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A new semi-symmetric non-metric connection on warped products

Une nouvelle connection semi-symétrique et non-métrique sur les produits déformés

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ABSTRACT. In this paper we study warped products endowed with a new semi-symmetric non-metric connection, which, we called Diallo-Massamba connection. We establish relationships between the Diallo-Massamba connection of the warped product to those of the base and the fiber. Also, we derive the curvature formulas for warped products with the Diallo-Massamba connection in terms of curvatures of its components. Examples of Diallo-Massamba connection are also given.
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1. Introduction

The idea of a semi-symmetric linear connection on a Riemannian manifold (M, g) was first proposed by Friedmann and Schouten [11]. If a linear connection's torsion tensor $\widetilde{T}$ has the following form, it is said to be semi symmetric connection

$$\widetilde{T}(X,Y) = u(Y)X - u(X)Y, \tag{1.1}$$

where u is a 1-form associated with the vector field U on M by

$$u(X) = g(X,U). \tag{1.2}$$

Hayden [13] developed the concept of semi-symmetric metric connection. A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if

$$\widetilde{\nabla} g = 0. \tag{1.3}$$

When the torsion of a metric connection is zero, it is a Levi-Civita connection, otherwise it is a Hayden connection [13]. As a result, metric connections include both Levi-Civita and Hayden connections. Yano [20] considered the semi-symmetric metric connection and studied some of its properties. When a Riemannian manifold admits a semi-symmetric metric connection and its is conformally flat, Yano demonstrated that the curvature tensor vanishes. The semi-symmetric metric connections was further studied by Imai [14], Mishra and Pandey [16], De and De [6], and several other mathematicians [15, 21].

Agashe and Chafle [1], introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold (M, g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric non-metric connection if

$$\widetilde{\nabla} g \neq 0. \tag{1.4}$$

Agashe and Chafle [2] studies some properties of submanifolds of a Riemannian manifold with semi-symmetric non-metric connections. Sengupta, De and Binh in [19], generalizes the semi-symmetric non-metric connection introduced in [1]. They determine the relationships between the curvature tensor $\widetilde{R}$ of M, with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ and the curvature tensor field R, with respect to the Riemannian connection. Further, the first and second Bianchi identities associated with the semi-symmetric non-metric connection are obtained. Some properties of the Weyl projective curvature tensor with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ are also studied.

Özgur and Sular [18] studied the warped products with semi-symmetric non-metric connection and established some results. Motivated by these results, we study warped products with a new semi-symmetric non-metric connection and we derive the curvature formulas for warped products with the semi-symmetric non-metric connection in terms of curvatures of its components.

The outline of the paper is as follows. In section 2, we prove the existence of a new semi-symmetric non-metric connection which generalize the semi-symmetric non-metric connections given by Agashe and Chafle [1], and Sengupta *et al.* [19]. We call this new connection as the Diallo-Massamba connection. Two examples of Diallo-Massamba connections are constructed. In section 3, we recall some basic notion on warped products [17]. In section 4, we establish the relationships between the Diallo-Massamba connection of the warped products and those of the base and the fiber. At the end, we construct an example of Diallo-Massamba connection on a warped product.

2. Diallo-Massamba connection

Diallo and Massamba [9] introduced a new semi-symmetric non-metric connection in a Riemannian manifold, which is a generalization of the semi-symmetric non-metric connection given by Agashe and Chafle [1] and the semi-symmetric non-metric connection given by J. Sengupta and al., [19]. They proved the following result.

Theorem 2.1. *[9] Let (M, g) be a Riemannian manifold of dimension n and ∇^g its Levi-Civita connection. Let U, V be two vector fields on M and f be a real smoot function on (M, g). Then there exists a unique affine connection ∇ on (M, g) associated with respect to (U, V, f) of torsion T^{∇} on M which satisfies*

$$T^{\nabla}(X, Y) = g(U, Y)X - g(U, X)Y, \tag{2.1}$$

and

$$\Big(\nabla_X g\Big)(Y, Z) = f\Big(g(V, Y)g(X, Z) + g(V, Z)g(X, Y)\Big), \tag{2.2}$$

for $X, Y, Z \in \Gamma(TM)$. This connection is given by

$$\nabla_X Y = \nabla^g_X Y + g(U, Y)X - g(X, Y)(U + fV), \tag{2.3}$$

for $X, Y, Z \in \Gamma(TM)$.

Proof. Any affine connection ∇ of torsion T^{∇} on (M,g) is characterized by the formula

$$\begin{aligned} 2g(\nabla_X YZ) &= 2g(\nabla^g_X Y,Z)+(\nabla_Z g)(X,Y)-(\nabla_Y g)(X,Z)\\ &\quad -(\nabla_X g)(Y,Z)+g(T^{\nabla}(X,Y),Z)+g(T^{\nabla}(Z,X),Y)\\ &\quad -g(T^{\nabla}(Y,Z),X), \end{aligned}$$

for $X,Y,Z\in\Gamma(TM)$. Formula (2.3), then follows from the two formulas (2.1) and (2.2). □

Next, we call this connection, the Diallo-Massamba connection. From (2.1) and (2.2), the Diallo-Massamba connection is a semi-symmetric non-metric connection.

Remark 2.2. *The Diallo-Massamba connection associated with a triplet of the form $(U;V;-1)$ is the Sengupta connection [19], in particular, if $U=V$, it is the Agashe and Chafle connection [1].*

The Diallo-Massamba connection associated with a triplet of the form $(U;V;0)$ is a semi-symmetric metric connection, in this case the vector field V plays no role, we can then speak of semi-symmetric metric associated with the vector field U on the Riemannian manifold (M,g).

Under the same notations of the Theorem 2.1, if R and R^g are the curvature tensors of ∇ and ∇^g on (M,g), respectively and $\xi=U+fV$, then R and R^g are related by:

$$\begin{aligned} R(X,Y)Z &= R^g(X,Y)Z+\Big[g(X,\xi)g(Y,Z)-g(Y,\xi)g(X,Z)\Big]\xi\\ &\quad +g(U,Z)\Big[g(U,Y)X-g(U,X)Y\Big]\\ &\quad +g(U,\xi)\Big[g(X,Z)Y-g(Y,Z)X\Big]\\ &\quad +g(\nabla^g_X U,Z)Y-g(\nabla^g_Y U,Z)X\\ &\quad +g(X,Z)\nabla^g_Y\xi-g(Y,Z)\nabla^g_X\xi, \end{aligned} \tag{2.4}$$

for $X,Y,Z\in\Gamma(TM)$. Let r and r^g the Ricci curvature of ∇ and ∇^g on (M,g), respectively. Then r and r^g are related by:

$$\begin{aligned} r(X,Y) &= r^g(X,Y)+(n-1)g(U,X)g(U,Y)+g(\nabla^g_X\xi,Y)\\ &\quad +\Big(\|\xi\|^2_g+(1-n)g(U,\xi)-\mathrm{div}_g(\xi)\Big)g(X,Y)\\ &\quad -g(X,\xi)g(Y,\xi), \end{aligned} \tag{2.5}$$

for $X,Y,Z\in\Gamma(TM)$.

We have the following example of Diallo-Massamba connection.

Example 2.3. *Let $M_1=]0,+\infty[\times]0,+\infty[=(]0,+\infty[)^2$ be a 2-dimensional manifold with standard coordinate (x_1,x_2) and a Riemannian metric g_1 on M_1 given by*

$$g_1(x_1,x_2)=\begin{pmatrix} \frac{1}{x_2^2} & 0\\ 0 & \frac{2}{x_2^2}\end{pmatrix}. \tag{2.6}$$

By a straightforward computation, we obtain the non zero components of the Levi-Civita connection ∇^{g_1} of g_1 are: $\nabla^{g_1}_{\partial_1}\partial_1=\frac{1}{2x_2}\partial_2$, $\nabla^{g_1}_{\partial_1}\partial_2=-\frac{1}{x_2}\partial_1$ and $\nabla^{g_1}_{\partial_2}\partial_2=-\frac{1}{x_2}\partial_2$. By a direct computation, we obtain the non zero components of the curvature R^{g_1} of g_1 are: $R^{g_1}(\partial_1,\partial_2)\partial_1=\frac{1}{2x_2^2}\partial_2$ and $R^{g_1}(\partial_1,\partial_2)\partial_2=$

$-\frac{1}{x_2^2}\partial_1$. *By taking* $U = \partial_1$ *and* $V = \partial_2$*, then the Diallo-Massamba connection with respect to the triplet* $(\partial_1, \partial_2, f)$ *on* M_1 *is:*

$$\begin{aligned} \nabla^1_{\partial_1}\partial_1 &= \frac{1}{2x_2}\Big(1 - \frac{2f}{x_2}\Big)\partial_2; \quad \nabla^1_{\partial_2}\partial_1 = -\frac{1}{x_2}\partial_1 + \frac{1}{x_2^2}\partial_2; \\ \nabla^1_{\partial_1}\partial_2 &= -\frac{1}{x_2}\partial_1; \quad \nabla^1_{\partial_2}\partial_2 = -\frac{2}{x_2^2}\partial_1 - \frac{1}{x_2}\Big(1 + \frac{2f}{x_2}\Big)\partial_2. \end{aligned}$$

The non zero component of torsion tensor of Diallo-Massamba connection is $T(\partial_1, \partial_2) = -\frac{1}{x_2^2}\partial_2$*. Hence* ∇^1 *is a semi symmetric non metric connection and the relation (2.1) is satisfied. Moreover, it is easy to see that* $\nabla^1 g_1$ *is not equal to zero.*

Assume $f = 1$*, then the non zero components of Diallo-Massamba curvature* R^1 *by making use of (2.4) with respect to the triplet* $(\partial_1, \partial_2, 1)$ *on* M_1 *are:*

$$\begin{aligned} R^1(\partial_1, \partial_2)\partial_1 &= -\frac{1}{2x^4}\partial_1 + \frac{1}{x_2^2}\Big(\frac{1}{2} - \frac{1}{x_2} - \frac{2}{x_2^2}\Big)\partial_2; \\ R^1(\partial_1, \partial_2)\partial_2 &= \frac{1}{x^2}\Big(\frac{2}{x_2} - 1\Big)\partial_1 + \frac{2}{x_2^4}\partial_2. \end{aligned}$$

We have also the following example of Diallo-Massamba connection.

Example 2.4. *Let* $M_2 =]0, +\infty[\times]0, +\infty[= (]0, +\infty[)^2$ *be a* 2*-dimensional manifold with standard coordinate* (x_3, x_4) *and a Riemannian metric* g_2 *on* M_2 *given by*

$$g_2(x_3, x_4) = \begin{pmatrix} 1 & 0 \\ 0 & x_3^2 \end{pmatrix}. \tag{2.7}$$

By a straightforward computation, we obtain the non zero components of the Levi-Civita connection ∇^{g_2} *of* g_2 *are:* $\nabla^{g_2}_{\partial_3}\partial_4 = \frac{1}{x_3}\partial_4$ *and* $\nabla^{g_2}_{\partial_4}\partial_4 = -x_3\partial_3$*. It is easy to shown that this connection is flat, that is the curvature is equal to zero. By taking* $U = \partial_3$ *and* $V = \partial_4$*, then the Diallo-Massamba connection with respect to the triplet* $(\partial_3, \partial_4, f)$ *on* M_2*, we find:*

$$\begin{aligned} \nabla^2_{\partial_3}\partial_3 &= -f\partial_4; \quad \nabla^2_{\partial_3}\partial_4 = \frac{1}{x_3}\partial_4; \\ \nabla^2_{\partial_4}\partial_3 &= \frac{x_3 + 1}{x_3}\partial_4; \quad \nabla^2_{\partial_4}\partial_4 = -(x_3 + x_3^2)\partial_3 - fx_3^2\partial_4. \end{aligned}$$

The non zero component of torsion tensor of Diallo-Massamba connection is $T(\partial_3, \partial_4) = -\partial_4$*. Hence* ∇ *is a semi symmetric non metric connection and the relation (2.1) is satisfied. Moreover, it is easy to see that* $\nabla^2 g_2$ *is not equal to zero.*

Now, we assume $f = 1$*, then the non zero components of Diallo-Massamba curvature* R^2 *by making use of (2.4) with respect to the triplet* $(\partial_3, \partial_4, 1)$ *on* M_2 *are:*

$$\begin{aligned} R^2(\partial_3, \partial_4)\partial_3 &= -(x_3 + x_3^2)\partial_3 + \Big(\frac{1}{x_3} + x_3^2\Big)\partial_4 \\ R^2(\partial_1, \partial_2)\partial_2 &= -x_3\partial_3 - (x_3 - x_3^2)\partial_4. \end{aligned}$$

3. Warped products

The concept of warped products was first developed by Bishop and O'Neil [5]. This notion was used to build a large class of complete Riemannian manifolds with negative sectional curvature. In studies relating to the solutions of Einstein's equations, warped products have important applications in general relativity [3, 4]. In addition to general relativity, warped product structures have attracted interest in many different fields of geometry, particularly due to their role in construction of new examples with interesting curvatures and symmetry properties. The existence of compact Einstein warped product manifolds is considered in [7]. The study of warped products has become a very active research subject (see [8, 10] and references therein).

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimensions n_1 and n_2 respectively and let F be a positive smooth function on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\mathrm{pr}_1 : M_1 \times M_2 \to M_1$ and $\mathrm{pr}_2 : M_1 \times M_2 \to M_2$. The warped product $M = M_1 \times_F M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian metric g given by

$$g = \mathrm{pr}_1^* g_1 + (F^2 \circ \mathrm{pr}_1)\mathrm{pr}_2^* g_2. \tag{3.1}$$

The function F is called the warping function. If the warping function F is constant, then the warped product $M = M_1 \times_F M_2$ is a direct product, which we call as trivial warped product.

The manifold M_1 is called the base and M_2 is the fiber of the warped product M . It is easy to see that the fibers $\{p\} \times M_2 = \mathrm{pr}_1^{-1}(p)$ and the leaves $M_1 \times \{q\} = \mathrm{pr}_2^{-1}(q)$ are submanifolds of M. Vectors tangent to leaves are horizontal, vectors tangent to fibers are verticals. We denote by $\mathcal{H}$ the orthogonal projection of $T_{(p,q)}M$ onto its horizontal subspace $T_{(p,q)}(M_1 \times \{q\})$, and by $\mathcal{V}$ the projection onto the vertical subspace $T_{(p,q)}(\{p\} \times M_2)$. The relation of a warped product to the base M_1 is almost as simple as in the special case of a Riemannian product. However, the relation of a warped product to the fiber M_2 often involves the warping function F.

Let X_1, Y_1, Z_1, U_1, V_1 (respectively X_2, Y_2, Z_2, U_2, V_2) be are vector fields on M_1 (respectively on M_2). We denote by X_1^h (respectively X_2^h) the lift to M of $X_1 \in \Gamma(TM_1)$ (respectively of $X_2 \in \Gamma(TM_2)$) , i.e., $X_1^h(x_1, x_2) = (X_1(x_1), 0_{T_{x_2}M_2})$ (respectively $X_2^v(x_1, x_2) = (0_{T_{x_1}M_1}, X_2(x_2))$), where $x_1 \in M_1$ respectively $x_2 \in M_2$). Also, we denote by $\varphi_1^h := \varphi_1 \circ \mathrm{pr}_1$ (respectively $\varphi_2^v := \varphi_2 \circ \mathrm{pr}_2$) the lift to M of $\varphi_1 \in \mathcal{C}^\infty(M_1)$ (respectively $\varphi_2 \in \mathcal{C}^\infty(M_2)$). Similarly, we can define the lifts to M of tensor fields defined on the base M_1 or on the fiber M_2.

We have the following lemmas for later use.

Lemma 3.1. *[17] If $X_i, Y_i \in \Gamma(TM_i)$, ∇^{g_i} is the Levi-Civita connection of (M_i, g_i) and ∇^g the Levi-Civita connection of the warped product $(M_1 \times M_2, g)$. Then we have:*

1. $\nabla^g_{X_1^h} Y_1^h = (\nabla^{g_1}_{X_1} Y_1)^h$,
2. $\nabla^g_{X_1^h} Y_2^v = (\frac{X_1 \cdot F}{F})^h Y_2^v$,
3. $\nabla^g_{X_2^v} Y_2^v = (\nabla^{g_2}_{X_2} Y_2)^v - F^h g_2(X_2, Y_2)^v (\mathrm{grad}_g F)^h$.

The following lemma provides the relation between the curvature R^g of the warped product $(M_1 \times M_2, g)$ in terms of the curvatures R^{g_1} and R^{g_2} of M_1 and M_2, respectively, and the warping function F.

Lemma 3.2. *[17] Let be the vector fields $X_i, Y_i, Z_i \in \Gamma(TM_i), i = 1, 2$. Let R^g the curvature of M and the curvatures R^{g_1} and R^{g_2} of M_1 and M_2, respectively. Then we have:*

1. $R^g(X_1^h, Y_1^h)Z_1^h = \Big(R^{g_1}(X_1, Y_1)Z_1\Big)^h$,
2. $R^g(Y_2^v, Y_1^h)Z_1^h = -\Big(\frac{H(Y_1,Z_1)}{F}\Big)^h Y_2^v$,
3. $R^g(X_1^h, Y_1^h)Y_2^v = R^g(Y_2^v, Z_2^v)X_1^h = 0$,
4. $R^g(X_1^h, Y_2^v)Z_2^v = -g_2(Y_2, Z_2)^v \Big(F\nabla^{g_1}_{X_1}(\mathrm{grad}(F)\Big)^h$,
5. $R^g(X_2^v, Y_2^v)Z_2^v = \Big(R^{g_2}(X_2, Y_2)Z_2\Big)^v + \frac{\|\mathrm{grad}F\|^2}{F^2}\Big[g_2(Y_2, Z_2)^v X_2^v - g_2(X_2, Z_2)^v Y_2^v\Big]$.

The next lemma provides the relation between the Ricci tensor of M in terms of the Ricci tensors of M_1 and M_2, respectively, and the warping function F.

Lemma 3.3. *[17] Given vector fields $X_i, Y_i \in \Gamma(TM_i)$, then we have:*

1. $r(X_1^h, Y_1^h) = \Big(r^{g_1}(X_1, Y_1)\Big)^h - n_2(\frac{H(X_1,Y_1)}{F})^h$, *where* $n_2 = \dim M_2$,
2. $r(X_1^h, X_2^v) = 0$,
3. $r(X_2^v, Y_2^v) = \Big(r^{g_2}(X_2, Y_2)\Big)^v - g(X_2, Y_2)^v\Big[\frac{\Delta F}{F} + \frac{(n_2-1)}{F^2}\|\mathrm{grad}F\|^2\Big]$,

where ΔF is the Laplacian of F on M_1.

Moreover, the scalar curvature Scal of M satisfies the condition

$$\mathrm{Scal} = \Big({}^{M_1}\mathrm{Scal}\Big)^h + \Big(\frac{1}{F^2}\Big)^h\Big({}^{M_2}\mathrm{Scal}\Big)^v - \Big(\frac{2n_2}{F}\Delta F - \frac{n_2(n_2-1)}{F^2}\|\mathrm{grad}F\|^2\Big)^h,$$

where ${}^{M_1}\mathrm{Scal}$ and ${}^{M_2}\mathrm{Scal}$ are scalar curvatures of M_1 and M_2, respectively.

4. Warped products with Diallo-Massamba connection

In this section, we consider warped products $M = M_1 \times_F M_2$ equipped with respect to the Diallo-Massamba connection (2.3).

Let $U_i, V_i \in \Gamma(TM_i), i = 1, 2, f \in \mathcal{C}^\infty(M_1 \times M_2)$. We denote by ∇^i the Diallo-Massamba connection with respect to the triplet $(U_i, V_i, 1)$ and by ∇ the associated with respect to the triplet (U_i, V_i, f), where $U = U_1^h + U_2^v$ and $V = V_1^h + V_2^v$. We set: $\xi = U + fV$. We have:

Proposition 4.1. *Under the same notations above, we have:*

$$\begin{aligned}
\nabla_{X_1^h}Y_1^h &= (\nabla^{g_1}_{X_1}Y_1)^h + g_1(U_1, Y_1)^h X_1^h - g_1(X_1, Y_1)^h\xi,\\
\nabla_{X_1^h}Y_2^v &= \Big(\frac{X_1(F)}{F}\Big)^h Y_2^v + (F^2)^h g_2(U_2, Y_2)^v X_1^h,
\end{aligned}$$

$$
\begin{aligned}
\nabla_{X_2^v} Y_1^h &= \Big[\frac{Y_1(F)}{F} + g_1(U_1, Y_1)\Big]^h X_2^v, \\
\nabla_{X_2^v} Y_2^v &= \big(\nabla^{g_2}_{X_2} Y_2\big)^v + (F^2)^h \Big[g_2(U_2, Y_2)^v X_2^v \\
&\quad - g_2(X_2, Y_2)^v \big(\xi + \frac{1}{F}\mathrm{grad}_g F\big)^h\Big],
\end{aligned}
$$

for $X_i, Y_i \in \Gamma(TM_i), i = 1, 2.$

Proof. After a long but no straightforward calculation using formula (2.3) and Lemma 3.1, we get the result. □

Corollary 4.2. *Under the same assumptions and notations of the proposition above, if* $f = 1$ *and* $U_i = -V_i, i = 1, 2 (\xi = 0)$*, i.e. the three connections* ∇^1, ∇^2 *and* ∇ *are connections of Agashe and Chafle. Then we have*

$$
\begin{aligned}
\nabla_{X_1^h} Y_1^h &= \big(\nabla^{g_1}_{X_1} Y_1\big)^h + g_1\big(U_1, Y_1\big)^h X_1^h, \\
\nabla_{X_1^h} Y_2^v &= \Big(\frac{X_1(F)}{F}\Big)^h Y_2^v + \big(F^2\big)^h g_2\big(U_2, Y_2\big)^v X_1^h, \\
\nabla_{X_2^v} Y_1^h &= \Big[\frac{Y_1(F)}{F} + g_1\big(U_1, Y_1\big)\Big]^h X_2^v, \\
\nabla_{X_2^v} Y_2^v &= \big(\nabla^{g_2}_{X_2} Y_2\big)^v + \big(F^2\big)^h \Big[g_2\big(U_2, Y_2\big)^v X_2^v \\
&\quad - g_2\big(X_2, Y_2\big)^v \big(\frac{1}{F}\mathrm{grad}_g F\big)^h\Big],
\end{aligned}
$$

for $X_i, Y_i \in \Gamma(TM_i), i = 1, 2.$

Next, we etablish a relation between the curvature tensors R, R^{g_1} and R^{g_2} be the curvature tensors with respect to the Diallo-Massamba connection (2.3) on M, M_1 and M_2, respectively.

Proposition 4.3. *Let* R, R^{g_1} *and* R^{g_2} *be the curvature tensors with respect to the Diallo-Massamba connection (2.3) on* M, M_1 *and* M_2*, respectively. If* $X_i, Y_i, Z_i \in \Gamma(TM_i)$ *and* $U = U_1^h + U_2^v$ *and* $V = V_1^h + V_2^v$*. We set* $\xi = U + fV$*. Then, we have:*

$$
\begin{aligned}
R(X_1^h, Y_1^h) Z_1^h &= \Big(R^{g_1}(X_1, Y_1) Z_1\Big)^h + \Big[g_1(X_1, \xi_1)^h g_1(Y_1, Z_1)^h \\
&\quad - g_1(Y_1, \xi_1)^h g_1(X_1, Z_1)^h\Big]\xi \\
&\quad + g_1(U_1, Z_1)^h \Big[g_1(U_1, Y_1)^h X_1^h - g_1(U_1, X_1)^h Y_1^h\Big] \\
&\quad + \Big(g_1(U_1, \xi_1)^h + (F^2)^h g_2(U_2, \xi_2)^v\Big)\Big[g_1(X_1, Z_1) Y_1^h \\
&\quad - g_1(Y_1, Z_1) X_1^h\Big] \\
&\quad + g_1(\nabla^{g_1}_{X_1} U_1, Z_1)^h Y_1^h - g_1(\nabla^{g_1}_{Y_1} U_1, Z_1)^h X_1^h \\
&\quad + g_1(X_1, Z_1)^h \nabla^g_{Y_1^h} \xi - g_1(Y_1, Z_1)^h \nabla^g_{X_1^h} \xi, \\
R(X_2^v, Y_1^h) Z_1^h &= -\Big(\frac{H(Y_1, Z_1)}{F}\Big)^h X_2^v + (F^2)^h g_1(Y_1, Z_1)^h \Big[g_2(X_2, \xi_2)^v \xi \\
&\quad - g_2(U_2, \xi_2)^v X_2^v\Big]
\end{aligned}
$$

$$
\begin{aligned}
& +g_1(U_1,Z_1)^h\Big[g_1(U_1,Y_1)^hX_2^v-(F^2)^hg(U_2,X_2)^vY_1^h\Big] \\
& -g(Y_1,Z_1)^h\Big[g_1(U_1,\xi_1)+\nabla^g_{X_2^V}\xi\Big], \\
R(X_1^h,Y_1^h)Z_2^v &= (F^2)^hg_2(U_2,Z_2)^v\Big[g_1(U_1,Y_1)^hX_1^h-g(U_1,X_1)^hY_1^h \\
& +\Big(\frac{X_1(F)}{F}\Big)^hY_1^h-\Big(\frac{Y_1(F)}{F}\Big)^hX_1^h\Big)\Big], \\
R(Y_2^v,Z_2^v)X_1^h &= (F^2)^hg_1(U_1,X_1)^h\Big[g_2(U_2,Z_2)^vY_2^v-g(U_2,Y_2)^vZ_2^v\Big] \\
& + g(\nabla^g_{Y_2}U_2,X_1)Z_2^v-g(\nabla^g_{Z_2}U_2,X_1)Y_2^v,
\end{aligned}
$$

$$
\begin{aligned}
R(X_1^h,Y_2^v)Z_2^v &= (F^2)^hg_2(Y_2,Z_2)^v\Big[g_1(X_1,\xi_1)^h\xi-\frac{\nabla_{X_1^h}(\mathrm{grad}F)}{F}-\nabla^g_{X_1^h}\xi\Big] \\
& +(F^2)^hg_2(U_2,Z_2)^v\Big(\frac{(X_1(F)}{F}\Big)^hY_2^v-(F^2)^hg_2(Y_2,Z_2)^v\Big(\frac{(U_1(F)}{F}\Big)^hX_1^h \\
& +(F^2)^hg_2(U_2,Y_2)^v\Big[(F^2)^hg_2(U_2,Y_2)^vX_1^h-g_1(U_1,X_1)^hY_2^v\Big] \\
& -(F^2)^hg_2(Y_2,Z_2)^vX_1^h\Big[g_1(U_1,\xi_1)^h+(F^2)^hg_2(U_2,\xi_2)^v\Big] \\
& -(F^2)^hg_2(\nabla^{g_2}_{Y_2}U_2,Z_2)^vX_1^h, \\
R(X_2^v,Y_2^v)Z_2^v &= \Big(R^{g_2}(X_2,Y_2)Z_2\Big)^v+\frac{\|\mathrm{grad}F\|^2}{F^2}\Big(g(Y_2,Z_2)^vX_2^v-g(X_2,Z_2)^vY_2^v\Big) \\
& +(F^4)^h\Big[g_2(X_2,\xi_2)^vg_2(Y_2,Z_2)^v-g_2(Y_2,\xi_2)^vg_2(X_2,Z_2)^v\Big]\xi \\
& +(F^4)^hg_2(U_2,Z_2)^v\Big[g(U_2,Y_2)^vX_2^v-g_2(U_2,X_2)^vY_2^v\Big] \\
& +(F^2)^h\Big[g_2(X_2,Z_2)^vY_2^v-g_2(Y_2,Z_2)^vX_2^v\Big]\Big[g_1(U_1,\xi_1)^h \\
& +(F^2)^hg_2(U_2,\xi_2)^v\Big] \\
& +g_2(\nabla^g_{X_2^v}U,Z_2^v)Y_2^v-g(\nabla^g_{Y_2^v}U,Z_2^v)X_2^v \\
& +(F^2)^h\Big[g_2(X_2,Z_2)^v\nabla^g_{Y_2^v}\xi-g(Y_2,Z_2)^v\nabla^g_{X_2^v}\xi\Big].
\end{aligned}
$$

Proof. After a long but no straightforward calculation using formula (2.4) and Lemma 3.2, we get the result. □

As a consequence of Proposition 4.3, by a contraction of the curvature tensors we obtain the Ricci tensors of the warped product with respect to the Diallo-Massamba connection as follows:

Corollary 4.4. *Let r and r^g denote the Ricci tensors of M with respect to the Levi-Civita connection and the Diallo-Massamba connection associated with the triplet $(U_i,V_i,1)$ and ∇ that associated with the triplet (U,V,f), where $U=U_1^h+U_2^v$ and $V=V_1^h+V_2^v$. we set $\xi=U+fV$.*

$$
\begin{aligned}
r(X_1^h,Y_1^h) &= \Big(r^{g_1}(X_1,Y_1)\Big)^h+(n-1)g_1(U_1,X_1)^hg_1(U_1,Y_1)^h \\
& +g_1(\nabla^{g_1}_{X_1}\xi_1,Y_1)^h+\Big(\|\xi\|_g^2+(1-n)g_1(U_1,\xi_1)^h \\
& -\mathrm{div}_g(\xi)\Big)g_1(X_1,Y_1)^h-g_1(X_1,\xi_1)^hg(Y_1,\xi_1)^h,
\end{aligned}
$$

$$
\begin{aligned}
r(X_1^h, Y_2^v) &= (n-1)(F^2)^h g_1(U_1, X_1)^h g_2(U_2, Y_2)^v \\
&\quad + (F^2)^h g_2(Y_2, \xi_2)^v \Big((\frac{X_1 F}{F}) - g_1(X_1, \xi_1) \Big)^h, \\
r(X_2^v, Y_1^h) &= (n-1)(F^2)^h g_1(U_1, Y_1)^h g_2(U_2, X_2)^v \\
&\quad + (F^2)^h g_2(X_2, \xi_2)^v \Big((\frac{Y_1 F}{F}) - g_1(Y_1, \xi_1) \Big)^h, \\
r(X_2^v, Y_2^v) &= \Big(r^{g_2}(X_2, Y_2) \Big)^v + (n-1)(F^4)^h g_2(U_2, X_2)^v g_2(U_2, Y_2)^v \\
&\quad + g(\nabla^g_{X_2^v} \xi_1, Y_2^v) + (F^2)^h (\|\xi\|_g^2 + (F^2)^h (1-n) g_2(U_2, \xi_2)^v \\
&\quad - \mathrm{div}_g(\xi)) g_2(X_2, Y_2)^v - (F^4)^h g_2(X_2, \xi_2)^v g(Y_2, \xi_2)^v.
\end{aligned}
$$

Proof. After a long but no straightforward calculation using formula (2.5) and Lemma 3.3, we get the result. □

Using examples 2.3 and 2.4, we can construct on a warped product a Diallo-Massamba connection.

Example 4.5. *Let M be a 4-dimensional manifold with standard coordinate (x_1, x_2, x_3, x_4). Let consider the warped product $g = g_1 + F^2 g_2$ where g_1 and g_2 defined as in (2.6) and (2.7) respectively, and $F = e^{x_1}$. By a straightforward computation, the non zero components of the Levi-Civita connection of the warped product ∇^g of g are given by:*

$$
\begin{aligned}
\nabla^g_{\partial_1}\partial_1 &= \frac{1}{2x_2}\partial_2, \quad \nabla^g_{\partial_2}\partial_2 = -\frac{1}{x_2}\partial_2, \quad \nabla^g_{\partial_1}\partial_3 = \nabla^g_{\partial_3}\partial_1 = \partial_3, \\
\nabla^g_{\partial_3}\partial_3 &= -e^{3x_1}\partial_1, \quad \nabla^g_{\partial_1}\partial_4 = \nabla^g_{\partial_4}\partial_1 = \partial_4, \quad \nabla^g_{\partial_3}\partial_4 = \frac{1}{x_3}\partial_4, \\
\nabla^g_{\partial_4}\partial_4 &= -x_3^2 e^{2x_1^2}\partial_1 - x_3\partial_3.
\end{aligned}
$$

By taking $U = \partial_1 + \partial_3$ and $V = \partial_2 + \partial_4$, we find te components of Diallo-Massamba connection on warped product:

$$
\begin{aligned}
\nabla_{\partial_1}\partial_1 &= \frac{1}{x_2}\Big(\frac{1}{2} - \frac{f}{x_2}\Big)\partial_2 - \frac{1}{x_2^2}\partial_3 - \frac{f}{x_2^2}\partial_4, \quad \nabla_{\partial_1}\partial_2 = -\frac{1}{x_2}\partial_1, \\
\nabla_{\partial_2}\partial_1 &= -\frac{1}{x_2}\partial_1 + \frac{1}{x_2^2}\partial_2, \quad \nabla_{\partial_2}\partial_2 = -\frac{2}{x_2^2}\partial_1 - \frac{1}{x_2}(1 + \frac{2f}{x_2})\partial_2 - \frac{2}{x_2^2}\partial_3 - \frac{2f}{x_2^2}\partial_4, \\
\nabla_{\partial_3}\partial_1 &= \Big(1 + \frac{1}{x_2^2}\Big)\partial_3, \quad \nabla_{\partial_3}\partial_3 = e^{2x_1}(-2\partial_1 - f\partial_2 - f\partial_4), \\
\nabla_{\partial_4}\partial_1 &= \Big(1 + \frac{1}{x_2^2}\Big)\partial_4, \quad \nabla_{\partial_1}\partial_4 = \partial_4, \nabla_{\partial_3}\partial_4 = \frac{1}{x_3}\partial_4 \\
\nabla_{\partial_4}\partial_3 &= (e^{2x_1} + \frac{1}{x_3})\partial_4, \quad \nabla_{\partial_2}\partial_3 = e^{2x_1}\partial_2, \quad \nabla_{\partial_1}\partial_3 = e^{2x_1}\partial_1 + \partial_3, \\
\nabla_{\partial_4}\partial_4 &= -2x_3^2 e^{2x_1}\partial_1 - f x_3^2 e^{2x_1}\partial_2 - x_3(1 + x_3 e^{2x_1})\partial_3 - f x_3^2 e^{2x_1}\partial_4.
\end{aligned}
$$

The non zero components of torsion tensor of Diallo-Massaba connection with respect to the warped product $g = g_1 + F^2 g_2$ are given by

$$
\begin{aligned}
T(\partial_1, \partial_2) &= -\frac{1}{x_2^2}\partial_2, \quad T(\partial_1, \partial_3) = e^{2x_1}\partial_1 - \frac{1}{x_2^2}\partial_3, \quad T(\partial_1, \partial_4) = -\frac{1}{x_2^2}\partial_4 \\
T(\partial_2, \partial_3) &= e^{2x_1}\partial_2, T(\partial_3, \partial_4) = -e^{2x_1}\partial_4.
\end{aligned}
$$

Hence ∇ is a semi symmetric non metric connection and the relation (2.1) is satisfied. Moreover, it is easy to see that ∇g is not equal to zero.

It is not difficult but requires a long calculation to find the components of the curvature tensor by applying Proposition 4.3 and that of the Ricci tensor by applying Corollary 4.4.

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