |f_{j+1}(t)|. \end{aligned}$$

As $|f_{j+1}(t)| \le 1$ and $|F_j(t)| \le E_\eta$, we get $|F_{j+1}(t)| \le E_\eta$. It remains to prove that $E_\eta \leqslant \dfrac{2}{\eta}$. To do so, it suffices to see that

$$e^{-y} \leqslant 1 - \left(\frac{e-1}{e}\right) y \qquad \text{for } 0 \leqslant y \leqslant 1,$$

yielding the result immediately. □

Lemma 3.13. *For* $0 \le \ell \le n$ *and* $j = 0, \ldots, k$, *let* $g_{j,k} = e^{-\eta H_{j,k}}$ *with*

$$H_{j,k} = \begin{cases} h_{j+1} + \ldots + h_k & \textit{when } j < k \\ 0 & \textit{when } j = k \end{cases}.$$

Then

$$F_k = \sum_{j=0}^{k} f_j g_{j,k}.$$

Proof. By induction on k, when $k = 0$, $H_{0,0} = 0$ thus $g_{0,0} = 1$ and, indeed, we have

$$F_0 = f_0 = f_0 g_{0,0}.$$

Assume now that the formula has been established at rank $k-1$ and let us show that $F_k = \sum_{j=0}^{k} f_j g_{j,k}$. By construction, we have

$$F_k = F_{k-1}e^{-\eta h_k} + f_k = \left(\sum_{j=0}^{k-1} f_j g_{j,k-1}\right) e^{-\eta h_k} + f_k.$$

with the induction hypothesis. It remains to notice that $g_{k,k} = e^{-\eta H_{k,k}} = 1$ and that, for $j = 0, \ldots, k-1$, $H_{j,k} = H_{j,k-1} + h_k$ thus $g_{j,k} = g_{j,k-1}e^{-\eta h_k}$ so that, indeed, we have $F_k = \sum_{j=0}^{k} f_j g_{j,k}$ as claimed. □

Recall that

$$T_0 = \sum_{j=0}^{m} f_j$$

and we set

$$T_1 = T_1^\eta = F_m = \sum_{j=0}^{m} f_j g_{j,m}$$

where the dependence on η comes from the definition of the $g_{j,n}$'s, in particular

$$\|T_1^\eta\|_\infty \leqslant E_\eta.$$

The first part of Lemma 3.9 is thus established and it remains to prove the second part. To do so, we start by some intermediary results;

Lemma 3.14. *If $H \in H^\infty$(Hardy space) and $Re(H) \geqslant 0$, then $e^{-H} \in H^\infty$ and*

$$\|e^{-H} - 1\|_2 \leqslant \|H\|_2.$$

Proof. Since H^∞ is a Banach algebra, the partial sums $\sum_{k=0}^{n}(-1)^k \frac{H^k}{k!}$ of e^{-H} are elements of H^∞. Moreover, since H is bounded, these sums converge uniformly toward e^{-H}, with $e^{-H} \in H^\infty$. Finally, if $z \in \mathbb{C}$ and $\Re(z) \geqslant 0$,

$$\left|e^{-z} - 1\right| = \left|\int_0^1 z e^{-tz} dt\right| \leqslant \int_0^1 |z| e^{-t\Re(z)} dt \leqslant |z|.$$

In our case $z = H(t)$, and we have

$$\left|e^{-H(t)} - 1\right| \leqslant |H(t)|$$

and by integration we have the desired inequality. □

Next let us introduce the following notations:

1. Let $f \in L^1([0,1])$, the spectrum of f, denoted by $\mathrm{spec}(f)$ is the set of indexes of non zero Fourier coefficients, that is

   $$\mathrm{spec}(f) = \mathrm{supp}\,\widehat{f} = \{n \in \mathbb{Z} \,:\, \widehat{f}(n) \neq 0\}.$$

   It is easy to show that, if $f, g \in L^2([0,1])$, then

   $$\mathrm{spec}(fg) \subset \mathrm{spec}(f) + \mathrm{spec}(g) = \{\lambda + \mu \,:\, \lambda \in \mathrm{spec}(f), \mu \in \mathrm{spec}(g)\}.$$

2. Let A be a subset of $[1, N[$, we denote by Λ_A the set of n_j's with $j \in A$ ***i.e***

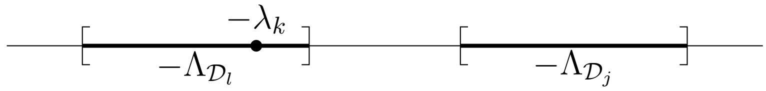
   $$\Lambda_A = \{n_j,\ j \in A\}.$$

Lemma 3.15. *Let $k \in \mathcal{D}_\ell = [2^\ell, 2^{\ell+1}[$ then $\widehat{f_j g_{j,m}}(-n_k) = 0$ if $j < \ell$.*

Proof. We must show that $-n_k \notin \mathrm{spec}(f_j g_{j,m})$. By contradiction, we suppose that $-n_k \in \mathrm{spec}(f_j g_j)$.

But since $\mathrm{spec}(f_j) \subset -\Lambda_{\mathcal{D}_j}$ and, from Lemma 3.14, $\mathrm{spec}(g_{j,m}) \subset \mathbb{N}$, thus

$$\mathrm{spec}(f_j g_{j,m}) \subset -\Lambda_{\mathcal{D}_j} + \mathbb{N}.$$



However, since $j < \ell$, $\mathcal{D}_j$ is to the left of $\mathcal{D}_\ell$, then $-\Lambda_{\mathcal{D}_\ell}$ is completetly to the left of $-\Lambda_{\mathcal{D}_j} + \mathbb{N}$ (since $k \in \mathcal{D}_\ell$, then $-n_k \in -\Lambda_{\mathcal{D}_\ell}$ by definition of $\Lambda_{\mathcal{D}_\ell}$). □

Proof of (2) *in Lemma 3.9.* Let $k \in [1, N[$ and $\ell \leqslant m$ such that $k \in I_\ell$. Hence

$$\widehat{T}_1(-n_k) = \sum_{j=0}^{m} \widehat{f_j g_{j,m}}(-n_k) = \sum_{j=\ell}^{m} \widehat{f_j g_{j,m}}(-n_k)$$

with Lemma 3.15.

On the other hand, $\widehat{T}_0(-n_k) = \dfrac{u_k}{k}$ while $\widehat{f}_j(-n_k) = \dfrac{u_k}{k}$ if $k \in I_j$ *i.e.* if $j = \ell$ and $\widehat{f}_j(-n_k) = 0$ otherwise. So we can write

$$\widehat{T}_0(-n_k) = \sum_{j=\ell}^{m} \widehat{f}_j(-n_k)$$

hence

$$\left|\widehat{T}_1(-n_k) - \widehat{T}_0(-n_k)\right| = \sum_{j=\ell}^{m} c_{-n_k}[f_j(g_{j,m} - 1)].$$

But

$$\begin{aligned} |c_{-n_k}[f_j(g_{j,m} - 1)]| &\leq \|f_j(g_{j,m} - 1)\|_{L^1([0,1])} \\ &\leq \|f_j\|_{L^2([0,1])}\|g_{j,m} - 1\|_{L^2([0,1])} = \|f_j\|_{L^2([0,1])}\|e^{-\eta H_{j,m}} - 1\|_{L^2([0,1])} \end{aligned}$$

by definition of $g_{j,m}$. Using Lemma 3.14 and then Lemma 3.13, we obtain

$$\begin{aligned} |c_{-n_k}[f_j(g_{j,m} - 1)]| &\leqslant \eta\|f_j\|_2\|H_{j,m}\|_2 \leqslant \eta\|f_j\|_2 \sum_{r=j+1}^{m} \|h_r\|_2 \\ &\leqslant \eta\|f_j\|_2\sqrt{2} \sum_{r=j+1}^{m} \|f_r\|_2 \end{aligned}$$

with Lemma 3.11. Then, from Lemma 3.10, we obtain

$$|c_{-n_k}[f_j(g_{j,m} - 1)]| \leq \eta 2^{-j/2}\sqrt{2} \sum_{r=j+1}^{+\infty} 2^{-r/2} = \eta 2^{-j/2}\sqrt{2}\frac{2^{-(j+1)/2}}{1 - 2^{-1/2}}.$$

We conclude that

$$\left|\widehat{T}_1(-\lambda_k) - \widehat{T}_0(-\lambda_k)\right| \leq 3\eta \sum_{j=\ell}^{m} 2^{-j} \leq 6\eta 2^{-\ell}.$$

On the other hand, as $k \in I_\ell = [2^\ell, 2^{\ell+1}[$,

$$|\widehat{T}_0(-\lambda_k)| = \frac{1}{k} > 2^{-\ell-1}$$

so that

$$\left|\widehat{T}_1(-\lambda_k) - \widehat{T}_0(-\lambda_k)\right| \leq 12\eta|\widehat{T}_0(-\lambda_k)|.$$

Choosing $\eta = \dfrac{1}{24}$ gives the result. □

Note that, when $\eta = \dfrac{1}{24}$, $E_\eta \leq 48$ so that we can take $C = 48$ in Lemma 3.9, leading to $C_{MPS} = \dfrac{1}{96}$ in Theorem 3.8.

3.3. *Strongly multidimensional sets*

Let $\delta > 0$ and $(m, n) \in \mathbb{N}^2$. A subset A of $\mathbb{Z}$ is $(\delta; m, n)$-strongly 2-dimensional if there exists numbers d and D with $D > (2 + \delta)d$ such that

$$A = \bigcup_{k \in I} (A_k + kD)$$

for some set I containing at least m integers and subsets $A_k \subseteq \{-d, \dots, d\}$ verifying $|A_k| \geqslant n$.

Theorem 3.16 (Hanson [10])**.** *Let $\delta > 0$ and m, n be two positive integers satisfying*

$$n \geqslant \pi^3 2^{21} C_{MPS}^3 \ln(n)^3 \quad \textit{and} \quad m \geqslant \pi^3 2^{21} C_{MPS}^3 \ln(n)^3 \ln(m)^3, \tag{3.11}$$

where C_{MPS} is the constant in Theorem B. Suppose A is $(\delta; m, n)$ strongly 2-dimensional subset of $\mathbb{Z}$. Then

$$\int_0^1 \left|\sum_{a \in A} e^{2i\pi a t}\right| \mathrm{d}t \geqslant \frac{C_{MPS}^2}{(2^9\pi)^2 \left(2 + \ln(1 + \frac{2}{\delta})\right)} \ln(m) \ln(n)$$

Combining this result with Theorem 3.3 in [27], we see that the estimate is also best possible up to the constant.

Given a set $I \subseteq \mathbb{Z}$, a positive integer q, and an arbitrary integer s, we define

$$I(q; s) = \{k \in I : k = s \text{ (mod q)}\}.$$

The proof of Theorem 3.16 is a direct consequence of two lemmas. The first one is the following:

Lemma 3.17. *Let I be a set of integers with $|I| \geqslant 8$. Then there are positive integers q and s such that*

$$\frac{|I|^{\frac{1}{3}}}{8} \leqslant |I(q; s)| \leqslant q^{1/2}.$$

Proof. For each $j \geqslant 1$, we choose any s_j such that $|I(4^j; s_j)|$ is maximal. But, on one hand,

$$I = \bigcup_{s=0}^{4^j-1} I(4^j, s)$$

and on the other hand, for j fixed, the sets $I(4^j; s)$ are disjoint, so at least one of them has cardinality larger than $4^{-j}|I|$. In particular, for some s_j,

$$|I(4^j, s_j)| \geqslant 4^{-j}|I|. \tag{3.12}$$

For $j = 1$, we thus have $|I(4; s_1)| \geqslant \dfrac{|I|}{4} \geqslant 2$. On the other hand, if $j = s \mod kp$ then $j = s \mod p$ so that, for any s

$$I(4^m; s) \subset I(4^\ell; s) \quad \text{for} \quad \ell < m,$$

and, for sufficiently large j we have $|I(4^j; s_j)| = 1 \leqslant 2^j$. Therefore, there exists a minimal j_0 such that $|I(4^{j_0}; s_{j_0})| \leqslant 2^{j_0}$. Let $q = 4^{j_0}$, and $s = s_{j_0}$ then using (3.12) and the definition of j_0

$$\frac{|I|}{q} \leqslant |I(q; s)| = |I(4^{j_0}; s_{j_0})| \leqslant 2^{j_0} = q^{\frac{1}{2}}.$$

In particular

$$|I|^{\frac{1}{3}} \leqslant q^{\frac{1}{2}}. \tag{3.13}$$

By minimality of $j_0 - 1$

$$\frac{q^{\frac{1}{2}}}{2} = 2^{j_0-1} \leqslant \left|I\left(4^{j_0-1}; s_{j_0-1}\right)\right| \leq \sum_{r=0}^{3} \left|I\left(4^{j_0}; s_{j_0-1} + r4^{j_0-1}\right)\right| \leq 4|I(4^{j_0}; s_{j_0})| = 4|I(q; s)|$$

by definition of s_{j_0}. We thus get

$$|I(q; s)| \geqslant \frac{q^{\frac{1}{2}}}{8} \geqslant \frac{|I|^{\frac{1}{3}}}{8},$$

with (3.13). □

Lemma 3.18. *Let $\delta > 0$ and let d and D be positive integers with $(2 + \delta)d < D$. Suppose I is a finite set of integers, and let*

$$F(t) = \sum_{k \in I} f_k(t) e^{2i\pi Dkt}$$

where

$$f_k(t) = \sum_{|n| \leqslant d} a_{n,k} e^{2i\pi nt}$$

Let q and s with $q > 4\pi$ and suppose $I(q; s) = \{k_1, \ldots, k_J\}$ then we have

$$\|F\|_{L^1([0,1]} \geqslant \frac{1}{32\pi(2 + \ln(1 + \frac{2}{\delta}))} \sum_{j=1}^{J} \|f_{k_j}\|_{L^1([0,1])} \left(\frac{C_{MPS}}{2j} - \frac{2\pi d}{qD}\right)$$

We split the remaining of this section into two parts. In the first one, we show that Lemmas 3.17 and 3.18 imply Theorem 3.11. In the second one, we prove Lemma 3.18.

Proof of Theorem 3.16

Let $\delta > 0$ m, n be two integers satisfying the conditions of the theorem and let A be strongly (δ, m, n)-regular. Thus, there are two integers d, D such that $D > (2+\delta)d$, we can write

$$A = \bigcup_{k \in I} (A_k + kD),$$

with $|I| \geqslant m$ and $A_k \subset \{-d, \dots, d\}$ with $|A_k| \geqslant n$. We can then write

$$F(t) := \sum_{a \in A} e^{2i\pi at} = \sum_{k \in I} f_k(t) e^{2i\pi Dkt}$$

with

$$f_k(t) = \sum_{a \in A_k} e^{2i\pi at} = \sum_{n=-d}^{d} a_{n,k} e^{2i\pi nt}$$

with $a_{n,k} = 1$ if $n \in A_k$ and $a_{n,k} = 0$ otherwise.

Assume first that there exists $k_1 \in I$ such that

$$\|f_{k_1}\|_{L^1([0,1])} \geqslant \frac{C_{MPS}}{2^9\pi} \ln(m) \ln(n). \tag{3.14}$$

We then choose $q \geqslant \dfrac{16\pi}{7C_{MPS}}$ in such a way that there is an s such that $I(q,s) = \{k_1\}$. Hence by Lemma 3.18,

$$\begin{aligned} \|F\|_{L^1([0,1])} &\geq \frac{\|f_{k_1}\|_{L^1([0,1])}}{32\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)}\left(\frac{C_{MPS}}{2} - \frac{2\pi d}{qD}\right) \\ &\geq \frac{C_{MPS}\|f_{k_1}\|_{L^1([0,1])}}{2^6\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)}\left(1 - \frac{7d}{8D}\right). \end{aligned}$$

As $D > 2d$, using (3.14), we conclude that, in this case

$$\|F\|_{L^1([0,1])} \geq \left(\frac{C_{MPS}}{2^9\pi}\right)^2 \frac{\ln n \ln m}{2+\ln\left(1+\frac{2}{\delta}\right)}$$

which establishes the theorem.

We will thus assume that, for each $k \in I$,

$$\|f_k\|_{L^1([0,1])} \leqslant \frac{C_{MPS}}{2^9\pi} \ln(m) \ln(n). \tag{3.15}$$

Note that, from Theorem B,

$$\|f_k\|_{L^1([0,1])} \geqslant C_{MPS} \ln(n)$$

so that $2^9\pi \leqslant \ln(m)$, in particular, $m \geq 8$. We then take q and s given by Lemma 3.17 applied to the set I so that $J = |I(q;s)|$ satisfies

$$\frac{m^{\frac{1}{3}}}{8} \leqslant J \leqslant q^{\frac{1}{2}}. \tag{3.16}$$

We write

$$I(q;s) = \{k_1 < \ldots < k_J\}$$

From Lemma 3.18, we get that

$$\begin{aligned} \|F\|_{L^1([0,1])} &\geqslant \frac{1}{2^5\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)} \sum_{j=1}^{J} \|f_{k_j}\|_{L^1} \left(\frac{C_{MPS}}{2j} - \frac{2\pi d}{qD}\right) \\ &= \frac{1}{2^5\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)}(T_1 - T_2) \end{aligned}$$

with

$$T_1 = \frac{C_{MPS}}{2}\sum_{j=1}^{J}\frac{\|f_{k_j}\|_{L^1}}{j} \quad \text{and} \quad T_2 = \frac{2\pi d}{qD}\sum_{j=1}^{J}\|f_{k_j}\|_{L^1}.$$

Next, as $\|f_{k_j}\|_{L^1} \geqslant C_{MPS}\ln(n)$,

$$T_1 \geq \frac{C_{MPS}^2\ln(n)}{2}\sum_{j=1}^{J}\frac{1}{j} \geq \frac{C_{MPS}^2\ln(n)}{2}\ln J \geq \frac{C_{MPS}^2\ln(n)\ln(m)}{8}$$

with (3.16).

On the other hand, from (3.15), we get

$$T_2 \leqslant \frac{2\pi J d}{qD}\frac{C_{MPS}}{2^9\pi}\ln(m)\ln(n) \leqslant \frac{2\pi}{q^{\frac{1}{2}}}\frac{C_{MPS}}{2^9\pi}\ln(m)\ln(n),$$

since $d \leqslant \dfrac{D}{2}$ and $J \leqslant q^{\frac{1}{2}}$ with (3.15). Further, (3.15) also implies that

$$q^{\frac{1}{2}} \geqslant \frac{m^{\frac{1}{3}}}{8} \geq 2^4\pi C_{MPS}\ln(m)\ln(n)$$

with (3.11), leading to $T_2 \leqslant \dfrac{1}{2^{13}\pi}$.

We have established that

$$\begin{aligned} \|F\|_{L^1([0,1])} &\geqslant \frac{1}{2^5\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)}\left(\frac{C_{MPS}^2\ln(m)\ln(n)}{2^3} - \frac{1}{2^{13}\pi}\right) \\ &= \frac{C_{MPS}^2}{(2^9)^2\pi\left(2+\ln\left(1+\frac{2}{\delta}\right)\right)}\left(2^{10}\ln(m)\ln(n) - \frac{1}{C_{MPS}^2}\right) \\ &\geqslant \frac{C_{MPS}^2}{(2^9\pi)^2\left(2+\ln(1+\frac{2}{\delta})\right)}\ln(m)\ln(n) \end{aligned}$$

since $\ln(m)\ln(n)C_{MPS}^2 \geqslant \dfrac{1}{2^{10}\pi - 1}$.

Proof of Lemma 3.18

The rest of this chapter consist in proving Lemma 3.18. The proof is divided into several lemmas. The first one is a simple lemma about numerical integration of trigonometric polynomials:

Lemma 3.19. *Let N be a positive integer. Then for any trigonometric polynomial f of degree d*

$$\left|\|f\|_{L^1([0,1])} - \frac{1}{N}\sum_{j=0}^{N-1}\left|f\left(\frac{j}{N}\right)\right|\right| \leqslant \frac{2\pi d}{N}\|f\|_{L^1([0,1])}.$$

Note that, as f is 1-periodic, writing $N = R+S$, $\displaystyle\frac{1}{N}\sum_{j=0}^{N-1}\left|f\left(\frac{j}{N}\right)\right| = \frac{1}{R+S}\sum_{j=-R}^{S-1}\left|f\left(\frac{j}{R+S}\right)\right|$.

Proof. We write, using the triangular and reverse triangular inequalities

$$\begin{aligned}
\left|\int_0^1 |f(t)|\,\mathrm{d}t - \frac{1}{N}\sum_{j=0}^{N-1}\left|f\left(\frac{j}{N}\right)\right|\right| &= \left|\sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}} |f(t)| - \left|f\left(\frac{j}{N}\right)\right|\mathrm{d}t\right| \\
&\leq \sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}\left|f(t) - f\left(\frac{j}{N}\right)\right|\mathrm{d}t = \sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{R}}\left|\int_{\frac{j}{N}}^{t} f'(s)\,\mathrm{d}s\right|\mathrm{d}t \\
&\leq \sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}\int_{\frac{j}{N}}^{t}|f'(s)|\,\mathrm{d}s\,\mathrm{d}t = \sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}|f'(s)|\int_{s}^{\frac{j+1}{N}}\mathrm{d}t\,\mathrm{d}s \\
&\leq \frac{1}{N}\sum_{j=0}^{N-1}\int_{\frac{j}{N}}^{\frac{j+1}{N}}|f'(s)|\,\mathrm{d}s = \int_0^1 |f'(s)|\,\mathrm{d}s
\end{aligned}$$

where we have used Fubini and the bound $\displaystyle\int_s^{\frac{j+1}{N}}\mathrm{d}t \leq \frac{1}{N}$ when $\dfrac{j}{N} \leq s \leq \dfrac{j+1}{N}$. We conclude with Bernstein's inequality, $\|f'\|_{L^1([0,1])} \leqslant 2\pi d\|f\|_{L^1([0,1])}$. □

For a finitely supported sequence $\big(A(k)\big)_{k\in\mathbb{Z}}$ we define its discrete Fourier transform (or **Z**-Fourier transform) as

$$\mathcal{F}_d[A](t) = \sum_{k\in\mathbb{Z}} A(k)e^{-2i\pi kt}.$$

If A, B are two finitely supported sequences, their convolution is the sequence $A * B$ defined by

$$A * B(k) = B * A(k) = \sum_{n\in\mathbb{Z}} A(k-n)B(n).$$

The Convolution Theorem is also valid here: $\mathcal{F}_d[A * B](t) = \mathcal{F}_d[A](t)\mathcal{F}_d[B](t)$. Two classical examples are

– the Dirichlet kernel: set $\mathbb{d}_L = \mathbb{1}_{-L,\dots,L}$ so that

$$D_L(t) = \mathcal{F}_d[\mathbb{d}_L](t) = \sum_{|k|\leqslant L} e^{2i\pi kt} = \frac{\sin(\pi(2L+1)t)}{\sin(\pi t)};$$

– the Fejer kernel: set $\mathbb{f}_L(k) = \left(1 - \frac{|k|}{L+1}\right) \mathbb{1}_{-L,\dots,L}(k)$ so that

$$F_L(t) = \mathcal{F}_d[\mathbb{f}_L](t) = \sum_{|k|\leqslant L} \left(1 - \frac{|k|}{L+1}\right) e^{2i\pi kt} = \frac{1}{L+1}\frac{(\sin(\pi(L+1)t))^2}{(\sin(\pi t))^2}.$$

Lemma 3.20. *Let M, N, R, S be integers with $2 \leqslant M < N$. Then there exists a function $K_{M,N}$ with the following properties:*

1. $K_{M,N}(k) = 1$ *for* $|k| \leqslant N$,
2. $K_{M,N}(k) = 0$ *for* $|k| \geqslant N + 2M$
3. *when* $R + S \geqslant 2N + 4M$, $\displaystyle\frac{1}{R+S}\sum_{j=-R}^{S-1}\left|\mathcal{F}_d[K_{M,N}]\left(\frac{j}{R+S}\right)\right| \leqslant 16\pi(2 + \ln(1 + N/M))$

Proof. Define

$$\begin{aligned} K_{M,N}(k) &= \frac{1}{M}\mathbb{d}_{N+M} * \mathbb{f}_{M-1}(k) = \frac{1}{M}\sum_{n\in\mathbb{Z}} \mathbb{1}_{\{-M-N,\dots,M+N\}}(k-n)\mathbb{1}_{\{-M+1,\dots,M+1\}}(k) \\ &= \frac{1}{M}\sum_{\substack{|n|\leqslant M-1 \\ |n-k|\leqslant N+M}} \left(1 - \frac{|n|}{M}\right). \end{aligned}$$

First, for $|k| \leqslant N$, if $|n| \leqslant M-1$, then $|n-k| \leqslant |n| + |k| \le N + M - 1$, so that

$$K_{M,N}(k) = \frac{1}{M}\sum_{|n|\leqslant M-1}\left(1 - \frac{|n|}{M}\right) = \frac{1}{M}\left(M - \frac{2}{M}\sum_{n=1}^{M-1} n\right) = \frac{1}{M}\left(M - \frac{(M-1)M}{M}\right) = 1.$$

On the other hand, if $|k| \geqslant N + 2M$ and $|n| \le M - 1$ then $|k-n| \ge |k| - |n| \ge N + M + 1$ so that the sum defining $K_{M,N}$ is empty and $K_{M,N} = 0$.

To prove the last item, the Convolution Theorem shows that

$$\mathcal{F}_d[K_{M,N}](t) = \frac{1}{M}D_{N+M}(t)F_{M-1}(t).$$

As D_{N+M} and F_{M-1} are both even, so is $K_{M,N}$ thus

$$\int_0^1 |\mathcal{F}_d[K_{M,N}](t)|\,\mathrm{d}t = 2\int_0^{\frac{1}{2}} |\mathcal{F}_d[K_{M,N}](t)|\,\mathrm{d}t = 2(I_1 + I_2 + I_3)$$

where

$$I_1 = \frac{1}{M}\int_0^{\frac{1}{N+M}} |D_{N+M}(t)F_{M-1}(t)|\,\mathrm{d}t$$

$$I_2 = \frac{1}{M} \int\limits_{\frac{1}{N+M}}^{\frac{1}{M}} |D_{N+M}(t) F_{M-1}(t)| \, \mathrm{d}t,$$

$$I_3 = \frac{1}{M} \int\limits_{\frac{1}{M}}^{\frac{1}{2}} |D_{N+M}(t) F_{M-1}(t)| \, \mathrm{d}t.$$

We have

$$|D_{N+M}(t)| \leqslant 2(N+M)+1, \; 0 \leqslant F_{M-1}(t) \leqslant M, \; \int\limits_0^1 F_{M-1}(t) \, \mathrm{d}t = 1.$$

It follows that

$$I_1 \leqslant \frac{1}{M} \int\limits_0^{\frac{1}{N+M}} (2(M+N)+1) M \, \mathrm{d}t = 2 + \frac{1}{M+N} \leqslant 3.$$

since M, N are positive integers.

Using the explicit expressions of D_{N+M} and F_{M-1}, we have

$$\begin{aligned} I_2 &= \frac{1}{M^2} \int\limits_{\frac{1}{N+M}}^{\frac{1}{M}} \frac{|\sin(\pi(2(N+M)+1)t)| \sin^2(\pi M t)}{\sin^3(\pi t)} \, \mathrm{d}t \leq \frac{1}{M^2} \int\limits_{\frac{1}{N+M}}^{\frac{1}{M}} \frac{\sin^2(\pi M t)}{\sin^3(\pi t)} \, \mathrm{d}t \\ &\leqslant \frac{\pi^2}{8} \int\limits_{\frac{1}{N+M}}^{\frac{1}{M}} \frac{\mathrm{d}t}{t} = \frac{\pi^2}{8} \ln\left(1 + \frac{N}{M}\right). \end{aligned}$$

using that $\sin \pi t \leq \pi t$ for $t \geq 0$ and that $\sin \pi t \geq 2t$ for $0 \leq t \leq \frac{1}{2}$.

Finally, for I_3, we do the same computation to bound

$$I_3 \leq \frac{1}{M^2} \int\limits_{\frac{1}{M}}^{\frac{1}{2}} \frac{\sin^2 \pi M t}{t^3} \, \mathrm{d}t = \int\limits_1^{\frac{M}{2}} \frac{\sin^2 \pi s}{s^3} \, \mathrm{d}s \leq \int\limits_1^{\frac{M}{2}} \frac{\mathrm{d}s}{s^3} \leq \frac{1}{2}.$$

Grouping all terms and slightly upper bounding the numerical constants, we obtain

$$\int\limits_0^1 |\mathcal{F}_d[K_{M,N}](t)| \, \mathrm{d}t \leqslant 8 \left(2 + \ln\left(1 + \frac{N}{M}\right)\right).$$

By Lemma 3.19 (and the 1-periodicity of $\mathcal{F}_d[K_{M,N}]$), we obtain

$$\frac{1}{R+S} \sum_{j=-R}^{S-1} \left| \mathcal{F}_d[K_{M,N}] \left(\frac{j}{2R+1}\right) \right| \leqslant \left(1 + \frac{2\pi d}{R+S}\right) \|\mathcal{F}_d[K_{M,N}]\|_{L^1([0,1])}.$$

But $d = N + 2M - 1$ and $R + S \geqslant 2N + 4M$ so

$$\frac{2\pi d}{N} \leqslant \frac{\pi(2N + 4M - 2)}{2N + 4M} \leqslant \pi.$$

Hence we get

$$\begin{aligned} \frac{1}{R+S} \sum_{j=-R}^{S-1} \left| \mathcal{F}_d[K_{M,N}] \left(\frac{j}{R+S} \right) \right| &\leqslant (1+\pi) \|\mathcal{F}_d[K_{M,N}]\|_{L^1([0,1])} \\ &\leqslant 16\pi \left(2 + \ln \left(1 + \frac{N}{M} \right) \right), \end{aligned}$$

concluding the proof. $\square$

Lemma 3.21. *Let R, S be positive integers and $K = \big(K(-R), \ldots, K(S-1)\big) \in \mathbb{C}^{R+S}$. We extend K into*

– an $R+S$-periodic sequence $K^{(p)}\big(j(R+S)+\ell\big) = K(\ell)$ for $\ell = -R, \ldots, S-1$ and $j \in \mathbb{Z}$;

– a finitely supported sequence $K^{(0)}$ by setting $K^{(0)}(\ell) = K(\ell)$ for $\ell = -R, \ldots, S-1$ and $K^{(0)}(\ell) = 0$ for $\ell \geq S$ as well as for $\ell \leq -R-1$.

Then

$$\int_0^1 \left| \sum_{m \in \mathbb{Z}} a_m K^{(p)}(m) e^{2i\pi mt} \right| \mathrm{d}t \leq \frac{1}{R+S} \sum_{\ell=-R}^{S-1} \left| \mathcal{F}_d[K^{(0)}] \left(\frac{\ell}{R+S} \right) \right| \int_0^1 \left| \sum_{m \in \mathbb{Z}} a_m e^{2i\pi mu} \right| \mathrm{d}u$$

Proof. Write elements of $\mathbb{C}^{R+S}$ as $(a_{-R}, \ldots, a_{S-1})$ and the scalar product $\langle a, b \rangle = \sum_{\ell=-R}^{S-1} a_\ell \overline{b_\ell}$. For $j = -R, \ldots, S-1$, denote by $e_k := \left[\frac{1}{\sqrt{R+S}} e^{2i\pi \frac{k\ell}{R+S}} \right]_{\ell=-R}^{S-1}$ so that $(e_k)_{k=-R,\ldots,S-1}$ is an orthonormal basis of $\mathbb{C}^{R+S}$.

Write $A_\ell = \sum_{n \in \mathbb{Z}} \overline{a_{n(R+S)+\ell}} e^{-2i\pi(n(R+S)+\ell)t}$ for $\ell = -R, \ldots, S-1$ and $A = (A_\ell) \in \mathbb{C}^{R+S}$. Then, by periodicity of $K^{(p)}$

$$\begin{aligned} \sum_{m \in \mathbb{Z}} a_m K^{(p)}(m) e^{2i\pi mt} &= \sum_{\ell=-R}^{S-1} K(\ell) \sum_{j \in \mathbb{Z}} a_{j(R+S)+\ell} e^{2i\pi(j(R+S)+\ell)t} = \langle K, A \rangle = \sum_{k=-R}^{S-1} \langle K, e_k \rangle \overline{\langle A, e_k \rangle} \\ &= \frac{1}{R+S} \sum_{k=-R}^{S-1} \left(\sum_{\ell=-R}^{S-1} K(\ell) e^{-2i\pi k \frac{\ell}{R+S}} \right) \left(\sum_{\ell=-R}^{S-1} \sum_{n \in \mathbb{Z}} a_{n(R+S)+\ell} e^{2i\pi(n(R+S)+\ell)t} e^{2i\pi k \frac{\ell}{R+S}} \right) \\ &= \frac{1}{R+S} \sum_{k=-R}^{S-1} \mathcal{F}_d[K_0] \left(\frac{k}{R+S} \right) \sum_{\ell=-R}^{S-1} \sum_{n \in \mathbb{Z}} a_{n(R+S)+\ell} e^{2i\pi[(n(R+S)+\ell)t + \ell \frac{k}{R+S}]}). \end{aligned}$$

Noticing that $e^{2i\pi\ell\frac{k}{R+S}} = e^{2i\pi[(n(R+S)+\ell)\frac{k}{R+S}]}$, we may write

$$\begin{aligned}\sum_{m\in\mathbb{Z}} a_m K^{(p)}(m) e^{2i\pi mt} &= \frac{1}{R+S}\sum_{k=-R}^{S-1}\mathcal{F}_d[K^{(0)}]\left(\frac{k}{R+S}\right)\sum_{\ell=-R}^{S-1}\sum_{n\in\mathbb{Z}} a_{n(R+S)+\ell} e^{2i\pi(n(R+S)+\ell)(t+\frac{k}{R+S})}\\ &= \frac{1}{R+S}\sum_{k=-R}^{S-1}\mathcal{F}_d[K^{(0)}]\left(\frac{k}{R+S}\right)\sum_{m\in\mathbb{Z}} a_m e^{2i\pi m(t+\frac{k}{R+S})}.\end{aligned}$$

From this, we deduce that

$$\begin{aligned}\int_0^1\left|\sum_{m\in\mathbb{Z}} a_m K^{(p)}(m)e^{2i\pi mt}\right|\,\mathrm{d}t &\leq \frac{1}{R+S}\sum_{k=-R}^{S-1}\left|\mathcal{F}_d[K^{(0)}]\left(\frac{k}{R+S}\right)\right|\int_0^1\left|\sum_{m\in\mathbb{Z}} a_m e^{2i\pi m(t+\frac{k}{R+S})}\right|\mathrm{d}t\\ &= \frac{1}{R+S}\sum_{k=-R}^{S-1}\left|\mathcal{F}_d[K^{(0)}]\left(\frac{k}{R+S}\right)\right|\int_0^1\left|\sum_{m\in\mathbb{Z}} a_m e^{2i\pi mu}\right|\mathrm{d}u,\end{aligned}$$

with the change of variable $u = t + \dfrac{j}{R+S}$ and periodicity of $u \longrightarrow \sum_m a_m e^{2i\pi mu}$. □

Lemma 3.22. *Let d, D and q be positive integers with $(2+2\delta)d+4 \leqslant D$ for some $\delta > 0$ and $q \geqslant 4\pi$.*

Suppose I is a finite set of integers and, for each $k \in I$ let f_k be a trigonometric polynomial of degree at most d. Then for any integer s, we have:

$$\int_0^1\left|\sum_{k\in I(q;s)} f_k(t)e^{2i\pi Dkt}\right|\,\mathrm{d}t \leqslant 32\pi(2+\ln(1+2/\delta))\int_0^1\left|\sum_{k\in I} f_k(t)e^{2i\pi Dkt}\right|\,\mathrm{d}t.$$

Proof. Assume we can prove the lemma for $s = 0$, that is, for any sequence $(f_k)_{k\in\mathbb{Z}}$ of trigonometric polynomials of degree at most d and any finite set I,

$$\int_0^1\left|\sum_{q\ell\in I} f_{\ell q}(t)e^{2i\pi D\ell qt}\right|\,\mathrm{d}t \leqslant 32\pi(2+\ln(1+2/\delta))\int_0^1\left|\sum_{k\in I} f_k(t)e^{2i\pi Dkt}\right|\,\mathrm{d}t.$$

Then, replacing I with $I-s$ and replacing (f_k) with (f_{k+s}) we get

$$\begin{aligned}\int_0^1\left|\sum_{k\in I(q;s)} f_k(t)e^{2i\pi D\ell qt}\right|\,\mathrm{d}t &= \int_0^1\left|\sum_{s+\ell q\in I} f_{s+\ell q}(t)e^{2i\pi D\ell qt}\right|\,\mathrm{d}t\\ &\leqslant 32\pi(2+\ln(1+2/\delta))\int_0^1\left|\sum_{k\in I-s} f_{s+k}(t)e^{2i\pi Dkt}\right|\,\mathrm{d}t\\ &= 32\pi(2+\ln(1+2/\delta))\int_0^1\left|e^{-2i\pi Dst}\sum_{\ell\in I} f_\ell(t)e^{2i\pi Dkt}\right|\,\mathrm{d}t\end{aligned}$$

which is the desired estimate since $|e^{-2i\pi Dst}| = 1$. So there is no loss of generality is assuming $s = 0$.

Next, we write $f_k = \sum_{-d\leqslant \ell\leqslant d} a_\ell^k e^{2i\pi(Dk+\ell)t}$ and

$$F(t) = \sum_{k\in I}\sum_{-d\leqslant \ell\leqslant d} a_\ell^k e^{2i\pi(Dk+\ell)t} = \sum_m a_m e^{2i\pi mt}$$

where

$$\begin{cases} a_m = a_\ell^k & \text{when } m = Dk+\ell, \quad k\in I \quad \text{and} \quad |\ell|\leqslant d \\ 0 & \text{otherwise} \end{cases}. \tag{3.17}$$

Let $N = d$, $M = \left\lceil \frac{\delta d}{2} \right\rceil$ the smallest integer larger than $\frac{\delta d}{2}$ and let $K_{M,N}$ the sequence from Lemma 3.20. Since

$$N + 2M \leqslant d + 2\left(\frac{\delta d}{2}+1\right) = d + \delta d + 2 \leqslant \frac{D}{2}$$

then we have

$$\operatorname{supp}(K_{M,N}) = [-N-2M, N+2M] \subseteq \left[-\frac{D}{2}, \frac{D}{2}\right].$$

Further $K_{M,N}(m) = 1$ for $m\in[-d,d]$. Next, $K_{M,N}^{(p)}$ to be the qD-periodic sequence defined by $K_{M,N}^{(p)}(jqD+\ell) = K_{M,N}(\ell)$ for $\ell = -N-2M,\ldots,N+2M$ and $K_{M,N}^{(p)}(k) = 0$ for all other k's. We take $R,S > N+2M$ such that $R+S = qD$ and then $K_{M,N}^{(p)}(k) = 0$ for $k = -R,\ldots,-N-2M-1$ and for $k = N+2M+1,\ldots,S-1$.

From Lemma 3.21, we get

$$\begin{aligned} \int_0^1 \left|\sum_m a_m K_{M,N}^{(p)}(m) e^{2i\pi mt}\right| \mathrm{d}t &\leq \frac{1}{R+S}\sum_{j=-R}^{S-1}\left|\mathcal{F}_d[K_{M,N}]\left(\frac{j}{R+S}\right)\right| \int_0^1 \left|\sum_{m\in\mathbb{Z}} a_m e^{2i\pi mu}\right| \mathrm{d}u \\ &\leq 16\pi\left(2+\ln\left(1+\frac{N}{M}\right)\right)\|F\|_{L^1([0,1])} \end{aligned}$$

with Lemma 3.20.

As $\operatorname{supp}(K_{M,N}) \subseteq \left[-\frac{D}{2}, \frac{D}{2}\right]$ and $K_{M,N}^{(p)}$ is periodic of period $R+S = qD$ then

$$K_{M,N}^{(p)}(m) \neq 0 \quad \text{if} \quad m = jqD+\ell' \quad \text{for} \quad j\in\mathbb{Z} \quad \text{and} \quad |\ell'|\leqslant \frac{D}{2}. \tag{3.18}$$

Combining (3.17) and (3.18) we have that $a_m K_{M,N}^{(p)}(m) \neq 0$ only when

$$m = Dk+\ell = jqD+\ell'.$$

Hence $|jqD - Dk| = |\ell-\ell'| \leqslant d + \frac{D}{2} < D$. But this can only happen when $jq = k$, which then also implies $\ell = \ell'$. In particular, $m = jqD+\ell$ with $|\ell|\leq d$ and then

$$a_m K_{M,N}^{(p)}(m) = a_{jqD+\ell} K_{M,N}^{(p)}(jqD+\ell) = a_{jqD+\ell} K_{M,N}^{(p)}(\ell) = a_\ell^{jqD}.$$

It follows that

$$\sum_{m\in\mathbb{Z}} a_m K_{M,N}(m) e^{2i\pi mt} = \sum_{j\in\mathbb{Z}}\sum_{-d\leqslant \ell\leqslant d} a_{jqD+\ell} e^{2i\pi[(jqD+\ell)t]}$$

$$\begin{aligned}
&= \sum_{\substack{k=0 \mod q \\ k\in I}} \sum_{-d\leqslant \ell\leqslant d} a_{kD+\ell} e^{2i\pi kDt} e^{2i\pi \ell t} \\
&= \sum_{k\in I(q;0)} e^{2i\pi kDt} \sum_{-d\leqslant \ell\leqslant d} a_{kD+\ell} e^{2i\pi \ell t} \\
&= \sum_{k\in I(q;0)} f_k(t) e^{2i\pi kDt}.
\end{aligned}$$

Finally we get

$$\begin{aligned}
\int_0^1 \left| \sum_{k\in I(q;0)} f_k(t) e^{2i\pi Kdt} \right| \mathrm{d}t &= \int_0^1 \left| \sum_m a_m K_{M,N}(m) e^{2i\pi mt} \right| \mathrm{d}t \\
&\leqslant 32\pi \left(2 + \ln(1 + N/M)\right) \|F\|_{L^1([0,1])} \\
&\leqslant 32\pi(2 + \ln(1 + 2/\delta)) \|F\|_{L^1([0,1])}
\end{aligned}$$

since $\dfrac{N}{M} \leqslant \dfrac{2d}{\delta D} \leqslant \dfrac{2}{\delta}$. □

We can now prove the lemma.

Proof of lemma 3.18. Write $I(q; s) = \{k_1, \ldots, k_J\}$ and write each k_j in the form $k_j = r_j q + s$. Applying lemma 3.22 yields

$$\begin{aligned}
\|F\|_{L^1([0,1])} &\geqslant \frac{1}{32\pi(2 + \ln(1 + 2/\delta))} \int_0^1 \left| \sum_{j=1}^J f_{k_j}(t) e^{2i\pi k_j Dt} \right| \mathrm{d}t = \int_0^1 \left| e^{2i\pi Dst} \sum_{j=1}^J f_{k_j}(t) e^{2i\pi r_j qDt} \right| \mathrm{d}t \\
&= \frac{1}{qD} \int_0^{qD} \left| \sum_{j=1}^J f_{k_j}\left(\frac{s}{qD}\right) e^{2i\pi r_j s} \right| \mathrm{d}s \\
&= \frac{1}{qD} \sum_{m=0}^{qD-1} \int_m^{m+1} \left| \sum_{j=1}^J f_{k_j}\left(\frac{s}{qD}\right) e^{2i\pi r_j s} \right| \mathrm{d}s.
\end{aligned}$$

But

$$\sum_{j=1}^J f_{k_j}\left(\frac{s}{qD}\right) e^{2i\pi r_j s} = \sum_{j=1}^J f_{k_j}\left(\frac{m}{qD}\right) e^{2i\pi r_j s} + \sum_{j=1}^J \left[f_{k_j}\left(\frac{s}{qD}\right) - f_{k_j}\left(\frac{m}{qD}\right) \right] e^{2i\pi r_j s}$$

so that

$$\|F\|_{L^1([0,1])} \geqslant \frac{1}{32(2 + \ln(1 + 2/\delta))} (T_1 - T_2)$$

with

$$T_1 = \frac{1}{qD} \sum_{m=0}^{qD-1} \int_m^{m+1} \left| \sum_{j=1}^J f_{k_j}\left(\frac{m}{qD}\right) e^{2i\pi r_j s} \right| \mathrm{d}s,$$

and

$$T_2 = \frac{1}{qD} \sum_{m=0}^{qD-1} \int_m^{m+1} \left| \sum_{j=1}^{J} \left(f_{k_j}\left(\frac{s}{qD}\right) - f_{k_j}\left(\frac{m}{qD}\right) \right) e^{2i\pi r_j s} \right| \mathrm{d}s.$$

It remains to show that

$$T_1 \geqslant \frac{C_{MPS}}{2} \sum_{j=1}^{J} \frac{\|f_{k_j}\|_{L^1}}{j} \quad \text{and} \quad T_2 \leq \frac{2\pi d}{qD} \sum_{j=1}^{J} \|f_{k_j}\|_{L^1}.$$

Let us start with T_1: using the 1-periodicity in s,

$$\begin{aligned} T_1 &= \frac{1}{qD} \sum_{m=0}^{qD-1} \int_0^1 \left| \sum_{j=1}^{J} f_{k_j}\left(\frac{m}{qD}\right) e^{2i\pi r_j s} \right| \mathrm{d}s, \\ &\geqslant \frac{C_{MPS}}{qD} \sum_{m=0}^{qD-1} \sum_{j=1}^{J} \frac{\left| f_{k_j}\left(\frac{m}{qD}\right) \right|}{j} = \frac{C_{MPS}}{qD} \sum_{j=1}^{J} \frac{1}{j} \sum_{m=0}^{qD-1} \left| f_{k_j}\left(\frac{m}{qD}\right) \right| \end{aligned}$$

with Theorem B. Applying Lemma 3.19 to f_{k_j} and using that $\frac{2\pi d}{qD} < \frac{1}{2}$ with our hypothesis on q and D, we get

$$\frac{1}{qD} \sum_{m=1}^{qD} \left| f_{k_j}\left(\frac{m-1}{qD}\right) \right| \geqslant \frac{\|f_{k_j}\|_{L^1}}{2}$$

and the desired estimate of T_1 follows immediately.

Let us now estimate T_2. For $s \in [m, m+1]$, we have

$$\begin{aligned} \left| \sum_{j=1}^{J} \left(f_{k_j}\left(\frac{s}{qD}\right) - f_{k_j}\left(\frac{m}{qD}\right) \right) e^{2i\pi r_j s} \right| &= \left| \sum_{j=1}^{J} \int_{\frac{m}{qD}}^{\frac{s}{qD}} f'_{k_j}(t) e^{2i\pi r_j s} \,\mathrm{d}t \right| \\ &\leqslant \int_{\frac{m}{qD}}^{\frac{m+1}{qD}} \left| \sum_{j=1}^{J} f'_{k_j}(t) e^{2i\pi r_j s} \right| \mathrm{d}t. \end{aligned}$$

From the 1-periodicity in s, the integral of this quantity over $[m, m+1]$ is the same as the integral over $[0, 1]$. Thus

$$\begin{aligned} T_2 &\leqslant \frac{1}{qD} \sum_{m=0}^{qD-1} \int_m^{m+1} \int_{\frac{m}{qD}}^{\frac{m+1}{qD}} \left| \sum_{j=1}^{J} f'_{k_j}(t) e^{2i\pi r_j s} \right| \mathrm{d}t \,\mathrm{d}s \\ &= \frac{1}{qD} \sum_{m=0}^{qD-1} \int_{\frac{m}{qD}}^{\frac{m+1}{qD}} \int_0^1 \left| \sum_{j=1}^{J} f'_{k_j}(t) e^{2i\pi r_j s} \right| \mathrm{d}s \,\mathrm{d}t \end{aligned}$$

$$= \frac{1}{qD}\int_0^1\int_0^1\left|\sum_{j=1}^J f'_{k_j}(t)e^{2i\pi r_j s}\right|\mathrm{d}s\,\mathrm{d}t$$
$$\leqslant \frac{1}{qD}\sum_{j=1}^J \|f'_{k_j}\|_{L^1([0,1])} \leqslant \frac{2\pi d}{qD}\sum_{j=1}^J \|f_{k_j}\|_{L^1([0,1])}$$

with Bernstein's inequality. □

3.4. *Newman's extremal trigonometric polynomial*

Finally, let us insist on the fact that all results in this section are worse case scenarios *i.e.* the lowest possible L^1 norm of a trigonometric polynomial with eventually a constraint on the spectrum or on the coefficients.

If one looks at the maximal possible value of the L^1-norm under the constraint that all coefficients have modulus 1, then, from Cauchy-Schwarz

$$\int_0^1\left|\sum_{j=0}^N a_j e^{2i\pi jt}\right|\mathrm{d}t \le \left(\int_0^1\left|\sum_{j=0}^N a_j e^{2i\pi jt}\right|^2\mathrm{d}t\right)^{\frac12} = \left(\sum_{j=0}^N |a_j|^2\right)^{\frac12} = \sqrt{N+1}$$

if $|a_j| = 1$ for all j's. However, if one wants to reach the bound $\sqrt{N+1}$, one would need that equality holds in the Cauchy-Schwarz Inequality so $\left|\sum_{j=0}^N a_j e^{2i\pi jt}\right|$ should be constant. But, it is easy to show that the only trigonometric polynomials with constant modulus are monomials (thus in our case, equality can only occur when $N = 0$). Indeed, consider the polynomial $P(z) = \sum_{j=0}^N a_j z^j$ with $a_N \neq 0$. Introduce $P^*(z) = \overline{P(\bar z)} = \sum_{j=0}^N \overline{a_j} z^j$. Then $|P(z)|^2 = c$ when $|z| = 1$ can be written in the form $P(z)P^*(1/z) = c$ for $|z| = 1$, which is now an equality between meromorphic functions over $\mathbb{C}\setminus\{0\}$. As this is true on $\{|z| = 1\}$ it is true over all of $\mathbb{C}\setminus\{0\}$, in particular, P has only 0 as zero thus $P(z) = a_N z^N$.

So the natural question is, how near to $\sqrt{N+1}$ can one get. A result by E. Beller and D. J. Newman [2] shows that one can obtain $\sqrt{N} - O(1)$. For sake of simplicity, we will only present a previous result by D. J. Newman [22] which contains similar ingredients:

Theorem 3.23 (Newman, [22])**.** *There is an absolute constant $c > 0$ and an integer N_0 such that, for every $N \ge N_0$ there is a sequence $a_0, \dots, a_N$ with $|a_j| = 1$ for all j and such that*

$$\int_0^1\left|\sum_{j=0}^N a_j e^{2i\pi jt}\right|\,\mathrm{d}t \ge \sqrt{N} - c.$$

Proof. The example is related to Gauss sums. We fix N and set $\omega = \exp\left(i\frac{\pi}{N+1}\right)$, $a_j = \omega^{j^2}$ and define

$$P(t) = \sum_{j=0}^{N} a_j e^{2i\pi jt} = \sum_{j=0}^{N} \exp\left(2i\pi\left(\frac{j^2}{2(N+1)} + jt\right)\right).$$

The theorem is based on the following lemma:

Lemma 3.24. *With the previous notation, when $N \to +\infty$*

$$\int_0^1 |P(t)|^4 \,\mathrm{d}t = N^2 + O(N^{3/2}).$$

We postpone the proof of the lemma and show how we can conclude with it.

Parseval's identity gives $\|P\|_2 = \sqrt{n+1}$ and then Hölder's inequality implies

$$N \leq N+1 = \int_0^1 |P(t)|^2 \,\mathrm{d}t \leq \left(\int_0^1 |P(t)| \,\mathrm{d}t\right)^{\frac{2}{3}} \left(\int_0^1 |P(t)|^4 \,\mathrm{d}t\right)^{\frac{1}{3}}.$$

From the lemma, we deduce that there is a constant $A > 0$ such that $\int_0^1 |P(t)|^4 \,\mathrm{d}t \leq N^2 + AN^{3/2}$ when N is large enough, so that

$$\begin{aligned} \int_0^1 |P(t)| \,\mathrm{d}t &\geq \frac{N^{3/2}}{(N^2 + AN^{3/2})^{1/2}} = N^{1/2}\frac{1}{(1 + AN^{-1/2})^{1/2}} = N^{1/2}\big(1 - \frac{A}{2}N^{-1/2} + o(N^{-1/2})\big) \\ &\geq N^{1/2} - A \end{aligned}$$

for N large enough. □

It remains to prove the lemma:

Proof of Lemma 3.24. We write

$$\left|\sum_{j=0}^{N} a_j e^{2i\pi jt}\right|^2 = \left(\sum_{j=0}^{N} a_j e^{2i\pi jt}\right)\left(\sum_{k=0}^{N} \overline{a_k} e^{-2i\pi kt}\right) := \sum_{\ell=-N}^{N} c_\ell e^{2i\pi \ell t}.$$

Note that $c_0 = \sum_{j=0}^{N} a_j \overline{a_j} = N+1$, $c_{-\ell} = \overline{c_\ell}$ and, for $\ell \geq 1$,

$$\begin{aligned} c_\ell &= \sum_{k=0}^{N-\ell} a_{\ell+k}\overline{a_k} = \sum_{k=0}^{N-\ell} \omega^{(\ell+k)^2}\omega^{-k^2} = \omega^{\ell^2}\sum_{k=0}^{N-\ell}\omega^{2k\ell} \\ &= \omega^{\ell^2}\frac{1 - \omega^{2(N-\ell+1)\ell}}{1 - \omega^{2\ell}} = -\omega^{-\ell}\frac{\omega^{\ell^2} - \omega^{-\ell^2}}{\omega^\ell - \omega^{-\ell}} \end{aligned}$$

since $\omega^{2(N+1)} = 1$. We thus obtain that

$$|c_\ell|^2 = \frac{\sin^2 \frac{\pi}{N+1}\ell^2}{\sin^2 \frac{\pi}{N+1}\ell}.$$

On the other hand, from Parseval,

$$\int_0^1 |P(t)|^4 \,\mathrm{d}t = \sum_{\ell=-N}^{N} |c_\ell|^2 = (N+1)^2 + 2\sum_{\ell=1}^{N} |c_\ell|^2$$

since $c_0 = N+1$ and $|c_{-\ell}| = |c_\ell|$.

To compute the last sum, we split it into 4 parts

$$\begin{aligned} S_1 &= \textstyle\sum_{1\leq\ell\leq\sqrt{N}} |c_\ell|^2 & S_2 &= \textstyle\sum_{\sqrt{N}<\ell\leq\frac{N+1}{2}} |c_\ell|^2 \\ S_3 &= \textstyle\sum_{\frac{N+1}{2}<\ell<N+1-\sqrt{N}} |c_\ell|^2 & S_4 &= \textstyle\sum_{N+1-\sqrt{N}\leq\ell\leq N} |c_\ell|^2. \end{aligned}$$

As $|c_{N+1-\ell}| = |c_\ell|$ we have $S_4 \leq S_1$ and $S_3 \leq S_2$ so that we only need to show that S_1 and S_3 are both $O(N^{3/2})$.

For S_1, we use the estimate $\left|\dfrac{\sin \ell\theta}{\sin\theta}\right| \leq \ell$ to bound $|c_\ell|^2 \leq \ell^2$, leading to

$$S_1 \leq \sum_{1\leq\ell\leq\sqrt{N}} \ell^2 \leq N \sum_{1\leq\ell\leq\sqrt{N}} 1 = N^{3/2}.$$

Further, as for $0 \leq \theta \leq \dfrac{\pi}{2}$, $\sin\theta \geq \dfrac{2}{\pi}\theta$, we bound $|c_\ell|^2 \leq \dfrac{(N+1)^2}{4\ell^2}$ so that

$$S_2 \leq \frac{(N+1)^2}{4} \sum_{\sqrt{N}<\ell\leq\frac{N+1}{2}} \frac{1}{\ell^2} \leq \frac{(N+1)^2}{4} \int_{\sqrt{N}-1}^{+\infty} \frac{\mathrm{d}t}{t^2} = \frac{(N+1)^2}{4(\sqrt{N}-1)} = O(N^{3/2})$$

as claimed. □

4. L^1 estimates with real frequencies

4.1. *Some results for the Besikovitch norm*

We will now present two results for the Besikovich norms of non-harmonic trigonometric polynomials.

We start with the analogue of Mc Gehee, Pigno and Smith's result. Let us present a nice argument by Hudson and Leckband that shows that the estimate for the Besikovich norm follows from Theorem B. The converse is of course true as well and the result was obtained directly in [15] and even allowed to slightly improve the constant.

Theorem 4.1 (Hudson & Leckband [12]). *For $(\lambda_j)_{j\geq 0}$ be real numbers with $\lambda_{j+1}-\lambda_j > 0$ and let $(a_j)_{j\geq 0}$ be a sequence of complex numbers with finite support. Then*

$$C_{MPS}\sum_{j=0}^{+\infty}\frac{|a_j|}{j+1}\leq \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{j=0}^{+\infty}a_je^{2i\pi\lambda_j t}\right|\,\mathrm{d}t$$

where C_{MPS} is the same constant as in Theorem B.

Moreover, assume that (λ_k) is q-lacunary i.e. $\lambda_0\geq 1$ and $\lambda_{k+1}\geq q\lambda_k$. Then, for $1\leq p<+\infty$,

$$A_{p,q}\left(\sum_{j=0}^{+\infty}|a_j|^2\right)^{1/2}\leq \lim_{T\to+\infty}\left(\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{j=0}^{+\infty}a_je^{2i\pi\lambda_j t}\right|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\leq B_{p,q}\left(\sum_{j=0}^{+\infty}|a_j|^2\right)^{1/2}$$

with $A_{p,q}$, $B_{p,q}$ the constants in Theorem 3.6.

Actually, only the first part is in [12]. The second part is here obtained with almost the same proof and seems to be new.

Proof. Let $a_0,\ldots,a_N$ be complex numbers, $\lambda_0<\lambda_1<\cdots<\lambda_N$ be real numbers and

$$\Phi(t)=\sum_{j=0}^{N}a_je^{2i\pi\lambda_j t}.$$

Let $\varepsilon>0$. By a lemma of Dirichlet ([29, p 235], [9]), there is an increasing sequence of integers $(M_n)_{n\geq 1}$ and, for each $n\geq 1$ a finite family of integers $(N_{j,n})_{j=0,\ldots,N}$ such that

$$\left|\lambda_j-\frac{N_{j,n}}{M_n}\right|<\frac{\varepsilon}{M_n}\qquad\text{for } j=0,\ldots,N$$

which implies that

$$\left|e^{2i\pi\lambda_j t}-e^{2i\pi\frac{N_{j,n}}{M_n}t}\right|\leq 2\pi\left|\lambda_j-\frac{N_{j,n}}{M_n}\right||t|\leq 2\pi\frac{\varepsilon}{M_n}|t|\qquad\text{for } j=0,\ldots,N.$$

Define the M_n-periodic function

$$\Psi_n(t)=\sum_{j=0}^{N}a_je^{2i\pi N_{j,n}t/M_n}$$

and note that, for $t\in[-M_n/2,M_n/2]$,

$$|\Phi(t)-\Psi_n(t)|\leq\sum_{j=0}^{N}|a_j|\left|e^{2i\pi\lambda_j t}-e^{2i\pi\frac{N_{j,n}}{M_n}t}\right|\leq 2\pi\varepsilon\sum_{j=0}^{N}|a_j|.$$

But then

$$\left|\left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Phi(t)|^p\,\mathrm{d}t\right)^{1/p}-\left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Psi_n(t)|^p\,\mathrm{d}t\right)^{1/p}\right|\leq\left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Phi(t)-\Psi_n(t)|^p\,\mathrm{d}t\right)^{1/p}$$

$$\leq\left(2\pi\sum_{j=0}^{N}|a_j|\right)^{1/p}\varepsilon^{1/p}.$$

We can now conclude as follows. First, in the general case, as Ψ_n is M_n-periodic, we may apply Theorem B to obtain

$$\begin{aligned} C_{MPS}\sum_{j=0}^{N}\frac{|a_k|}{j+1} &\leq \frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Psi_n(t)|\,\mathrm{d}t \\ &\leq \frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Phi(t)|\,\mathrm{d}t+\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Phi(t)-\Psi_n(t)|\,\mathrm{d}t \\ &\leq \frac{1}{M_n}\int_{-M_n/2}^{M_n/2}|\Phi(t)|\,\mathrm{d}t+2\pi\varepsilon\sum_{j=0}^{N}|a_j| \end{aligned}$$

Letting $n\to+\infty$ and then $\varepsilon\to 0$ we obtain Theorem 4.1:

$$C_{MPS}\sum_{k=0}^{N}\frac{|a_k|}{k+1}\leq \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=0}^{N}a_k e^{2i\pi\lambda_k t}\right|\,\mathrm{d}t.$$

Let us now assume further that $\lambda_{k+1}\geq q\lambda_k$. Let $\tilde{q}$ be such that $1<\tilde{q}<q$ and, assume that ε has been chosen such that $q-(1+q)\varepsilon>\tilde{q}$. Observe that

$$\begin{aligned} \frac{N_{j+1,n}}{M_n} &\geq \lambda_{j+1}-\left|\lambda_{j+1}-\frac{N_{j+1,n}}{M_n}\right|\geq q\lambda_j-\frac{\varepsilon}{M_n} \\ &\geq q\frac{N_{j,n}}{M_n}-q\left|\lambda_j-\frac{N_{j,n}}{M_n}\right|-\frac{\varepsilon}{M_n} \\ &\geq q\frac{N_{j,n}}{M_n}-(1+q)\frac{\varepsilon}{M_n}, \end{aligned}$$

that is $N_{j+1,n}\geq qN_{j,n}-(1+q)\varepsilon\geq\tilde{q}N_{j,n}$.

Applying (3.6) to $n_k=N_{k,n}$ and $M=M_n$ we obtain

$$A_{p,\tilde{q}}\left(\sum_{j=0}^{+\infty}|a_j|^2\right)^{\frac12}\leq\left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}\left|\sum_{j\geq0}a_j e^{2i\pi\frac{N_{j,n}}{M_n}t}\right|^p\mathrm{d}t\right)^{\frac1p}\leq B_{p,\tilde{q}}\left(\sum_{j=0}^{+\infty}|a_j|^2\right)^{\frac12}.$$

From (**??**) we conclude that

$$\begin{aligned} A_{p,\tilde{q}}\left(\sum_{j=0}^{+\infty}|a_j|^2\right)^{\frac12}-\left(2\pi\sum_{j=0}^{N}|a_j|\right)^{1/p}\varepsilon^{1/p} &\leq \left(\frac{1}{M_n}\int_{-M_n/2}^{M_n/2}\left|\sum_{j\geq0}a_j e^{2i\pi\frac{N_{j,n}}{M_n}t}\right|^p\mathrm{d}t\right)^{\frac1p} \\ &\leq B_{p,\tilde{q}}\left(\sum_{j=0}^{+\infty}|a_j|^2\right)^{\frac12}+\left(2\pi\sum_{j=0}^{N}|a_j|\right)^{1/p}\varepsilon^{1/p}. \end{aligned}$$

The result follows by letting $\varepsilon\to0$ and then $\tilde{q}\to q$. □

4.2. *Ingham's estimate*

In [13], Ingham also proved an L^1 estimate for trigonometric sums that he improved further in [14]. The inequality was further improved by Mordell [20].

Theorem 4.2 (Ingham [14])**.** *Let* $\gamma > 0$*. Let* $(\lambda_j)_{j\in\mathbb{Z}}$ *be a sequence of real numbers such that* $\lambda_{j+1} - \lambda_j \geq \gamma$*. Let* $(c_j)_{j\in\mathbb{Z}}$ *be a sequence of complex numbers. Then, for* $T \geq \dfrac{1}{\gamma}$ *and every* N*,*

$$\frac{1}{2}\max_{j=-N,\ldots,N}|c_j| \leq \frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{j=-N}^{N} c_j e^{2i\pi\lambda_j t}\right|\mathrm{d}t.$$

Ingham's estimate is a bit weaker than Nazarov's estimate (Theorem C) when the sum is one sided, *i.e.* if $c_j = 0$ for $j = -N, \ldots, -1$. On the other hand, the estimate by McGehee, Pigno and Smith and thus also the one in Nazarov's inequality are not valid for every two sided trignonmetric polynomial (*see* Section 5).

Proof. We will present the slightly simpler proof from [13] which leads to worse constants and is only valid form $T > \dfrac{1}{\gamma}$. We will only prove the result for $\gamma > 1$ and $T = 1$. A scaling argument allows to conclude.

Thus, take a sequence $(\lambda_j)_{j\in\mathbb{Z}}$ such that $\lambda_{j+1} - \lambda_j \geq \gamma$ thus for $j \neq \ell$,

$$|\lambda_j - \lambda_\ell| \geq \gamma|j-\ell| > 1. \tag{4.1}$$

We then fix $N \geq 1$ and a finite sequence $(c_j)_{j=-N,\ldots,N}$. We take ℓ so that $|c_\ell| = \max_{j=-N,\ldots,N}|c_j|$.

Note that if $h \in L^1(\mathbb{T})$,

$$\begin{aligned}\int_{\mathbb{R}} h(t)\sum_{j=-N}^{N} c_j e^{2i\pi\lambda_j t}e^{-2i\pi\lambda_\ell t}\,\mathrm{d}t &= \int_{\mathbb{R}}\sum_{j=-N}^{N} c_j h(t) e^{2i\pi(\lambda_j-\lambda_\ell)t}\,\mathrm{d}t \\ &= \sum_{j=-N}^{N} c_j\widehat{h}(\lambda_j-\lambda_\ell) \\ &= c_\ell\widehat{h}(0) + \sum_{j\in\{-N,\ldots,N\}\setminus\{\ell\}} c_j\widehat{h}(\lambda_j-\lambda_\ell).\end{aligned}$$

As for Ingham's L^2 estimate, we consider $h(t) = \mathbb{1}_{[-1/2,1/2]}(t)\cos\pi t$. As h is supported in $[-1/2,1/2]$ and $|h| \leq 1$, and as $|c_j| \leq |c_\ell|$ we obtain

$$|c_\ell|\left(|\widehat{h}(0)| - \sum_{j\in\{-N,\ldots,N\}\setminus\{\ell\}}|\widehat{h}(\lambda_j-\lambda_\ell)|\right) \leq \int_{-1/2}^{1/2}\left|\sum_{j=-N}^{N} c_j e^{2i\pi\lambda_j t}\right|\mathrm{d}t.$$

On the other hand, by definition of f, $\widehat{h}(0)=\dfrac{2}{\pi}$ and, for $|t|\geq 1$, $|\widehat{h}(t)|\leq\dfrac{\widehat{h}(0)}{4t^2-1}$. With (4.1) we thus get

$$\begin{aligned}\sum_{j\in\{-N,\ldots,N\}\setminus\{\ell\}}|\widehat{h}(\lambda_j-\lambda_\ell)| &\leq \widehat{h}(0)\sum_{j\in\{-N,\ldots,N\}\setminus\{\ell\}}\frac{1}{4\gamma_1^2(j-\ell)^2-1}\leq\frac{2\widehat{h}(0)}{\gamma^2}\sum_{k=1}^{+\infty}\frac{1}{4k^2-1}\\ &= \frac{\widehat{h}(0)}{\gamma^2}\sum_{k=1}^{+\infty}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)=\frac{\widehat{h}(0)}{\gamma^2}.\end{aligned}$$

We thus obtain

$$\frac{2(\gamma^2-1)}{\pi\gamma^2}\max_{j\in\{-N,\ldots,N\}}|c_j|\leq\int_{-1/2}^{1/2}\left|\sum_{j=-N}^{N}c_je^{2i\pi\lambda_jt}\right|\mathrm{d}t$$

which □

Letting $T\to+\infty$ and using the constant in the proof (which is slightly better for large T) we obtain:

Corollary 4.3. *Let $\gamma>0$. Let $(\lambda_j)_{j\in\mathbb{Z}}$ be a sequence of real numbers such that $\lambda_{j+1}-\lambda_j\geq\gamma$. Let $(c_j)_{j\in\mathbb{Z}}$ be a sequence of complex numbers. Then, for every N,*

$$\max_{j\in\{-N,\ldots,N\}}|c_j|\leq\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{j=-N}^{N}c_je^{2i\pi\lambda_jt}\right|\mathrm{d}t.$$

In particular, for every N,

$$\frac{2}{\pi}\max_{j\in\{-N,\ldots,N\}}|c_j|\leq\int_{-1/2}^{1/2}\left|\sum_{j=-N}^{N}c_je^{2i\pi jt}\right|\mathrm{d}t.$$

A second corollary is obtained by interpolating the L^1 and L^2 inequalities:

Corollary 4.4. *Let $\gamma>0$. Let $(\lambda_j)_{j\in\mathbb{Z}}$ be a sequence of real numbers such that $\lambda_{j+1}-\lambda_j\geq\gamma$. Let $(c_j)_{j\in\mathbb{Z}}$ be a sequence of complex numbers. Then, for $T>\dfrac{1}{\gamma}$ and every N, $1<p<2$ and p' the dual index $\dfrac{1}{p}+\dfrac{1}{p'}=1$,*

$$\left(\frac{2(T^2\gamma^2-1)}{\pi T^2\gamma^2}\right)^{\frac{2-p}{p}}\big(C(T,\gamma)\big)^{2\frac{p-1}{p}}\left(\sum_{j=-N}^{N}|c_j|^{p'}\right)^{\frac{1}{p'}}\leq\left(\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{j=-N}^{N}c_je^{2i\pi\lambda_jt}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}$$

with

$$C(T,\gamma)=\begin{cases}\dfrac{\pi^2}{8}\dfrac{(\gamma T)^2-1}{(\gamma T)^3} & \text{for } \dfrac{1}{\gamma}<T\leq\dfrac{2}{\gamma}\\ \dfrac{\pi^2}{64} & \text{for } T\geq\dfrac{2}{\gamma}\end{cases}.$$

In particular, letting $T \to +\infty$,

$$\frac{\pi^{\frac{5p-6}{p}}}{2^{4\frac{2p-1}{p}}}\left(\sum_{j=-N}^{N}|c_j|^{p'}\right)^{\frac{1}{p'}} \leq \left(\int_{-1/2}^{1/2}\left|\sum_{j=-N}^{N}c_je^{2i\pi jt}\right|^p \mathrm{d}t\right)^{\frac{1}{p}}.$$

4.3. *Nazarov's theorem*

Theorem 4.5 (Quantitative version of Nazarov theorem for small δ)**.** *There exists positive constants c_*, δ_* such that, for $0 < \delta < \delta_*$ the following holds:*

for every $\lambda_0 < ... < \lambda_N$ of real numbers satisfying $\lambda_{k+1} - \lambda_k \geqslant 1$ and $a_0, ..., a_N$ complex numbers,

$$\sum_{k=0}^{N}\frac{|a_k|}{k+1} \leqslant \frac{c_*}{\delta^{15/2}}\int_{-\frac{1+\delta}{2}}^{\frac{1+\delta}{2}}\left|\sum_{k=0}^{N}a_ke^{2i\pi\lambda_kt}\right|\mathrm{d}t. \tag{4.2}$$

First, we start by some preliminary notations and results. Let $I = I_\delta := [-\frac{1+\delta}{2}, \frac{1+\delta}{2}]$.

Let us fix $(\lambda_k)_{k=0,\dots,N} \subset \mathbb{R}$ with $\lambda_{k+1} - \lambda_k \geq 1$ for every k, $(a_k)_{k=0,\dots,N}$ a sequence of complex numbers and write $|a_k| = a_ku_k$ with $|u_k| = 1$. Let

$$S = \sum_{k=0}^{N}\frac{|a_k|}{k+1} \qquad \text{and} \qquad \phi(t) = \sum_{k=0}^{N}a_ke^{2i\pi\lambda_kt},$$

so that we must find C_δ such that:

$$\|\phi\|_{L^1(I)} \geq C_\delta S. \tag{4.3}$$

We define

$$S_\delta = \sum_{k=0}^{N}\frac{|a_k|}{k+N_\delta} \quad \text{and} \quad T_\delta(t) = \sum_{k=0}^{N}\frac{u_k}{k+N_\delta}e^{-2i\pi\lambda_kt}$$

where N_δ is a large integer that we will adjust through the proof. This integer will be of the form $N_\delta = 2^{m_\delta}$. We will prove that

$$\|\phi\|_{L^1(I)} \geq B_\delta S_\delta \tag{4.4}$$

and, as $k + N_\delta \leq (k+1)N_\delta$, $S_\delta \geq \dfrac{S}{N_\delta}$ so that we obtain the desired inequality (4.3) with a constant $C_\delta = \dfrac{B_\delta}{N_\delta}$.

The meaning of N_δ is the following. Consider

– a new sequence of frequencies $(\tilde\lambda_j)_{j\in\mathbb{Z}}$ such that $\tilde\lambda_j = \lambda_{j-N_\delta}$ for $j = N_\delta, \dots, N_\delta + N$. and then $\tilde\lambda_j = \lambda_0 + j - N_\delta$ for $j < N_\delta$ and $\tilde\lambda_j = \lambda_N + j - N$ for $j > N$. In particular, we still have $|\tilde\lambda_j - \tilde\lambda_k| \geq 1$. In other words, the sequence $(\lambda_j)_{j=0,\dots,N}$ is completed into a sequence $(\lambda_j)_{j\in\mathbb{Z}}$ that is still 1-separated and then shifting it by N_δ.

– A new sequence of complex numbers $(\tilde{a}_j)_{j\in\mathbb{Z}}$ with $\tilde{a}_j = a_{j-N_\delta}$ for $j = N_\delta, \dots, N_\delta + N$ and $\tilde{a}_j = 0$ for other j's. In other words, the sequence $(a_j)_{j=0,\dots,N}$ is completed into a sequence $(a_j)_{j\in\mathbb{Z}}$ by 0-padding it and then shifting it by N_δ.

Then (4.4) reads

$$\int_{I_\delta} \left| \sum_{j\in\mathbb{Z}} \tilde{a}_j e^{2i\pi\tilde{\lambda}_j t} \right| \mathrm{d}t \geq B_\delta \sum_{j\in\mathbb{Z}} \frac{|\tilde{a}_j|}{j + N_\delta}$$

with the convention that $0/0 = 0$.

Note that, up to adding 0 terms at the end of the sequence a_j, we may assume that $N + N_\delta$ is of the form $2^{n_\delta} - 1$ for some integer n_δ. This will allow us to write

$$\sum_{k=0}^{N} = \sum_{j=m_\delta}^{n_\delta} \sum_{2^j \leq r+N_\delta < 2^{j+1}} .$$

Next, as in Ingham's proof, we will introduce an auxillary function. Again, we consider

$$h(t) = \begin{cases} \cos(\pi t) & \text{if } |t| \leqslant \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

whose Fourier transform is given by $\widehat{h}(\lambda) = \dfrac{2}{\pi} \dfrac{\cos(\pi\lambda)}{1 - 4\lambda^2}$. We will need to smooth a bit this function to obtain a better decay of the Fourier transform and thereby slightly enlarge its support. More precisely, let $p = 11$, $q = 8$ and let

$$f_\delta(t) = \frac{p+q}{\delta} \mathbb{1}_{[-\frac{\delta}{2(p+q)}, \frac{\delta}{2(p+q)}]}(t)$$

and define its $p+q$-fold self convolution[1]

$$g_\delta = *_{p+q} f_\delta.$$

Clearly ψ_δ is non-negative, even and with support $[-\frac{\delta}{2}, \frac{\delta}{2}]$. Finally, we define φ_δ as

$$\varphi_\delta = \frac{\pi}{2} h * g_\delta. \tag{4.5}$$

In the following lemma, we list the properties needed on φ_δ. They are all established via easy calculus and straight forward Fourier analysis.

Lemma 4.6. *There is a $c_0 > 0$ and a $\delta_0 > 0$ such that, if $0 < \delta < \delta_0$ then,*

1. $\widehat{\varphi_\delta}(\lambda) = \dfrac{\cos(\pi\lambda)}{1 - 4\lambda^2} \operatorname{sinc}^{p+q}\left(\dfrac{\pi\delta\lambda}{p+q}\right)$,
2. $\|\varphi_\delta\|_\infty \leqslant \dfrac{\pi}{2}$.

1If χ is a function, we write $*_2\psi = \psi * \psi$ for the convolution of ψ with itself and then define inductively $*_{k+1}\psi = (*_k\psi) * \psi$.

3. *Let* $D_\delta = c_0\delta^{-\frac{p+q}{p+2}} \geq 1$ *and* $\nu > 0$ *then, for* $|\lambda| \geqslant \max(1, D_\delta \nu^{\frac{1}{p+2}})$,

$$|\widehat{\varphi}(\lambda)| \leqslant \frac{\nu}{|\lambda|^q}$$

4. *Let* $\gamma_\delta = \left(\operatorname{sinc}\dfrac{\pi\delta}{p+q}\right)^{p+q}$ *then, for* $|\lambda| \geq 1$,

$$|\widehat{\varphi_\delta}(\lambda)| \leqslant \frac{\gamma_\delta}{4\lambda^2 - 1}.$$

From now on, we will assume that $0 < \delta < \frac{7}{3\pi} < 1$ so that $D_\delta \geq 1$. In particular, when $|\lambda| \geq D_\delta$, we have $|\widehat{\varphi}(\lambda)| \leqslant |\lambda|^{-3}$.

Proof of Lemma 4.6. The first one is simple Fourier analysis.

For the second one, we write

$$\|\varphi_\delta\|_\infty \leqslant \frac{\pi}{2}\|g_\delta\|_1\|h\|_\infty \leqslant \frac{\pi}{2}\|f_\delta\|_1^{p+q}\|h\|_\infty$$

and use simple calculus to conclude.

For the third one, we notice that $4\lambda^2 - 1 \geq 3\lambda^2$ when $\lambda \geq 1$ thus

$$|\widehat{\varphi_\delta}(\lambda)| = \left|\frac{\cos(\pi\lambda)}{4\lambda^2-1}\left(\frac{\sin(\frac{\pi\delta\lambda}{p+q})}{\frac{\pi\delta\lambda}{p+q}}\right)^{2p}\right| \leqslant \frac{1}{3}\left(\frac{p+q}{\pi}\right)^{p+q}\delta^{-(p+q)}\frac{1}{|\lambda|^{p+2}}\frac{1}{|\lambda|^q} \leqslant \frac{\nu}{|\lambda|^q}$$

since $|\lambda| \geq \dfrac{1}{3^{\frac{1}{p+2}}}\left(\dfrac{p+q}{\pi}\right)^{\frac{p+q}{p+2}}\delta^{-\frac{p+q}{p+2}}\nu^{\frac{1}{p+2}}$. Thus this fact is established with $c_0 = \dfrac{1}{3^{\frac{1}{p+2}}}\left(\dfrac{p+q}{\pi}\right)^{\frac{p+q}{p+2}}$.

For the last one, we take δ_0 small enough to have $\operatorname{sinc}\dfrac{\pi\delta_0}{p+q} = \sup_{t\geq\delta_0}|\operatorname{sinc} t|$ and then $|\widehat{g_\delta}(\lambda)| \leq \gamma_\delta$ when $|\lambda| \geq 1$. The first identity allows to conclude. □

We can now state the first crucial result in this proof.

Lemma 4.7. *There exist* $\delta_1 > 0$, $c_1 > 0$ *such that, if* $0 < \delta < \delta_1$, $0 < c_1\delta^2 < 1 - \sqrt{\gamma_\delta}$.

Moreover, let

$$\alpha_\delta = 1 - \left(c_1\delta^2 + \frac{\gamma_\delta}{1 - c_1\delta^2}\right)$$

and let m_δ *be such that*

$$N_\delta = 2^{m_\delta} \geq \delta^{-7/2}.$$

Then, for $0 \leqslant k \leqslant N$,

$$\sum_{\substack{0\leqslant j\leqslant N\\ j\neq k}} \frac{|\widehat{\varphi}(\lambda_j - \lambda_k)|}{j + N_\delta} \leqslant \frac{1 - \alpha_\delta}{k + N_\delta}.$$

Proof. The Taylor expension when $\delta \to 0$ of $1-\sqrt{\gamma_\delta}$ is of the form

$$1-\sqrt{\gamma_\delta} = A\delta^2 + O(\delta^4)$$

with $A > 0$. Thus, if $c_1 < A$, for δ small enough, $0 < c_1\delta^2 < 1-\sqrt{\gamma_\delta}$. Next, notice that $0 < 1-\left(\beta+\dfrac{\gamma_\delta}{1-\beta}\right) < 1$ if $0<\beta<1-\sqrt{\gamma_\delta}$ which shows that $0<\alpha_\delta<1$. For future use, note that there is a $\kappa > 0$ such that

$$\alpha_\delta = \kappa\delta^2 + O(\delta^4). \tag{4.6}$$

We will further assume that δ is small enough for

$$\delta^{-7/2} \geq \max\left(c_1^{-\frac{q+1}{q-2}}\delta^{-2\frac{q+1}{q-2}}, c_2\delta^{-2-\frac{p+q}{p+2}}\right) = \max\left(c_1^{-\frac{q+1}{q-2}}\delta^{-3}, \frac{c_0}{c_1}\delta^{-2-19/13}\right)$$

since we chose $q=8$ and $p=11$.

Note that, for every $\varepsilon > 0$, the power $7/2$ could be reduced to $3+\varepsilon$ by taking p large enough, but can not be reduced below 3 with this construction.

We can now turn to the estimate itself. Set $\beta = c_1\delta^2$ and split the sum in the left hand side of the main inequality into two sums

$$E := \sum_{\substack{0\leqslant j\leqslant N\\ j\neq k}} \frac{|\widehat{\varphi}(\lambda_j-\lambda_k)|}{j+N_\delta} = E_1+E_2$$

where

$$E_1 = \sum_{j+N_\delta<(1-\beta)(k+N_\delta)} \frac{|\widehat{\varphi}(\lambda_j-\lambda_k)|}{j+N_\delta}$$

and

$$E_2 = \sum_{\substack{j+N_\delta\geqslant(1-\beta)(k+N_\delta)\\ j\neq k}} \frac{|\widehat{\varphi}(\lambda_j-\lambda_k)|}{j+N_\delta}.$$

The result is obtained if we prove the two estimates

$$E_1 \leqslant \frac{\beta}{k+N_\delta} \quad \text{et} \quad E_2 \leqslant \frac{\gamma_\delta/(1-\beta)}{k+N_\delta}.$$

Now, as $N_\delta \geq \dfrac{c_0}{c_1}\delta^{-2-\frac{p+q}{p+2}}$, $\beta N_\delta \geqslant D_\delta$. Then, if j is an index corresponding to E_1,

$$|\lambda_k-\lambda_j| \geqslant |k-j| = (k+N_\delta)-(j+N_\delta) \geqslant \beta(k+N_\delta) \geqslant \beta N_\delta \geqslant D_\delta.$$

Hence, from Lemma 4.63 with $\nu = 1$,

$$\begin{aligned} E_1 &\leqslant \sum_{j+N_\delta<(1-\beta)(k+N_\delta)} |\widehat{\varphi}(\lambda_j-\lambda_k)| \leqslant \sum_{j+N_\delta<(1-\beta)(k+N_\delta)} \frac{1}{|\lambda_j-\lambda_k|^q} \\ &\leqslant \sum_{j+N_\delta<(1-\beta)(k+N_\delta)} \frac{1}{\big(\beta(k+N_\delta)\big)^q}. \end{aligned}$$

But, E_1 contains less than k terms so

$$E_1 \leqslant \frac{k}{k+N_\delta}\frac{\beta^{-(q+1)}}{(k+N_\delta)^{q-2}}\frac{\beta}{k+N_\delta} \leqslant \frac{\beta}{k+N_\delta} \tag{4.7}$$

since $N_\delta \geqslant \beta^{-\frac{q+1}{q-2}} = c_1^{-\frac{q+1}{q-2}}\delta^{-2\frac{q+1}{q-2}}$.

We shall now bound E_2. In this sum,

$$j+N_\delta \geqslant (1-\beta)(k+N_\delta) \quad \text{and} \quad |\lambda_j-\lambda_k| \geqslant 1$$

then

$$\begin{aligned} E_2 &\leqslant \frac{1}{(1-\beta)(k+N_\delta)}\sum_{\substack{1\leqslant j\leqslant N\\ j\neq k}}\frac{\gamma_\delta}{4(\lambda_j-\lambda_k)^2-1} \\ &\leqslant \frac{\gamma_\delta}{(1-\beta)(k+N_\delta)}\sum_{\substack{1\leqslant j\leqslant N\\ j\neq k}}\frac{1}{4(j-k)^2-1} \\ &\leqslant \frac{\gamma_\delta}{(1-\beta)(k+N_\delta)}\sum_{\ell=1}^{\infty}\frac{2}{4\ell^2-1}. \end{aligned}$$

Since $\dfrac{2}{4\ell^2-1} = \dfrac{1}{2\ell-1} - \dfrac{1}{2\ell+1}$, we obtain the expected bound $E_2 \leqslant \dfrac{\gamma_\delta/(1-\beta)}{k+N_\delta}$. □

The following lemma is a first step towards proving Theorem 4.5 and is a consequence of Lemma 4.7.

Lemma 4.8. *Let us use the notations of the lemma 4.7. Then,*

$$\left|\int_{I_\delta} T_\delta(t)\left(\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right)\varphi(t)\,\mathrm{d}t\right| \geq \alpha_\delta \sum_{k=0}^{N}\frac{|a_k|}{k+N_\delta}. \tag{4.8}$$

Proof. By definition of T_δ,

$$\begin{aligned} \int_{I_\delta} T_\delta(t)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t &= \sum_{j=0}^{M}\frac{u_j}{j+N_\delta}\int_{I_\delta} e^{-2i\pi\lambda_j t}e^{2i\pi\lambda_k t}\varphi(t)dt \\ &= \sum_{j=0}^{M}\frac{u_j}{j+N_\delta}\widehat{\varphi}(\lambda_j-\lambda_k) \\ &= \frac{u_k}{k+N_\delta} + \sum_{\substack{0\leqslant j\leqslant M\\ j\neq k}}\frac{u_j}{j+N_\delta}\widehat{\varphi}(\lambda_j-\lambda_k) \end{aligned}$$

thus

$$\left|\int_{I_\delta} T_\delta(t)e^{2i\pi\lambda_k t}\varphi(t)dt - \frac{u_k}{k+N_\delta}\right| \leq \sum_{\substack{0\leqslant j\leqslant M\\ j\neq k}}\frac{1}{j+N_\delta}\widehat{\varphi}(\lambda_j-\lambda_k).$$

By applying the lemma 4.7, that

$$\left| \int_{I_\delta} T_\delta(t) e^{2i\pi\lambda_k t}\varphi(t)dt - \frac{u_k}{k+N_\delta}\right| \leqslant \frac{1-\alpha_\delta}{k+N_\delta}.$$

It follows that

$$\left| \int_{I_\delta} T_\delta(t) a_k e^{2i\pi\lambda_k t}\varphi(t)dt - \frac{u_k a_k}{k+N_\delta}\right| \leqslant \frac{1-\alpha_\delta}{k+N_\delta}|a_k|.$$

Using the fact that $u_k a_k = |a_k|$ and the triangular inequality, we obtain

$$\left| \int_{I_\delta} T_\delta(t) \left(\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right)\varphi(t)dt - \sum_{k=0}^{N}\frac{|a_k|}{k+N_\delta}\right| \leqslant (1-\alpha_\delta)\sum_{k=0}^{N}\frac{|a_k|}{k+N_\delta}$$

from which the lemma follows immediately. □

Construction of $\tilde{T}_\delta$

The construction shares several features with the proof of the Littlewood conjecture by McGehee, Pigno and Smith. In particular, we again use a dyadic decomposition and the same procedure to obtain a bounded function.

Recall that, from our assumption on N_δ and N, we can write

$$T_\delta(t) = \sum_{k=0}^{M} \frac{u_k}{k+N_\delta} e^{-2i\pi\lambda_k t} = \sum_{j=m_\delta}^{n_\delta} f_j(t)$$

where we set $\mathcal{D}_j = \{k \in \mathbb{N} \,:\, 2^j \leq k < 2^{j+1}\}$ and

$$f_j(t) = \sum_{r+N_\delta\in\mathcal{D}_j} \frac{u_r}{r+N_\delta} e^{-2i\pi\lambda_r t}.$$

Before we estimate the norms of the f_j's, let us recall Hilbert's inequality (*see e.g.* [4, Chapter 10]).

Lemma 4.9 (Hilbert's inequality)**.** *Let $\lambda_1,\ldots,\lambda_N$ be real numbers with $|\lambda_k-\lambda_\ell| \geq 1$ when $k\neq\ell$, and let $z_1,\ldots,z_N$ be complex numbers. We have*

$$\left| \sum_{\substack{1\leq k,\ell\leq N\\ k\neq\ell}} \frac{z_k\overline{z_\ell}}{\lambda_k-\lambda_\ell}\right| \leqslant \pi\sum_{k=1}^{N}|z_k|^2.$$

We can now prove the following:

Lemma 4.10. *For $m_\delta \leqslant j \leqslant n_\delta$,*

1. $\|f_j\|_\infty \leqslant 1$
2. $\|f_j\|_{L^2(I_\delta)} \leqslant 2^{-\frac{j}{2}}\sqrt{|I_\delta|+1}.$

Proof. The first estimate is obtained by straightforward computations: for all t,

$$|f_j(t)| \leqslant \sum_{r+N_\delta \in \mathcal{D}_j} \frac{1}{r+N_\delta} \leqslant \frac{|\mathbb{D}_j|}{2^j} = 1$$

hence $\|f_j\|_\infty \leqslant 1$.

For the second one, set $v_r = \dfrac{u_r}{r+N_\delta}$. Then

$$\begin{aligned}
\|f_j\|^2_{L^2(I_\delta)} &= \int_{I_\delta} f_j(t)\overline{f_j(t)}\,\mathrm{d}t = \int_{I_\delta} \sum_{r+N_\delta, s+N_\delta \in \mathcal{D}_j} v_r\overline{v_s} e^{-2i\pi(\lambda_r-\lambda_s)t}\,\mathrm{d}t \\
&= |I_\delta| \sum_{r+N_\delta} |v_r|^2 + \sum_{\substack{r+N_\delta, s+N_\delta \in \mathcal{D}_j \\ r\neq s}} v_r\overline{v_s} \int_{-|I_\delta|/2}^{|I_\delta|/2} e^{-2i\pi(\lambda_r-\lambda_s)t}\,\mathrm{d}t \\
&= |I_\delta| \sum_{r+N_\delta,} |v_r|^2 + \sum_{\substack{r+N_\delta, s+N_\delta \in \mathcal{D}_j \\ r\neq s}} v_r\overline{v_s} \left(\frac{e^{i|I_\delta|\pi(\lambda_r-\lambda_s)} - e^{-i|I_\delta|\pi(\lambda_r-\lambda_s)}}{2i\pi(\lambda_r-\lambda_s)} \right).
\end{aligned}$$

It follows that

$$\|f_j\|^2_{L^2(I_\delta)} \leqslant |I_\delta| \sum_{r+N_\delta} |v_r|^2 + \frac{1}{2\pi} \left| \sum_{\substack{r,s \\ r\neq s}} \frac{v_r e^{-i\pi\lambda_r(1+\delta)}\overline{v_s e^{-i\pi\lambda_s(1+\delta)}}}{\lambda_s-\lambda_r} \right| + \frac{1}{2\pi} \left| \sum_{\substack{r,s \\ r\neq s}} \frac{v_r e^{i\pi\lambda_r(1+\delta)}\overline{v_s e^{i\pi\lambda_s(1+\delta)}}}{\lambda_s-\lambda_r} \right|$$

We then apply Hilbert's inequality 4.9 to the last two sums to obtan

$$\begin{aligned}
\|f_j\|^2_{L^2(I_\delta)} &\leqslant |I_\delta| \sum_{r+N_\delta} |v_r|^2 + \frac{1}{2} \sum_{r+N_{\delta,\beta} \in \mathcal{D}_j} |v_r|^2 + \frac{1}{2} \sum_{r+N_\delta \in \mathcal{D}_j} |v_r|^2 \\
&= (|I_\delta|+1) \sum_{r+N_\delta} |v_r|^2.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{r+N_\delta \in \mathcal{D}_j} |v_r|^2 &= \sum_{r+N_\delta \in \mathcal{D}_j} \frac{|u_r|^2}{(r+N_\delta)^2} = \sum_{r+N_\delta \in \mathcal{D}_j} \frac{1}{(r+N_\delta)^2} \\
&\leqslant \sum_{r+N_\delta \in \mathcal{D}_j} \frac{1}{2^{2j}} \leqslant \frac{2^j}{2^{2j}} = 2^{-j},
\end{aligned}$$

we get $\|f_j\|^2_{L^2(I_\delta)} \leqslant (|I_\delta|+1)2^{-j}$ as claimed. □

We then write the $|I_\delta|$-periodic Fourier series expansion of $|f_j| \in L^2(I_\delta)$ as

$$|f_j(t)| = \sum_{s\in\mathbb{Z}} a_{s,j} e^{\frac{2i\pi}{|I_\delta|}st},$$

and again define $h_j \in L^2(I_\delta)$ via its Fourier series expansion

$$h_j(t) = a_{0,j} + 2\sum_{s=1}^{\infty} a_{s,j} e^{\frac{2is\pi}{|I_\delta|}t}.$$

As in Lemma 3.11, $\mathrm{Re}(h_j) = |f_j|$ and $\|h_j\|_2 \leqslant \sqrt{2}\|f_j\|_2$.

We again define a sequence $(F_j)_{j \geqslant m_\delta}$ inductively through

$$F_{m_\delta} = f_{m_\delta} \quad \text{and} \quad F_{j+1} = F_j e^{-\eta h_{j+1}} + f_{j+1}$$

where $0 < \eta < 1$ is a small parameter that we will adjust later. As in Lemma 3.12,

$$\|F_j\|_\infty \leqslant \frac{2}{\varepsilon}. \tag{4.9}$$

Lemma 4.11. *Let $m_\delta \leqslant n \leqslant n_\delta$. For $j = m_\delta, \ldots, n$ we define $g_{j,n} = e^{-\varepsilon H_{j,n}}$ with*

$$H_{j,n} = \begin{cases} h_{j+1} + \ldots + h_n & \text{if } j < n \\ 0 & \text{if } j = n \end{cases}.$$

Then $F_n = \sum_{j=m_\delta}^{n} f_j g_{j,n}$. Moreover $\|H_{j,n}\|_2 \leqslant \frac{\sqrt{2(|I_\delta|+1)}}{\sqrt{2}-1} 2^{-\frac{j}{2}}$.

Proof. The first part is the same as Lemma 3.13. Moreover, Lemma 4.10 implies

$$\begin{aligned} \|H_{j,n}\|_2 &= \left\| \sum_{r=j+1}^{n} h_r \right\|_2 \leqslant \sum_{r=j+1}^{n} \|h_r\|_2 \leqslant \sqrt{2} \sum_{r=j+1}^{n} \|f_r\|_2 \\ &\leqslant \sqrt{2(|I_\delta|+1)} \sum_{r=j+1}^{\infty} 2^{-\frac{r}{2}} = \frac{\sqrt{2(|I_\delta|+1)}}{\sqrt{2}-1} 2^{-\frac{j}{2}}, \end{aligned}$$

as stated. □

Lemma 4.12. *Assume that $0 < \eta \leqslant \frac{\sqrt{|I_\delta|}(\sqrt{2}-1)}{\sqrt{2(|I_\delta|+1)}}$. Then, for $m_\delta \leqslant j \leqslant n \leqslant n_\delta$,*

1. $\|g_{j,n} - 1\|_2 \leqslant \eta \|H_{j,n}\|_2 \leqslant \frac{\sqrt{2(|I_\delta|+1)}}{\sqrt{2}-1} 2^{-\frac{j}{2}} \eta$
2. *The Fourier series of $g_{j,n}(t) - 1$ writes $g_{j,n}(t) - 1 = \sum_{s \geqslant 0} c_{s,j} e^{\frac{2is\pi}{|I|}t}$, with $\sum_{s=0}^{+\infty} |c_{s,j}|^2 \leqslant 1$.*

Proof. By Lemma 3.14, $\|g_j - 1\|_2 \leqslant \varepsilon \|H_j\|_2$. Then, since $g_{j,n}$ is analytic, its Fourier series writes

$$g_j(t) - 1 = \sum_{s \geqslant 0} c_{s,j} e^{\frac{2i\pi}{|I_\delta|} st}.$$

But then, with Parseval

$$\left(\sum_{s \geqslant 0} |c_{s,j}|^2 \right)^{\frac{1}{2}} = \frac{1}{\sqrt{|I_\delta|}} \|g_j - 1\|_2 \leqslant \frac{\eta}{\sqrt{|I_\delta|}} \|H_j\|_2 \leqslant \frac{\eta}{\sqrt{|I_\delta|}} \frac{\sqrt{2(|I_\delta|+1)}}{\sqrt{2}-1} 2^{-\frac{j}{2}} \leqslant 2^{-\frac{j}{2}} < 1$$

which implies the claimed bound. □

Now recall that

$$T_\delta = \sum_{m_\delta \leqslant j \leqslant n_\delta} f_j$$

and define

$$\tilde{T}_\delta = F_{n_\delta}.$$

Recall that in (4.9) we proved that this function has a controlled L^∞-norm: $\|\tilde{T}_\delta\|_\infty \leq \dfrac{2}{\eta}$.

The key estimates here is the following;

Lemma 4.13. *Once again, we use the notations of the lemma 4.7. There exists* $\delta_2 > 0$ *such that, if* $0 < \delta < \delta_2$ *and* $N_\delta \geq \delta^{-7/2}$ *then*

$$\left| \int_{I_\delta} (\tilde{T}_\delta - T_\delta)(t) \left(\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t} \right) \varphi(t)\,\mathrm{d}t \right| \leqslant \frac{2}{3}\alpha_\delta \sum_{k=0}^{N} \frac{|a_k|}{k+N_\delta}. \tag{4.10}$$

where φ *is the function defined in* (4.5).

Proof. It is enough to prove that, under the conditions of the lemma, for $0 \leqslant k \leqslant N$ we have

$$\left| \int_{I_\delta} (\tilde{T}_\delta - T_\delta)(t) e^{2i\pi\lambda_k t} \varphi(t)\,\mathrm{d}t \right| \leqslant \frac{2}{3}\frac{\alpha_\delta}{k+N_\delta}. \tag{4.11}$$

Once (4.11) is established, it will then be enough to multiply the left hand side by a_k and to use the triangular inequality

We fix $k \in [0, N]$ and let ℓ the index such that $k + N_\delta \in \mathcal{D}_\ell$. We define R, R_1 and R_2 as follows

$$\begin{aligned} R &= \int_{I_\delta} (\tilde{T}_{\delta,\beta} - T_{\delta,\beta})(t) e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t \\ &= \int_{I} \sum_{m_\delta \leqslant j \leqslant \ell-2} f_j(t)\big(g_j(t)-1\big)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t + \int_{I_\delta} \sum_{\ell-1 \leqslant j \leqslant n_\delta} f_j(t)\big(g_j(t)-1\big)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t \\ &:= R_1 + R_2 \end{aligned}$$

We will first bound R_1. Note that if $s \in \mathbb{Z}$,

$$\begin{aligned} \int_{I_\delta} f_j(t)\varphi(t) e^{2i\pi\lambda_k t} e^{\frac{2i\pi}{|I_\delta|}st}\,\mathrm{d}t &= \int_{I_\delta} \sum_{r+N_\delta \in \mathcal{D}_j} \frac{u_r}{r+N_\delta}\varphi(t) e^{2i\pi(-\lambda_r+\lambda_k+\frac{s}{|I_\delta|})}\,\mathrm{d}t \\ &= \sum_{r+N_\delta \in \mathcal{D}_j} \frac{u_r}{r+N_\delta}\widehat{\varphi}\left(\lambda_r - \lambda_k - \frac{s}{|I_\delta|}\right). \end{aligned}$$

From there, we obtain

$$\int_{I_\delta} f_j(t)\big(g_j(t)-1\big)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t = \int_{I_\delta} f_j(t)\varphi(t)e^{2i\pi\lambda_k t}\sum_{s\geqslant 0} c_{s,j} e^{\frac{2is\pi}{|I_\delta|}t}\,\mathrm{d}t$$

$$
\begin{aligned}
&= \sum_{s=0}^{+\infty} c_{s,j} \int_{I_\delta} f_j(t)\varphi(t) e^{2i\pi\lambda_k t} e^{\frac{2i\pi}{|I_\delta|}st}\,\mathrm{d}t \\
&= \sum_{s=0}^{+\infty} c_{s,j} \sum_{r+N_\delta\in\mathcal{D}_j} \frac{u_r}{r+N_\delta}\widehat{\varphi}\left(\lambda_r-\lambda_k-\frac{s}{|I_\delta|}\right) \\
&= \sum_{r+N_\delta\in\mathcal{D}_j} \frac{u_r}{r+N_\delta}\sum_{s=0}^{\infty} c_{s,j}\widehat{\varphi}\left(\lambda_r-\lambda_k-\frac{s}{|I_\delta|}\right).
\end{aligned}
$$

So finally we get

$$
R_1 = \sum_{m_\delta\leqslant j\leqslant \ell-2}\ \sum_{r+N_\delta\in\mathcal{D}_j} \frac{u_r}{r+N_\delta}\sum_{s=0}^{\infty} c_{s,j}\widehat{\varphi}\left(\lambda_r-\lambda_k-\frac{s}{|I_\delta|}\right).
$$

Let $c_s(r)=c_{s,j}$ if $r+N_\delta\in I_j$. Since $\widehat{\varphi}$ is an even function, we can write

$$
R_1 = \sum_{2^{m_\delta}\leqslant r+N_\delta<2^{\ell-1}} \frac{u_r}{r+N_\delta}\sum_{s=0}^{\infty} c_s(r)\widehat{\varphi}\left(\lambda_k-\lambda_r+\frac{s}{|I_\delta|}\right) := \sum_{2^{m_\delta}\leqslant r+N_\delta<2^{\ell-1}} \frac{u_r}{r+N_\delta}E_r.
$$

From Lemma 4.6, recall that, with $D_\delta = c_0\delta^{-\frac{p+q}{p+2}}$ and $\nu=\dfrac{\alpha_\delta}{|I_\delta|^{1/q}}$ then, for

$$
|\lambda| \geqslant \max(1, D_\delta \nu^{-\frac{1}{p+2}}),
$$

we have

$$
|\widehat{\varphi}(\lambda)| \leqslant \frac{\alpha_\delta}{(|I_\delta||\lambda|)^q}. \tag{4.12}
$$

Now, $s\geqslant 0$, $|I_\delta \geq 1$ and, as $r+N_\delta<2^{\ell-1}$, $2^\ell\leqslant k+N_\delta<2^{\ell+1}$, $\lambda_k>\lambda_r$ thus

$$
\begin{aligned}
|I_\delta|\left|\lambda_k-\lambda_r+\frac{s}{|I_\delta|}\right| &= |I_\delta|(\lambda_k-\lambda_r)+s \geqslant \lambda_k-\lambda_r+s\geqslant k-r+s \qquad (4.13)\\
&= (k+N_\delta)-(r+N_\delta) > 2^\ell-2^{\ell-1}=2^{\ell-1}\geqslant 2^{m_\delta-1}.
\end{aligned}
$$

Further, from (4.6) and $1\leq |I_\delta|\leq 2$, $D_\delta\nu^{-\frac{1}{p+2}} \leq c_3\delta^{-\frac{p+q+2}{p+2}} = c_3\delta^{-\frac{20}{12}}$. Thus, choosing m_δ sufficiently large for $N_\delta = 2^{m_\delta}\delta^{-7/2}$ and $\delta\leq\delta_2$ for some $\delta_2>0$ small enough, we are able to apply (4.12) and obtain, with (4.13)

$$
\left|\widehat{\varphi}\left(\lambda_k-\lambda_r+\frac{s}{|I_\delta|}\right)\right| \leqslant \frac{\alpha_\delta}{(k-r+s)^q}.
$$

We can now bound E_r. Since $\sum_{s=0}^{\infty}|c_s(r)|^2\leqslant 1$, the previous bound and Cauchy-Schwarz give us

$$
\begin{aligned}
|E_r| &= \sum_{s=0}^{\infty}|c_s(r)|\left|\widehat{\varphi}\left(\lambda_k-\lambda_r+\frac{s}{1+\delta}\right)\right| \leqslant \alpha_\delta\left(\sum_{s=0}^{\infty}\frac{1}{(k-r+s)^{2q}}\right)^{1/2}\\
&\leqslant \alpha_\delta\left(\sum_{n=k-r}^{\infty}\sum_{n\geqslant k-r}\frac{1}{n^{2q}}\right)^{1/2}
\end{aligned}
$$

$$\leqslant \alpha_\delta \left(\int\limits_{k-r-1}^{\infty} \frac{dt}{t^{2q}} \right)^{1/2} = \frac{\sqrt{2q-1}\alpha_\delta}{(k-r-1)^{q-1/2}}.$$

But $k - r > 2^{\ell-1}$ then $k - r - 1 \geqslant 2^{\ell-1}$. Since $k + N_\delta \in \mathcal{D}_\ell$ *i.e* $2^\ell \leqslant k + N_\delta \leqslant 2^{\ell+1}$ we get $\dfrac{1}{k-r-1} \leqslant \dfrac{4}{k+N_\delta}$ and then

$$|E_r| \leqslant \frac{4^{q-1/2}\sqrt{2q-1}\alpha_\delta}{25(k+N_\delta)^{q-1/2}}$$

Finally, we deduce that

$$\begin{aligned} |R_1| &= \left| \sum_{2^{m_\delta} \leqslant r+N_\delta < 2^{\ell-1}} \frac{u_r}{r+N_\delta} E_r \right| \leqslant \sum_{2^{m_\delta} \leqslant r+N_\delta < 2^{\ell-1}} \frac{|E_r|}{r+N_\delta} \\ &\leqslant \frac{4^{q-1/2}\sqrt{2q-1}\alpha_\delta}{(k+N_\delta)^{q-1/2}} \end{aligned}$$

since last sum has at most $2^{\ell-1} \leqslant k + N_\delta$.

We will now bound R_2.

$$\begin{aligned} |R_2| &\leqslant \sum_{\ell-1 \leqslant j \leqslant n_\delta} \int\limits_I |f_j(t)|\,|g_j(t)-1|\,|\varphi(t)|\,\mathrm{d}t \\ &\leqslant \|\varphi\|_\infty \sum_{\ell-1 \leqslant j \leqslant n_\delta} \|f_j\|_2 \|g_j - 1\|_2. \end{aligned}$$

According to the lemmas 4.10 (1) and 4.12 (1) we get

$$|R_2| \leqslant \|\varphi\|_\infty \eta \frac{\sqrt{2}(2+\delta)}{\sqrt{2}-1} \sum_{l-1 \leqslant j \leqslant m} 2^{-j}.$$

since

$$\sum_{\ell-1 \leqslant j \leqslant n_\delta} 2^{-j} \leqslant \sum_{j=\ell-1}^{\infty} 2^{-j} = 2^{-\ell+2} = 8.2^{-(\ell+1)}$$

and $k + N_\delta \in I_l$ then $\dfrac{1}{2^{\ell+1}} \leqslant \dfrac{1}{k+N_\delta}$. Consequently

$$\sum_{\ell-1 \leqslant j \leqslant n_\delta} 2^{-j} \leqslant \frac{8}{k+N_\delta}.$$

Finally, we deduce that

$$R_2 \leqslant \eta \|\varphi\|_\infty \frac{\sqrt{2}(|I_\delta|+1)}{\sqrt{2}-1} \frac{8}{k+N_\delta}$$

and we obtain $R_2 \leqslant \dfrac{\alpha_\delta}{3} \dfrac{1}{k+N_\delta}$ when $\eta \leqslant \dfrac{(\sqrt{2}-1)}{24\|\varphi\|_\infty(|I_\delta|+1)\sqrt{2}} \alpha_\delta$. □

Note that, from (4.6) and $|I_\delta| \le 2$, we can take $\eta = c_4\delta^2$ for some $c_4 > 0$.

It is now easy to deduce the theorem 4.5 using the 2 inequalities (4.8) and (4.10).

Proof of theorem 4.5. Let $S_\delta = \sum_{k=0}^{N} \frac{|a_k|}{k+N_\delta}$ and $\Phi(t) = \sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}$ as previously defined. Recall that in (4.8) , we have shown that

$$\begin{aligned}
\alpha_\delta S_\delta &\le \left|\int_{I_\delta} T_\delta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right| \\
&\le \left|\int_{I_\delta} \tilde{T}_\delta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right| + \left|\int_{I_\delta} \big(\tilde{T}_\delta(t) - T_\delta(t)\big)\phi(t)\varphi(t)\,\mathrm{d}t\right| \\
&\le \left|\int_{I_\delta} \tilde{T}_\delta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right| + \frac{2}{3}\alpha_\delta S_\delta
\end{aligned}$$

with (4.10).

It follows that

$$\begin{aligned}
S_\delta &\le \frac{3}{\alpha_\delta}\left|\int_{I} \tilde{T}_\delta(t)\phi(t)\varphi(t)\,\mathrm{d}t\right| \\
&\le \frac{3\|\tilde{T}\|_\infty\|\varphi\|_\infty}{\alpha_\delta}\int_{I_\delta} |\phi(t)|\,\mathrm{d}t.
\end{aligned}$$

But, if δ is small enough and $N_\delta \ge \delta^{-7/2}$,

– from Lemma 4.6, $\|\varphi\|_\infty \le \frac{\pi}{2}$;

– from (4.6), $\alpha_\delta = \kappa\delta^2 + O(\delta^4)$

– as $\eta = c_4\delta^2$, from (4.9), $\|\tilde{T}\|_\infty = \frac{2}{c_4\delta^2}$.

Therefore, there are two absolute constants δ_* and c_* such that, if $\delta \le \delta_*$,

$$S_\delta \le \frac{c_*}{\delta^4}\int_{I_\delta} |\phi(t)|\,\mathrm{d}t.$$

As noticed at the start of the proof, this implies that

$$\sum_{n=0}^{N} \frac{|a_k|}{k+1} \le \frac{c_*}{\delta^{\frac{15}{2}}}\int_{-\frac{1+\delta}{2}}^{\frac{1+\delta}{2}} \left|\sum_{k=0}^{N} a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t$$

for every N, every sequence of real numbers $(\lambda_j)_{j\ge 0}$ with $\lambda_{j+1} - \lambda_j \ge 1$ and every complex sequence $(a_j)_{j\ge 0}$. □

We have not fully optimised the proof, by taking q sufficiently large and p/q sufficiently large, one can replace $\delta^{15/2}$ by $\delta^{7+\eta}$ for any fixed η.

5. Some open problems

The L^2 theory of exponential sums is rather well understood. This is not the case for L^1 theory for which there are still many open questions. Let us mention a few of them.

1. There is a major difference between the sums that appear in Ingham's Theorem and those that appear in Mc Geehe, Pigno, Smith and Nazarov's Theorems. In the L^2 case, the sums can be two sided and not in the L^1 case. The proof given here does not work in this case (this is due to the construction of T_1) and we are tempted to conjecture the following

   Question 1. *Let $T > 1$ and $(\lambda_k)_{k\in\mathbb{Z}}$ a real sequence such that $\lambda_{k+1} - \lambda_k \geq 1$ for every k, λ_k has same sign as k and $\sum \dfrac{1}{1+|\lambda_k|}$ converges.*

   Does there exist a constant C such that, for every N and every sequence $(c_j)_{j=-N,\ldots,N}$ of complex numbers,

$$C\sum_{k=-N}^{N}\frac{|c_k|}{|k|+1} \leq \int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N} c_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t.$$

   So far, the best known two sided inequality is Ingham's L^1-Inequality presented in Section 4.2. Note that this inequality implies that, for every $\varepsilon > 0$, there is a C_ε such that

$$C_\varepsilon\sum_{k=-N}^{N}\frac{|c_k|}{(|k|+1)^{1+\varepsilon}} \leq \int_{-T/2}^{T/2}\left|\sum_{k=-N}^{N} c_k e^{2i\pi\lambda_k t}\right| \mathrm{d}t.$$

   Note also that some condition on the growth of λ_k is needed. Indeed, if we consider the Féjer kernel $F_N(t) = \displaystyle\sum_{k=-N}^{N}\left(1-\frac{|k|}{N+1}\right)e^{2i\pi kt}$ then, for $T > 1$, $\displaystyle\int_{-T/2}^{T/2}|F_N(t)|\,\mathrm{d}t \leq T+1$ while

$$\sum_{k=-N}^{N}\frac{1}{|k|+1}\left(1-\frac{|k|}{N+1}\right) \geq \frac{1}{2}\sum_{|k|\leq N/2}\frac{1}{|k|+1} \to +\infty.$$

2. The second question is the optimality of the condition $T > 1$ in Nazarov's Theorem. In view of Ingham's L^2 counterexample, it is tempting to conjecture that $T > 1$ is requested (as already observed by Nazarov, his proof requires this). On the other hand, Ingham's L^1-inequality is valid for $T = 1$, so that the same might be true in Nazarov's result.

   Question 2. *Is the condition $T > 1$ necessary in Nazarov's Theorem. If yes, what is the right behaviour of the constants.*

   We think the estimate shown here is not optimal.

3. Another question related to the previous one comes from Haraux's Theorem. When $\lambda_{k+1}-\lambda_k\to+\infty$, T can be chosen arbitrarily small. A key element of Haraux's strategy is that the lower and upper bounds in Ingham's inequality are both multiples of the ℓ^2-norm of the coefficients. This is no longer the case in the L^1 setting.

   In forthcomming work, we have been able to use a compactness argument to show that one may take T arbitrarily small in Nazarov's Theorem when $\lambda_{k+1}-\lambda_k\to+\infty$. However, in doing so, we loose control of constants. This leads to the following:

   Question 3. *Prove a quantitative version Nazarov's Theorem with T arbitrarily small when $\lambda_{k+1}-\lambda_k\to+\infty$.*

4. Are $L^1([-T,T])$-norms of lacunary non-harmonic Fourier series comparable to the ℓ^2-norms of the coefficients? This is the case for Besikovich norms.

   Note that

$$\sum_{j=0}^{\infty}\frac{|a_j|}{j+1}\leq\left(\sum_{j=0}^{\infty}\frac{1}{(j+1)^2}\right)^{1/2}\left(\sum_{j=0}^{\infty}|a_j|^2\right)^{1/2}=\frac{\pi}{\sqrt{6}}\left(\sum_{j=0}^{\infty}|a_j|^2\right)^{1/2}.$$

   The question is then

   Question 4. *Find a (gap) condition on (Λ_k) and on T that implies that there is a constant $C>0$ such that, for every (a_k) with finite support*

$$\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{j=0}^{+\infty}a_je^{2i\pi\lambda_jt}\right|\mathrm{d}t\geq C\left(\sum_{j=0}^{\infty}|a_j|^2\right)^{1/2}.$$

   One may also go the other way and ask whether one can still obtain a result like Theorem B when $\lambda_{k+1}-\lambda_k\to 0$ but with a smaller power on the denominator. For instance, when $\lambda_k=\ln k$ then the Fourier series become Dirichlet series

$$\sum_{k=1}^{+\infty}\frac{a_k}{k^{2i\pi t}}=\sum_{k=1}^{+\infty}a_ke^{2i\pi(\ln k)t}.$$

   The following question was asked in [3]:

   Question 5. *Is it true that, for every sequence $(a_k)_{k\geq1}$ with finite support,*

$$C\left(|a_1|+\sum_{k\geq2}\frac{|a_k|}{\sqrt{k}\ln k}\right)\leq\lim_{T\to+\infty}\int_{-T/2}^{T/2}\left|\sum_{k=1}^{+\infty}\frac{a_k}{k^{2i\pi t}}\right|\mathrm{d}t.$$

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