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ε-Pseudo Weak-Demicompactness for 2×2 Block Operator Matrices

εPseudo faible-demi-compacité pour les opérateurs matriciels 2×2 en block

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ABSTRACT. The purpose of this paper is to give some properties for the so-called ε-pseudo weakly demicompact linear operators acting on Banach spaces. Some sufficient conditions on the entries of an unbounded 2×2 block operator matrix $\mathcal{L}_0$ ensuring its ε-pseudo weak demicompactness are provided. In addition, we develop, in the bounded case, the class of ε-pseudo Fredholm perturbation to investigate the essential pseudo-spectra of $\mathcal{L}_0$. The results are formulated in terms of some denseness conditions on the topological dual space.
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1. Introduction

This paper is devoted to some spectral properties related with the so-called ε-pseudo weak demicompactness, for a 2×2 block operator matrix (in short, B.O.M) with a domain $\mathcal{D}(\mathcal{L}_0) = (\mathcal{D}(A_1) \cap \mathcal{D}(A_3)) \times (\mathcal{D}(A_2) \cap \mathcal{D}(A_4))$ and that is represented by the following form:

$$\mathcal{L}_0 = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

The operator $\mathcal{L}_0$ acts on the product of two Banach spaces X and Y with entries A_1, A_2, A_3 and A_4. The operators A_i are linear, closed, and densely defined. Their domains are respectively denoted by $\mathcal{D}(A_i)$, $i = 1, ..., 4$.

In this work, central items of interest include the class of demicompactness [21, 22]. Note that this class was introduced by Petryshyn W. V. to discuss an iterative method of fixed points for nonlinear operators acting on Hilbert spaces. The definition asserts that if X is a Banach space and $T : \mathcal{D}(T) \subset X \to X$ is a linear operator, then T is called demicompact if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that the sequence $(I - T)x_n$ converges in X, there exists a convergent subsequence of $(x_n)_n$. The family of demicompact operators on X is denoted by $\mathcal{DC}(X)$. For more details on this subject, the reader can see [4, 5, 13, 14, 15, 16, 17]. In Fredholm theory, the first two papers were developed by Petryshyn W.V. in 1972 [23] and by Akashi W. Y. in 1984 [2]. Note that the demicompactness class plays an important role in the theory of perturbations since it contains compact and more general the Fredholm perturbation operators. In this direction several papers were recently developed. Here, we cite some of those which are related with our interest. In [5], Chaker W., Jeribi A. and Krichen B. have utilized demicompact operators in order to investigate the essential spectra of linear operators. In 2014, Krichen B. [15] defined, as a generalization the demicompactness, the class of relative demicompact linear operators with respect to a given linear operator. This definition asserts that if $T : \mathcal{D}(T) \subset X \longrightarrow X$ and $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ are two linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$, then T is said to be S_0-demicompact (or relatively demicompact with respect to S_0), if every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(S_0x_n - Tx_n)_n$ converges in X, has a convergent subsequence. In 2018, Krichen B. and O'Regan D. [16] elaborated the class of relative weak demicompactness. If $T : \mathcal{D}(T) \subset X \longrightarrow X$ and $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ are two linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$, T is said to be weakly S_0-demicompact (or weakly relatively demicompact with respect to S_0), if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(S_0x_n - Tx_n)_n$ converges weakly in X, then there is a weakly convergent subsequence of $(x_n)_n$. the symbol $\mathcal{WDC}(S_0)(X)$ will denote the family of all weakly S_0-demicompact operators on X, and $\mathcal{WDC}(I)(X) = \mathcal{WDC}(X)$. Note that, the class of demicompact operators acting on a Banach space contains the class of weakly compact operators. Lately, Ben Brahim F., Jeribi A. and Krichen B. [4] developed the notion of pseudo demicompactnesss. For $\varepsilon > 0$, $T : \mathcal{D}(T) \subset X \longrightarrow X$ is said to be pseudo demicompact if for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $((I - T - D)x_n)_n$ converges in X, there exists a convergent subsequence of $(x_n)_n$. Recently, Chtourou I. and Krichen B. [6] introduced the notion of a relatively ε-pseudo weakly demicompact operator as follows: Let $\varepsilon > 0$ and let $T : \mathcal{D}(T) \subset X \to X$, $S_0 : \mathcal{D}(S_0) \subset X \to X$ be two linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$, then T is said to be ε-pseudo weakly S_0-demicompact (relative ε-pseudo weakly demicompact with respect to S_0), if for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and for all bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(S_0 - T - D)x_n$ converges weakly in X, then $(x_n)_n$ has a weakly convergent subsequence of $(x_n)_n$. We denoted by $\mathcal{WDC}_\varepsilon(S_0)(X)$ the family of ε-pseudo weakly S_0-demicompact operators on X. When $S_0 = I, T$ is simply said ε-pseudo weakly demicompact. The aim of this project is to ghather some characterizations which are related to this concept. More specifically, we are interested in the description of this class by means of ε-pseudo Fredholm and upper ε-pseudo semi-Fredholm operators. Another important interest of this paper is the study of pseudo-spectra which hold more informations than spectra, especially, about transient instead of just asymptotic behavior of dynamical systems. Historically, this concept was firstly introduced by Varah J. M. [26] in 1967 and has been subsequently employed by others mathematicians for example Landau H. J. [18], Trefethen L. N. [24] and Davies E. B. [7]. More precisely, the definition of pseudo-spectrum of a closed linear operator T is given for every $\varepsilon > 0$ by:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{ \lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\}.$$

By convention, we write $\|(\lambda - T)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$, (spectrum of T). In [7], Davies E. B. has defined equivalently the pseudo-spectrum of any closed operator T as follows: for every $\varepsilon > 0$,

$$\sigma_\varepsilon(T) := \bigcup_{\|D\| < \varepsilon} \sigma(T + D).$$

Similarly as the Schechter' essential spectrum, the authors in [3], studied some properties of the essential pseudo-spectrum of a densely defined, closed linear operator T acting on a Banach space X. This essential pseudo-spectrum is given by:

$$\sigma_{e5,\varepsilon}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(T + K).$$

The following useful result characterizes $\sigma_{e5,\varepsilon}(T)$ by means of the class of Fredholm perturbation $\mathcal{F}(X)$ (see Definition 2.3).

Theorem 1.1. [12] Let X be a Banach space, T be a closed densely defined linear operator on X and $\varepsilon > 0$. Then,

$$\sigma_{e5,\varepsilon}(T) = \bigcap_{K \in \mathcal{F}(X)} \sigma_\varepsilon(T + K). \qquad \diamondsuit$$

We organize the paper in the following way: In section 2, we recall some preliminary results needed in the sequel. In section 3, we establish some results concerning the class of ε-pseudo weakly demicompact operators. In section 4, we provide some sufficient conditions on the inputs of the block operator matrix $\mathcal{L}_0$ to ensure the ε-pseudo weak demicompactness. Finally, we show the relationship between the essential pseudo-spectra of $\mathcal{L}_0$ and the essential pseudo-spectra of its entries.

2. Preliminary results

In the beginning of this section, we recall some standard definitions and notations from Fredholm theory needed in the sequel. Let X and Y be two Banach spaces. We denote $\to$ for the strong convergence (i.e. norm convergence in X) and $\rightharpoonup$ for the weak convergence (with respect to the weak topology of X). Throughout this paper, we consider $V : \mathcal{D}(V) \subset X \longrightarrow Y$ as a linear operator with domain $\mathcal{D}(V)$ and range $\mathcal{R}(V) \subset Y$. If the graph of V is a closed subset of $X \times Y$, then V is closed. The set of all closed (resp. bounded) linear operators acting from X into Y is denoted by $\mathcal{C}(X, Y)$ (resp. $\mathcal{L}(X, Y)$). We denote by $\mathcal{K}(X, Y)$ the subset of compact operators of $\mathcal{L}(X, Y)$. For $V \in \mathcal{C}(X, Y)$, we use notations $\alpha(V)$ for the dimension of the kernel $\mathcal{N}(V)$ and $\beta(V)$ for the codimension of the range $\mathcal{R}(V)$ in Y. The graph norm of $x \in \mathcal{D}(V)$ is defined by

$$\|x\|_V := \|x\| + \|Vx\|.$$

It follows from the closedness of V that $X_V := (\mathcal{D}(V), \|.\|_V)$ is a Banach space. Clearly, we have

$$\|Vx\| \leq \|x\|_V, \text{ for every } x \in \mathcal{D}(V)$$

and consequently,

$$V \in \mathcal{L}(X_V, X).$$

Definition 2.1. Let X, Y and Z be three Banach spaces. Let $V : \mathcal{D}(V) \subset X \longrightarrow Y$ and $U : \mathcal{D}(U) \subset X \longrightarrow Z$ be two linear operators. U is said to be V-bounded, if $\mathcal{D}(V) \subset \mathcal{D}(U)$ and there exist constant $a, b \geq 0$ such that

$$\|Ux\| \leq a\|x\| + b\|Vx\|, \text{ for all } x \in \mathcal{D}(V).$$

The greatest lower bound of all possible values $b \geq 0$ is called the relative bound of U with respect to V or the V-bound of U.

Now a linear operator $U : X \longrightarrow Y$ is said to be V-defined if $\mathcal{D}(V) \subset \mathcal{D}(U)$. We denote by $\widehat{U}$ the restriction of U to $\mathcal{D}(V)$. Besides, If $\widehat{U}$ is bounded from X_V into Y, we say that U is V-bounded. We can see that, if U is closed, then U is V-bounded. Therefore, we have the obvious relations:

(i) $\alpha(\widehat{V}) = \alpha(V),\ \beta(\widehat{V}) = \beta(V),\ \mathcal{R}(\widehat{U}) = \mathcal{R}(U)$.
(ii) $\alpha(\widehat{V} + \widehat{U}) = \alpha(V + U),\ \beta(\widehat{V} + \widehat{U}) = \beta(V + U),\ \mathcal{R}(\widehat{U} + \widehat{V}) = \mathcal{R}(U + V)$.

Definition 2.2. Let X be a Banach space. An operator $V \in \mathcal{L}(X,Y)$ is said to be weakly compact if $V(B)$ is relatively weakly compact in Y for every bounded set $B \subset X$.

The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$, the family of weakly compact operators on X is simply denoted by $\mathcal{W}(X) := \mathcal{W}(X,X)$ The set $\mathcal{W}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [8, 9]).

Now, we define the sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from X into Y, respectively, by

$$\Phi_+(X,Y) = \{V \in \mathcal{C}(X,Y) \text{ such that } \alpha(V) < \infty \text{ and } \mathcal{R}(V) \text{ closed in } Y\},$$
$$\Phi_-(X,Y) = \{V \in \mathcal{C}(X,Y) \text{ such that } \beta(V) < \infty \text{ and } \mathcal{R}(V) \text{ closed in } Y\},$$
$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y),$$

and

$$\Phi_\pm(X,Y) := \Phi_-(X,Y) \cup \Phi_+(X,Y).$$

For $V \in \Phi_\pm(X,Y)$, we define the index of V by the following difference

$$i(V) := \alpha(V) - \beta(V).$$

By the index theorem we have

$$i(UV) = i(U) + i(V).$$

If $X = Y$, then $\mathcal{L}(X,Y), \mathcal{C}(X,Y), \mathcal{K}(X,Y), \mathcal{W}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\mathcal{W}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively. If $V \in \mathcal{C}(X)$, $\rho(V)$ denotes the resolvent set of V, $\sigma(V)$ the spectrum of V.

Definition 2.3. [10] Let X and Y be two Banach spaces and let $U \in \mathcal{L}(X,Y)$. The operator U is called:
(i) Fredholm perturbation if $V + U \in \Phi(X,Y)$, whenever $V \in \Phi(X,Y)$.
(ii) Upper semi-Fredholm perturbation if $V + U \in \Phi_+(X,Y)$, whenever $V \in \Phi_+(X,Y)$.
(iii) Lower semi-Fredholm perturbation if $V + U \in \Phi_-(X,Y)$, whenever $V \in \Phi_-(X,Y)$.

The set of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively.

In general, we have

$$\mathcal{K}(X,Y) \subset \mathcal{F}_+(X,Y) \subset \mathcal{F}(X,Y).$$

$$\mathcal{K}(X,Y) \subset \mathcal{F}_-(X,Y) \subset \mathcal{F}(X,Y).$$

If $X = Y$, $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$ are respectively replaced by $\mathcal{F}(X)$, $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$, respectively.

Lemma 2.1. [11] Let X and Y be two Banach spaces and let $V \in \mathcal{C}(X,Y)$ and $U : X \to Y$ be a linear operator.
(i) If $V \in \Phi(X,Y)$ and $U \in \mathcal{F}(X,Y)$, then $V + U \in \Phi(X,Y)$ and $i(V + U) = i(V)$.
(ii) If $V \in \Phi_+(X,Y)$ and $U \in \mathcal{F}_+(X,Y)$, then $V + U \in \Phi_+(X,Y)$ and $i(V + U) = i(V)$.
(iii) If $V \in \Phi_-(X,Y)$ and $U \in \mathcal{F}_-(X,Y)$, then $V + U \in \Phi_-(X,Y)$ and $i(V + U) = i(V)$.

Theorem 2.1. [19, 20] Let X, Y and Z be three Banach spaces, $V \in \mathcal{L}(Y,Z)$ and $U \in \mathcal{L}(X,Y)$.
(i) If $VU \in \Phi_+(X,Z)$, then $U \in \Phi_+(X,Y)$.
(ii) If $VU \in \Phi_-(X,Z)$, then $V \in \Phi_-(Y,Z)$.
(iii) If $X = Y = Z$, $VU \in \Phi(X)$ and $UV \in \Phi(X)$, then $V \in \Phi(X)$ and $U \in \Phi(X)$.
(iv) If $V \in \Phi_+(Y,Z)$ and $U \in \Phi_+(X,Y)$, then $VU \in \Phi_+(X,Z)$.
(v) If $V \in \Phi_-(Y,Z)$ and $U \in \Phi_-(X,Y)$, then $VU \in \Phi_-(X,Z)$.
(vi) If $V \in \Phi(Y,Z)$ and $U \in \Phi(X,Y)$, then $VU \in \Phi(X,Z)$ and $i(V + U) = i(V) + i(U)$.

Definition 2.4. Let X and Y be two Banach spaces and let $V \in \mathcal{C}(X,Y)$.
(i) An operator V is said to have a left Fredholm inverse if there exists $V_l \in \mathcal{L}(Y, X_V)$ such that $I_{X_V} - V_l\widehat{V} \in \mathcal{K}(X_V)$.
(ii) An operator V is said to have a right Fredholm inverse if there exists $V_r \in \mathcal{L}(Y, X_V)$ such that $I_Y - \widehat{V}V_r \in \mathcal{K}(Y)$.

Definition 2.5. Let X and Y be two Banach spaces and let $V \in \mathcal{C}(X,Y)$ and $\varepsilon > 0$.
(i) V is called a ε-pseudo upper (resp. lower) semi-Fredholm operator if $V + D$ is an upper (resp. lower) semi-Fredholm operator for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$.
(ii) V is called a ε-pseudo semi-Fredholm operator if $V + D$ is a semi-Fredholm operator for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$.
(iii) V is called a ε-pseudo Fredholm operator if $V + D$ is a Fredholm operator for all $D \in \mathcal{L}(X,Y)$ such that $\|D\| < \varepsilon$.

The sets of all ε-pseudo Fredholm, ε-pseudo upper Fredholm and ε-pseudo lower Fredholm are, respectively, denoted by $\Phi^\varepsilon(X,Y)$, $\Phi^\varepsilon_+(X,Y)$ and $\Phi^\varepsilon_-(X,Y)$.

If $X = Y$, the sets $\Phi^\varepsilon(X,Y)$, $\Phi^\varepsilon_+(X,Y)$ and $\Phi^\varepsilon_-(X,Y)$ are replaced by $\Phi^\varepsilon(X)$, $\Phi^\varepsilon_+(X)$ and $\Phi^\varepsilon_-(X)$, respectively.

Moreover, we have the following inclusions

$$\Phi^\varepsilon_+(X,Y) \subsetneq \Phi_+(X,Y),$$

$$\Phi^\varepsilon_-(X,Y) \subsetneq \Phi_-(X,Y), \text{ and}$$

$$\Phi^\varepsilon(X,Y) \subsetneq \Phi(X,Y).$$

Note that the previous inclusions are strict. To see this, we consider the following shift operator on l^2.

$$Tx = (x_2, x_3, x_4, \ldots)$$

Since $\|T\| = 1$, we take $\varepsilon > 1$ and $D = -T$. Then, we have $T + D = 0 \notin \Phi(X)$.

Lemma 2.2. Let X and Y be two Banach spaces and $\varepsilon > 0$. Let $V \in \mathcal{L}(Y, X)$ and $U \in \mathcal{L}(X, Y)$.
(i) If $V \in \Phi(Y, X)$, $U \in \Phi^{\varepsilon}(X, Y)$ and $(I - V)D \in \mathcal{F}(X, Y)$, then $VU \in \Phi^{\varepsilon}(X, Y)$ and $i(VU + D) = i(V) + i(U + D)$ for all $D \in \mathcal{L}(X, Y)$ satisfying $\|D\| < \varepsilon$.
(ii) If $V \in \Phi_+(Y, X)$, $U \in \Phi_+^{\varepsilon}(X, Y)$ and $(I - V)D \in \mathcal{F}_+(X, Y)$, then $VU \in \Phi_+^{\varepsilon}(X, Y)$.

Proof. (i) For each $D \in \mathcal{L}(X, Y)$ satisfying $\|D\| < \varepsilon$, we have

$$VU + D = V(U + D) + (I - T)D. \tag{2.1}$$

Since $V \in \Phi(Y, X)$ and $U + D \in \Phi(X, Y)$, then applying Theorem 2.1 (v) on Eq. (2.1) and using the fact that $(I - V)D \in \mathcal{F}(X, Y)$, we get $VU \in \Phi^{\varepsilon}(X, Y)$ and $i(VU + D) = i(V) + i(U + D)$.
(ii) We reason in the same way as the proof of (i). Q.E.D.

Definition 2.6. Let X and Y be two Banach spaces and let $U \in \mathcal{C}(X, Y)$ and $\varepsilon > 0$.
(i) U is said to have an ε-pseudo left weak-Fredholm inverse if there exists $U_l^w \in \mathcal{L}(Y, X_U)$ and $W \in \mathcal{W}(X_U)$ such that $U_l^w(U + D) = I_{X_U} - W$, for all $D \in \mathcal{L}(X, Y)$ such that $\|D\| < \varepsilon$. The operator U_l^w is called ε-pseudo left weak-Fredholm inverse of U.
(ii) U is said to have an ε-pseudo right weak-Fredholm inverse if there exists $U_r^w \in \mathcal{L}(Y, X_U)$ and $W \in \mathcal{W}(Y)$ such that $(U + D)U_r^w = I_Y - W$, for all $D \in \mathcal{L}(X, Y)$ such that $\|D\| < \varepsilon$. The operator U_r^w is said an ε-pseudo right weak-Fredholm inverse of U.
(iii) U is said to have an ε-pseudo weak-Fredholm inverse if there exists a map which is both an ε-pseudo left and an ε-pseudo right weak-Fredholm inverse of U.

In this research work, we are basically interested in the following essential pseudo-spectra

$$\begin{aligned}
\sigma_{e1,\varepsilon}(V) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi_+^{\varepsilon}(X)\},\\
\sigma_{e2,\varepsilon}(V) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi_-^{\varepsilon}(X)\},\\
\sigma_{e3,\varepsilon}(V) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi_\pm^{\varepsilon}(X)\},\\
\sigma_{e4,\varepsilon}(V) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - V \notin \Phi^{\varepsilon}(X)\},\\
\sigma_{e5,\varepsilon}(V) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_{\varepsilon}(V + K),
\end{aligned}$$

Note that if ε tends to 0, we recover the well-known definitions of essential spectra of V.

3. The class $\mathcal{WDC}_{\varepsilon}(X, Y)$

We start this section by defining ε-pseudo weakly demicompact linear operators.

Definition 3.1. Let $(Y, \|.\|_Y)$ be a Banach space and let X be a subspace of Y endowed with a norm $\|.\|_X$ such $(X, \|.\|_X)$ is a Banach space. Let $T \in \mathcal{C}(X, Y)$ be a closed linear operator from X into Y and $\varepsilon > 0$. Then, T is called ε-pseudo weakly demicompact if for every sequence $(x_n)_n$ in $\mathcal{D}(T)$ and $D \in \mathcal{L}(X, Y)$ with $\|D\| < \varepsilon$ such that $(x_n - Tx_n - Dx_n)_n$ converges weakly in Y, then there exists a weakly convergent subsequence of $(x_n)_n$ in X.

We denote by $\mathcal{WDC}_\varepsilon(X,Y)$, the set of all ε-pseudo weakly demicompact operators from X into Y. If $(X,\|.\|_X)=(Y,\|.\|_Y)$, we simply denote by $\mathcal{WDC}_\varepsilon(X)$.

Theorem 3.1. Let X be a Banach space and let $T:\mathcal{D}(T)\subset X\to X$ be a closed linear operator. Assume that $X^*+X^*\circ T$ is dense in $(X_T)^*$, where X^* and $(X_T)^*$ denote the topological dual spaces of X and $X_T=(\mathcal{D}(T),\|.\|_T)$ respectively. Then, for every $\varepsilon>0$ the following equivalence holds.

$T\in\mathcal{WDC}_\varepsilon(X)$ if, and only if, $\widehat{T}\in\mathcal{WDC}_\varepsilon(X_T,X)$.

Proof. Let $\varepsilon>0$, $D\in\mathcal{L}(X)$ such that $\|D\|<\varepsilon$ and let $(x_n)_n$ be a bounded sequence of X_T such that $x_n-\widehat{T}x_n-Dx_n\rightharpoonup y$, in X. Clearly, $(x_n)_n$ is bounded in X and $x_n-Tx_n-Dx_n\rightharpoonup y$. Since $T\in\mathcal{WDC}_\varepsilon(X)$, then there exists a subsequence $(x_{\varphi(n)})_n\subset\mathcal{D}(T)$ such that $x_{\varphi(n)}\rightharpoonup x,\ x\in X$. We have to show that $x_{\varphi(n)}\rightharpoonup x$ in X_T. For this purpose, let $f\in(X_T)^*$, it follows that there exists $(f_m)_m$ with $f_m=g_m+h_m\circ T$, $m\in\mathbb{N}$. Where $(g_m)_m\subset X^*$, $(h_m)_m\subset X^*$ and $\|f_m-f\|_{(X_T)^*}\to 0$, as $m\to+\infty$. Clearly, $g_m(x_{\varphi(n)})\to g_m(x)$ for all $m\in\mathbb{N}$. Now,

$$Tx_{\varphi(n)}=Tx_{\varphi(n)}+Dx_{\varphi(n)}-x_{\varphi(n)}-Dx_{\varphi(n)}+x_{\varphi(n)}\rightharpoonup -y+x-Dx.$$

It follows from the closedness of T that $x\in\mathcal{D}(T)$ and $x-Dx-y=Tx$. Consequently, $Tx_{\varphi(n)}\rightharpoonup Tx$ in X. Which implies that $h_m(Tx_{\varphi(n)})\rightharpoonup h_m(Tx)$ for all $m\in\mathbb{N}$. It follows that $f_m(x_{\varphi(n)})\to f_m(x)$, for all $m\in\mathbb{N}$. Now, write

$$\begin{aligned}|f(x_{\varphi(n)})-f(x)| &\le |f(x_{\varphi(n)})-f_m(x_{\varphi(n)})|+|f_m(x_{\varphi(n)})-f_m(x)|+|f_m(x)-f(x)|\\ &\le \|f-f_m\|_{X_T^*}\|x_{\varphi(n)}\|_T+|f_m(x_{\varphi(n)})-f_m(x)|+\|f_m-f\|_{X_T^*}\|x\|_T\end{aligned}$$

Since $(x_n)_n$ is a bounded sequence of X_T, then there exists $M>0$ such that $\|x_{\varphi(n)}\|\le M$ and $\|Tx_{\varphi(n)}\|\le M$. Let $\delta>0$ then there exists $m_0\in\mathbb{N}$ such that for all $m\ge m_0$,

$$\|f-f_m\|_{X_T^*}<\frac{\delta}{3M}.$$

It follows that

$$|f_m(x_{\varphi(n)})-f(x)|\le\frac{\delta}{3}+|f_{m_0}(x_{\varphi(n)})-f_{m_0}(x)|+\frac{\delta}{3}.$$

Now, from the fact that $f_{m_0}(x_{\varphi(n)})\to f_{m_0}(x)$, as $n\to+\infty$, we deduce that there exists $n_0\in\mathbb{N}$ such that for all $n\ge n_0$,

$$|f_{n_0}(x_{\varphi(n)})-f_{n_0}(x)|\le\frac{\delta}{3}.$$

Consequently,

$$|f(x_{\varphi(n)})-f(x)|<\frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3}<\delta.$$

Hence, $x_{\varphi(n)}\rightharpoonup x$ in X_T.

Conversely, let $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $(x_n)_n$ be a bounded sequence of X such that $x_n - Tx_n - Dx_n \rightharpoonup y$ in X. Then, there exists $M > 0$ such that $\|x_n\| \leq M$, and $\|Tx_n - x_n + Dx_n\| \leq M$ for all $n \geq 0$. It follows that

$$\|x_n\|_T = \|Tx_n\| + \|x_n\| \leq (3 + M)\varepsilon.$$

Then $(x_n)_n$ is bounded in X_T. Since $x_n - \widehat{T}x_n - Dx_n \rightharpoonup y$ in X and $\widehat{T} \in \mathcal{WDC}_\varepsilon(X_T, X)$, then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ and $x \in X$ such that $x_{\varphi(n)} \rightharpoonup x$, in X_T. Which achieves the proof. Q.E.D.

Theorem 3.2. Let X be a Banach space and let $T : \mathcal{D}(T) \subset X \to X$ be a closed linear operator. Assume that $X^* + X^* \circ T$ is dense in $(X_T)^*$. Fix $\varepsilon > 0$ and $S \in \mathcal{L}(X)$. If $T \in \mathcal{WDC}_\varepsilon(X)$ and the operator $I - T$ has a left (resp. right) ε-pseudo weakly Fredholm inverse T_l (resp. T_r) such that ST_l (resp. T_rS) $\in \mathcal{WDC}(X)$, then $T + S \in \mathcal{WDC}_\varepsilon(X)$.

Proof. Let $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, then there exist $T_l \in \mathcal{L}(X, X_T)$ (resp. $T_r \in \mathcal{L}(X, X_T)$) and $K \in \mathcal{W}(X_T)$ (resp. $K' \in \mathcal{W}(X)$) such that:

$$T_l(I - \widehat{T} - D) = I - K, \text{ on } X_T.$$

$$(\text{resp. } (I - \widehat{T} - D)T_r = I - K', \text{ on } Y).$$

Then, the operator $I - \widehat{T} - S - D$ can be written as follows

$$I - \widehat{T} - S - D = (I - ST_l)(I - \widehat{T} - D) - SK. \tag{3.1}$$

$$(\text{resp. } I - \widehat{T} - S - D = (I - \widehat{T} - D)(I - T_rS) - K'S). \tag{3.2}$$

Now, let $(x_n)_n$ be a bounded sequence of X_T satisfying $(I - \widehat{T} - S - D)x_n$ converges weakly to an element of X. It follows from Eq. (3.1) (resp. Eq. (3.2)) together with the weak compactness of SK (resp. $K'S$), the weak demicompactness of ST_l (resp. T_rS) and the boundedness of $(I - \widehat{T} - D)x_n$ that $(I - \widehat{T} - D)x_n$ has a weakly convergent subsequence. Since T is ε-pseudo weakly demicompact, according to Theorem 3.1, $\widehat{T}$ is ε-pseudo weakly demicompact. Therefore, $(x_n)_n$ admits a weakly convergent subsequence in X_T and this shows the ε-pseudo weak demicompactness of $\widehat{T} + S$ and consequently the ε-pseudo weak demicompactness of $T + S$. Q.E.D.

4. ε-Pseudo weak-demicompactness for B.O.M

Proposition 4.1. Let X be a Banach space, $\varepsilon > 0$ and A: $\mathcal{D}(A) \subset X \longrightarrow X$ be a closed linear operator and D: $X \longrightarrow X$ be a bounded linear operator. Let $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} := \widetilde{\mathcal{A}} + \widetilde{\mathcal{D}}$ with $\widetilde{\mathcal{A}} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $\widetilde{\mathcal{D}} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$. Assume that $X^* + X^* \circ A$ is dense in X_A^*. If $\widetilde{\mathcal{A}}$ is ε-pseudo weakly demicompact matrix and $\mathcal{I} - \widetilde{\mathcal{A}}$ has a left (resp. right) ε-pseudo weakly Fredholm inverse $\widetilde{\mathcal{A}}_l$ (resp. $\widetilde{\mathcal{A}}_r$) such that $\widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l \in \mathcal{DC}(X \times X)$. Then $\mathcal{A} \in \mathcal{WDC}_\varepsilon(X \times X)$.

Proof. First, let us prove that $(X\times X)^*+(X\times X)^*\circ\widetilde{\mathcal{A}}$ is dense in $(X\times X)^*_{\widetilde{\mathcal{A}}}$. Let $f:(\mathcal{D}(A)\times X,\|.\|_{\widetilde{\mathcal{A}}})\longrightarrow\mathbb{R}$ a bounded linear form. Then there exists $f_1:X_A\longrightarrow\mathbb{R}$ and $f_2:X\longrightarrow\mathbb{R}$ two bounded linears forms such that $f(x,y)=f_1(x)+f_2(y)$ (put $f_1(x)=f(x,0)$ and $f_2(y)=f(0,y)$). Since $X^*+X^*\circ A$ is dense in X^*_A, there exists two sequence $(h_{1n})_n,(k_{1n})_n$ in X^* such that $h_{1n}+k_{1n}\circ A\to f_1$. Set $H_n(x,y)=h_{1n}(x)+f_2(y)$ and $K_n(x,y)=k_{1n}(x)$ for all $(x,y)\in X\times X$. Observe that H_n and K_n are linears. Moreover,

$$\begin{aligned}|H_n(x,y)| &\le \|h_{1n}\|\|x\|+\|f_2\|\|y\|\\ &\le (\|h_{1n}\|+\|f_2\|)\|(x,y)\|,\end{aligned}$$

and

$$|K_n(x,y)|\le\|k_{1n}\|\|x\|.$$

Since $H_n(x,y)+K_n\circ\widetilde{\mathcal{A}}(x,y)=h_{1n}(x)+f_2(y)+k_{1n}(Ax)$. Then,

$$H_n+K_n\circ\widetilde{\mathcal{A}}\to f.$$

Consequently, $(X\times X)^*+(X\times X)^*\circ\widetilde{\mathcal{A}}$ is dense in $(X\times X)^*_{\widetilde{\mathcal{A}}}$.

Let $\mathcal{P}=\begin{pmatrix}P_1 & P_2\\ P_3 & P_4\end{pmatrix}\in\mathcal{L}(X\times X)$ such that $\|\mathcal{P}\|<\varepsilon$, then there exist $\mathcal{K}\in\mathcal{W}((X\times X)_{\widetilde{\mathcal{A}}})$ and $\widetilde{\mathcal{A}}_l\in\mathcal{L}(X\times X,(X\times X)_{\widetilde{\mathcal{A}}})$ such that:

$$\widetilde{\mathcal{A}}_l(I-\widehat{\widetilde{\mathcal{A}}}-\mathcal{P})=I-\mathcal{K}.$$

Then, the matrix $I-\widehat{\mathcal{A}}-\mathcal{P}$ can be written as follows

$$I-\widehat{\mathcal{A}}-\mathcal{P}=(I-\widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l)(I-\widehat{\widetilde{\mathcal{A}}}-\mathcal{P})-\widetilde{\mathcal{D}}\mathcal{K}. \tag{4.1}$$

Now, let $(x_n,y_n)_n$ be a bounded sequence of $(X\times X)_{\widetilde{\mathcal{A}}}$ such that $(I-\mathcal{A}-\mathcal{P})(x_n,y_n)_n$ converges weakly to an element of $X\times X$. It follows from Eq. 4.1 together with the weak compactness of $\widetilde{\mathcal{D}}\mathcal{K}$, the weak demicompactness of $\widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l$ and the boundedness of $(I-\widehat{\widetilde{\mathcal{A}}}-\mathcal{P})(x_n,y_n)$ that $(I-\widehat{\widetilde{\mathcal{A}}}-\mathcal{P})(x_n,y_n)_n$ admits a weakly convergent subsequence. Since $\widetilde{\mathcal{A}}$ is ε-pseudo weakly demicompact, then by applying Theorem 3.1, we infer that $\widehat{\widetilde{\mathcal{A}}}$ is ε-pseudo weakly demicompact. Therefore, $(x_n,y_n)_n$ admits a weakly convergent subsequence in $(X\times X)_{\widetilde{\mathcal{A}}}$ and this shows the ε-pseudo weak demicompactness of $\widehat{\mathcal{A}}$. So, $\mathcal{A}\in\mathcal{WDC}_\varepsilon(X\times X)$. Q.E.D.

Proposition 4.2. Let X be two Banach space, $\varepsilon>0$. Let A: $\mathcal{D}(A)\subset X\longrightarrow X$ and D: $\mathcal{D}(D)\subset X\longrightarrow X$ be two closed linear operators. Let B: $X\longrightarrow X$ and C: $X\longrightarrow X$ be two bounded linear operators.

Let $\mathcal{B}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}:=\widetilde{\mathcal{A}}+\widetilde{\mathcal{B}}$ with $\widetilde{\mathcal{A}}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}$, $\widetilde{\mathcal{B}}=\begin{pmatrix}0 & B\\ C & 0\end{pmatrix}$. Assume that $X^*+X^*\circ A$ is dense in X^*_A and $X^*+X^*\circ D$ is dense in X^*_D. If $\widetilde{\mathcal{A}}$ is ε-pseudo weakly demicompact matrix and $\mathcal{I}-\widetilde{\mathcal{A}}$ has a left (resp. right) ε-pseudo weakly Fredholm inverse $\widetilde{\mathcal{A}}_l$ (resp. $\widetilde{\mathcal{A}}_r$) such that $\widetilde{\mathcal{B}}\widetilde{\mathcal{A}}_l\in\mathcal{DC}(X\times X)$. Then $\mathcal{B}\in\mathcal{WDC}_\varepsilon(X\times X)$.

Proof. First, let us prove that $(X \times X)^* + (X \times X)^* \circ \widetilde{\mathcal{A}}$ is dense in $(X \times X)^*_{\widetilde{\mathcal{A}}}$. Let $f : (\mathcal{D}(A) \times X, \|.\|_{\widetilde{\mathcal{A}}}) \longrightarrow \mathbb{R}$ a bounded linear form. Then there exists $f_1 : X_A \longrightarrow \mathbb{R}$ and $f_2 : X \longrightarrow \mathbb{R}$ two bounded linears forms such that $f(x, y) = f_1(x) + f_2(y)$ (put $f_1(x) = f(x, 0)$ and $f_2(y) = f(0, y)$). Since $X^* + X^* \circ A$ is dense in X^*_A, there exists two sequence $(h_{1n})_n, (k_{1n})_n$ in X^* such that $h_{1n} + k_{1n} \circ A \to f_1$ and $X^* + X^* \circ D$ is dense in X^*_D, there exists two sequence h_{2n}, k_{2n} in X^* such that $h_{2n} + k_{2n} \circ D \to f_2$. Set $W_n(x, y) = H_n(x, y) + K_n \circ \widetilde{\mathcal{A}}(x, y)$ where $H_n(x, y) = h_{1n}(x) + h_{2n}(y)$ and $K_n(x, y) = k_{1n}(x) + k_{2n}(y)$ for all $(x, y) \in X \times X$. Observe that W_n, H_n and k_n are linears. moreover,

$$\begin{aligned} |H_n(x, y)| &\leq \|h_{1n}\|\|x\| + \|h_{2n}\|\|y\| \\ &\leq (\|h_{1n}\| + \|h_{2n}\|)\|(x, y)\|, \end{aligned}$$

and

$$\begin{aligned} |K_n(x, y)| &\leq \|k_{1n}\|\|x\| + \|k_{2n}\|\|y\| \\ &\leq (\|k_{1n}\| + \|k_{2n}\|)\|(x, y)\|, \end{aligned}$$

Therefore,

$$W_n(x, y) = h_{1n}(x) + h_{2n}(y) + k_{1n}(Ax) + k_{2n}(Dx) \to f_1(x) + f_2(y) = f(x, y).$$

Consequently, $(X \times X)^* + (X \times X)^* \circ \widetilde{\mathcal{A}}$ is dense in $(X \times X)^*_{\widetilde{\mathcal{A}}}$.

Let $\mathcal{P} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathcal{L}(X \times X)$ such that $\|\mathcal{P}\| < \varepsilon$, then there exist $\mathcal{K} \in \mathcal{W}((X \times X)_{\widetilde{\mathcal{A}}})$ and $\widetilde{\mathcal{A}}_l \in \mathcal{L}(X \times X, (X \times X)_{\widetilde{\mathcal{A}}})$ such that:

$$\mathcal{A}_l(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P}) = I - \mathcal{K}.$$

Then, the matrix $I - \widehat{\mathcal{B}} - \mathcal{P}$ can be written as follows

$$I - \widehat{\mathcal{B}} - \mathcal{P} = (I - \widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l)(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P}) - \widetilde{\mathcal{D}}\mathcal{K}. \tag{4.2}$$

Now, let $(x_n, y_n)_n$ be a bounded sequence of $(X \times X)_{\widetilde{\mathcal{A}}}$ such that $(I - \widehat{\mathcal{B}} - \mathcal{P})(x_n, y_n)_n$ converges weakly to an element of $X \times X$. It follows from Eq. (4.2) together with the weak compactness of $\widetilde{\mathcal{D}}\mathcal{K}$, the weak demicompactness of $\widetilde{\mathcal{D}}\widetilde{\mathcal{A}}_l$ and the boundedness of $(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P})(x_n, y_n)$ we infer that $(I - \widehat{\widetilde{\mathcal{A}}} - \mathcal{P})(x_n, y_n)_n$ admits a weakly convergent subsequence. Since $\widetilde{\mathcal{A}}$ is ε-pseudo weakly demicompact, then by applying Theorem 3.1, we infer that $\widehat{\widetilde{\mathcal{A}}}$ is ε-pseudo weakly demicompact. Therefore, $(x_n, y_n)_n$ admits a weakly convergent subsequence in $(X \times X)_{\widetilde{\mathcal{A}}}$ and this shows the ε-pseudo weak demicompactness of $\widehat{\mathcal{B}}$. So, $\mathcal{B} \in \mathcal{WDC}_\varepsilon(X \times X)$. Q.E.D.

In the rest of this section, we will investigate the essential pseudo-spectrum of a bounded 2×2 matrix operator.

Definition 4.1. Let X and Y be two Banach spaces, $\varepsilon > 0$ and let $T \in \mathcal{L}(X, Y)$. The operator T is called:

(i) ε-pseudo Fredholm perturbation if $T + D \in \mathcal{F}(X, Y)$ for all $D \in \mathcal{L}(X, Y)$ such that $\|D\| < \varepsilon$.
(ii) ε-pseudo upper semi-Fredholm perturbation if $T + D \in \mathcal{F}_+(X, Y)$ for all $D \in \mathcal{L}(X, Y)$ such that $\|D\| < \varepsilon$.
(iii) ε-pseudo lower semi-Fredholm perturbation if $T + D \in \mathcal{F}_-(X, Y)$ for all $D \in \mathcal{L}(X, Y)$ such that $\|D\| < \varepsilon$.

The set of ε-pseudo Fredholm perturbation, ε-pseudo upper semi-Fredholm perturbation and ε-pseudo lower semi-Fredholm perturbation are denoted by $\mathcal{F}^\varepsilon(X, Y)$, $\mathcal{F}_+^\varepsilon(X, Y)$ and $\mathcal{F}_-^\varepsilon(X, Y)$, respectively.

When $X = Y$, $\mathcal{F}^\varepsilon(X, X)$, $\mathcal{F}_+^\varepsilon(X, X)$ and $\mathcal{F}_-^\varepsilon(X, X)$ will simply denoted $\mathcal{F}^\varepsilon(X)$, $\mathcal{F}_+^\varepsilon(X)$ and $\mathcal{F}_-^\varepsilon(X)$, respectively.

The set $\mathcal{F}^\varepsilon(X, Y)$ (resp. $\mathcal{F}_+^\varepsilon(X, Y)$ and $\mathcal{F}_-^\varepsilon(X, Y)$) is a subspace of $\mathcal{L}(X, Y)$. Moreover, we have the following inclusions

$$\mathcal{F}^\varepsilon(X, Y) \subsetneq \mathcal{F}(X, Y),$$

$$\mathcal{F}_+^\varepsilon(X, Y) \subsetneq \mathcal{F}_+(X, Y), \text{ and}$$

$$\mathcal{F}_-^\varepsilon(X, Y) \subsetneq \mathcal{F}_-(X, Y).$$

Theorem 4.1. Let X and Y be two Banach spaces and let $T \in \mathcal{C}(X, Y)$ and $F : X \to Y$ be a linear operator. Then:
(i) $T + F \in \Phi^\varepsilon(X, Y)$ whenever $T \in \Phi^\varepsilon(X, Y)$ and $F \in \mathcal{F}^\varepsilon(X, Y)$.
(ii) $T + F \in \Phi_+^\varepsilon(X, Y)$ whenever $T \in \Phi_+^\varepsilon(X, Y)$ and $F \in \mathcal{F}^\varepsilon(X, Y)$.
(iii) $T + K \in \Phi_-^\varepsilon(X, Y)$ whenever $T \in \Phi_-^\varepsilon(X, Y)$ and $F \in \mathcal{F}^\varepsilon(X, Y)$.

Proof. (i) Let $D \in \mathcal{L}(X, Y)$ such that $\|D\| < \varepsilon$. we can write $D = D_1 + D_2$ such that $\|D_1\| < \frac{\varepsilon}{2}$ and $\|D_2\| < \frac{\varepsilon}{2}$. By using Lemma 2.1, we get

$$T + F + D = T + D_1 + F + D_2 \in \Phi(X, Y).$$

Consequently,

$$T + F \in \Phi^\varepsilon(X, Y).$$

Q.E.D.

Lemma 4.1. Let X_1 and X_2 be two Banach spaces. Let

$$\mathcal{F} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where $F_{ij} \in \mathcal{L}(X_i, X_j)$, with $i, j = 1, 2$. Then

(i) $\mathcal{F} \in \mathcal{F}^\varepsilon(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}^\varepsilon(X_i, X_j)$, with $i, j = 1, 2$.
(ii) $\mathcal{F} \in \mathcal{F}_+^\varepsilon(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}_+^\varepsilon(X_i, X_j)$, with $i, j = 1, 2$.
(iii) $\mathcal{F} \in \mathcal{F}_-^\varepsilon(X_1 \times X_2)$ if and only if $F_{ij} \in \mathcal{F}_-^\varepsilon(X_i, X_j)$ with $i, j = 1, 2$.

Proof. (i) Suppose that $F_{ij} \in \mathcal{F}^{\varepsilon}_{-}(X_i, X_j)$ with $i, j = 1, 2$ and we will prove that $\mathcal{F} \in \mathcal{F}^{\varepsilon}_{-}(X_1 \times X_2)$. Let $\mathcal{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in \mathcal{L}(X_1 \times X_2)$ such that $\|\mathcal{P}\| < \varepsilon$. First, let us consider the following decomposition:

$$\mathcal{F} + \mathcal{P} = \begin{pmatrix} F_{11} + P_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & F_{12} + P_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ F_{21} + P_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F_{22} + P_{22} \end{pmatrix}.$$

It is sufficient to prove that $F_{ij} \in \mathcal{F}^{\varepsilon}_{-}(X_i, X_j)$ with $i, j = 1, 2$, then each operator in the right side of the previous equality is ε-pseudo Fredholm perturbation on $X_1 \times X_2$. For example, we will prove the result for the first operator. The proof for the other operators will be similarly achieved. Let $\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Phi(X_1 \times X_2)$ and let us denote $\widetilde{F} := \begin{pmatrix} F_{11} + P_{11} & 0 \\ 0 & 0 \end{pmatrix}$. From Atkinson Theorem [19] , it follows that there exist

$$\mathcal{L}_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \mathcal{L}(X_1 \times X_2)$$

and

$$\mathcal{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \in \mathcal{K}(X_1 \times X_2),$$

such that

$$\mathcal{L}\mathcal{L}_0 = I - \mathcal{K} \text{ on } X_1 \times X_2.$$

Then

$$(\mathcal{L} + \widetilde{F})\mathcal{L}_0 = I - \mathcal{K} + \widetilde{F}\mathcal{L}_0 = \begin{pmatrix} I - K_{11} + (F_{11} + P_{11})A_0 & K_{12} + (F_{11} + P_{11})B_0 \\ -K_{21} & I - K_{22} \end{pmatrix}.$$

Since $F_{11} \in \mathcal{F}^{\varepsilon}(X_1)$, then $F_{11} + P_{11} \in \mathcal{F}(X_1)$. By using Theorem 2.3.1 in [12], we will have

$$I - K_{11} + (F_{11} + P_{11})A_0 \in \Phi(X_1).$$

This together with the fact that $I - K_{22} \in \Phi(X_2)$, allows us to deduce, from Lemma 11.5.1 in [12], that

$$(\mathcal{L} + \widetilde{F})\mathcal{L}_0 - \begin{pmatrix} 0 & 0 \\ -K_{21} & D \end{pmatrix} \in \Phi(X_1 \times X_2).$$

Since K_{21} is compact and using the Theorem 2.1, we have $\mathcal{L} + \widetilde{F} \in \Phi(X_1 \times X_2)$.

Conversely, assume that $\mathcal{F} \in \mathcal{F}^{\varepsilon}_{-}(X_1 \times X_2)$ and we will prove that $F_{11} \in \mathcal{F}^{\varepsilon}_{-}(X_1)$. Let $A \in \Phi(X)$ and let define the operator

$$\mathcal{L}_1 := \begin{pmatrix} A & -F_{12} \\ 0 & I \end{pmatrix}.$$

From Lemma 11.5.1 in [12], it follows that

$$\mathcal{L}_1 \in \Phi(X_1 \times X_2).$$

Hence,

$$F + \mathcal{L}_1 = \begin{pmatrix} A + F_{11} & 0 \\ F_{21} & I + F_{22} \end{pmatrix} \in \Phi(X_1 \times X_2).$$

Using Remark 11.5.2 [12], we have

$$A + F_{11} \in \Phi_-(X_1). \tag{4.3}$$

In the same way, we may consider the Fredholm operator

$$\begin{pmatrix} A & 0 \\ -F_{21} & I \end{pmatrix} \in \Phi(X_1 \times X_2).$$

Using Remarks 11.5.1 and 11.5.2 in [12], it is easy to deduce that

$$A + F_{11} \in \Phi_-(X_1). \tag{4.4}$$

From Eqs. (4.3) and (4.4), it follows that $F_{11} + P_{11} \in \mathcal{F}(X_1)$. In the same way, we can prove that

$$F_{22} \in \mathcal{F}^{\varepsilon}(X_2).$$

Now, we have to prove that $F_{12} \in \mathcal{F}^{\varepsilon}(X_2, X_1)$ and $F_{21} \in \mathcal{F}^{\varepsilon}(X_1, X_2)$. For this, let us consider $A \in \Phi(X_2, X_1)$ and $B \in \Phi(X_1, X_2)$. Then,

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \Phi(X_1 \times X_2).$$

Using the fact that $F_{11} + P_{11} \in \mathcal{F}^{\varepsilon}(X_1)$ and $F_{22} + P_{22} \in \mathcal{F}^{\varepsilon}(X_2)$, we can deduce that

$$F + \begin{pmatrix} -F_{11} & 0 \\ 0 & -F_{22} \end{pmatrix} \in \mathcal{F}^{\varepsilon}(X_1 \times X_2).$$

Hence,

$$\begin{pmatrix} 0 & A + F_{12} \\ B + F_{21} & 0 \end{pmatrix} \in \Phi^{\varepsilon}(X_1 \times X_2).$$

So,

$$A + F_{12} \in \Phi^{\varepsilon}(X_2\, X_1)$$

and

$$B + F_{21} \in \Phi^{\varepsilon}(X_1, X_2).$$

Q.E.D.

Theorem 4.2. Let X, Y be two Banach spaces, $\varepsilon > 0$ and A: $X \longrightarrow X$, B: $Y \longrightarrow X$, C: $X \longrightarrow Y$ and D: $Y \longrightarrow Y$ are four bounded operators. Let $L := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and let $\mathcal{P} := \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathcal{L}(X \times Y)$ such that $\|\mathcal{P}\| < \varepsilon$.

(i) If $CA \in \mathcal{F}^{\varepsilon}(X,Y),\ CB \in \mathcal{F}^{\varepsilon}(Y)$ and $C \in \mathcal{F}(X,Y)$ then,

$$\sigma_{ei,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{ei,\varepsilon}(A) \cup \sigma_{ei,\varepsilon}(D)]\backslash\{0\},\ i = 4,5.$$

(ii) If $CA \in \mathcal{F}_{+}^{\varepsilon}(X,Y),\ CB \in \mathcal{F}_{+}^{\varepsilon}(Y)$ and $C \in \mathcal{F}_{+}(X,Y)$ then,

$$\sigma_{e1,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e1,\varepsilon}(A) \cup \sigma_{e1,\varepsilon}(D)]\backslash\{0\}.$$

(iii) If $CA \in \mathcal{F}_{-}^{\varepsilon}(X,Y),\ CB \in \mathcal{F}_{-}^{\varepsilon}(Y)$ and $C \in \mathcal{F}_{-}(X,Y)$ then,

$$\sigma_{e2,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e2,\varepsilon}(A) \cup \sigma_{e2,\varepsilon}(D)]\backslash\{0\}.$$

Proof. Let $\lambda \in \mathbb{C}\backslash\{0\}$. Then, we have

$$\begin{aligned} \lambda - L &= \begin{pmatrix} \lambda - A & -B \\ -C & \lambda - D \end{pmatrix} \\ &= \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ -CA & -CB \end{pmatrix} + \begin{pmatrix} I & 0 \\ \frac{-C}{\lambda} & I \end{pmatrix}\begin{pmatrix} \lambda - A & -B \\ 0 & \lambda - D \end{pmatrix}. \end{aligned}$$

Suppose $\lambda \notin [\sigma_{e5,\varepsilon}(A) \cup \sigma_{e5,\varepsilon}(D)]\backslash\{0\}$, then by Lemma 6.6.1 in [11],

$$\begin{pmatrix} \lambda - A & -B \\ 0 & \lambda - D \end{pmatrix} \in \Phi^{\varepsilon}(X \times Y) \text{ and } i\begin{pmatrix} \lambda - A - P_1 & -B - P_2 \\ -P_3 & \lambda - D - P_4 \end{pmatrix} = 0$$

for all $\mathcal{P} := \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathcal{L}(X \times Y)$ such that $\|\mathcal{P}\| < \varepsilon$.

Since $\begin{pmatrix} I & 0 \\ \frac{-C}{\lambda} & I \end{pmatrix}$ is invertible then the operator matrix

$\begin{pmatrix} I & 0 \\ \frac{-C}{\lambda} & I \end{pmatrix}$ is Fredholm and $i\begin{pmatrix} I & 0 \\ \frac{-C}{\lambda} & I \end{pmatrix} = 0.$

Moreover by hypothesis and by applying Theorem 2.4 in [1], we get

$$\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I & 0 \\ \frac{-C}{\lambda} & I \end{pmatrix}\right]\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{-CP_1}{\lambda} & 0 \end{pmatrix} \in \mathcal{F}(X,Y),$$

for all $\mathcal{P} := \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathcal{L}(X \times Y)$ such that $\|\mathcal{P}\| < \varepsilon$.

Consequently, by using Lemma 2.2, we get

$$\begin{pmatrix} I & 0 \\ \frac{-C}{\lambda} & I \end{pmatrix}\begin{pmatrix} \lambda - A & -B \\ 0 & \lambda - D \end{pmatrix}$$

is ε-pseudo Fredholm matrix and

$$i\left[\begin{pmatrix} I & 0 \\ \frac{-C}{\lambda} & I \end{pmatrix}\begin{pmatrix} \lambda - A & -B \\ 0 & \lambda - D \end{pmatrix} + \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}\right]$$

$$= i\begin{pmatrix} I & 0 \\ \frac{-C}{\lambda} & I \end{pmatrix} + i\begin{pmatrix} \lambda - A - P_1 & -B - P_2 \\ -P_3 & \lambda - D - P_4 \end{pmatrix}$$

$$= 0.$$

On the other hand, since $CA \in \mathcal{F}^{\varepsilon}(X, Y), CB \in \mathcal{F}^{\varepsilon}(Y)$, it follows from the Lemma 4.1 that

$$\begin{pmatrix} 0 & 0 \\ -CA & -CB \end{pmatrix} \in \mathcal{F}^{\varepsilon}(X \times Y).$$

So, applying Theorem 4.1, we get

$$\lambda - L \in \Phi^{\varepsilon}(X \times Y) \text{ and } i(\lambda - L - \mathcal{P}) = 0,$$

for all $\mathcal{P} := \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathcal{L}(X \times Y)$ such that $\|\mathcal{P}\| < \varepsilon$.

Thus,

$$\lambda \notin \sigma_{e5,\varepsilon}(L)\backslash\{0\}.$$

Hence,

$$\sigma_{ei,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{ei,\varepsilon}(A) \cup \sigma_{ei,\varepsilon}(D)]\backslash\{0\},\ i = 4, 5.$$

The proof of (ii) and (iii) may be checked in the same way as the proof of (i). Q.E.D.

Theorem 4.3. Let X, Y be two Banach spaces, $\varepsilon > 0$ and A: $X \longrightarrow X$, B: $Y \longrightarrow X$, C: $X \longrightarrow Y$ and D: $Y \longrightarrow Y$ are four bounded operators. Let $L := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

(i) If $C \in \mathcal{F}^{\varepsilon}(X, Y)$ then,

$$\sigma_{ei,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{ei,\varepsilon}(A) \cup \sigma_{ei,\varepsilon}(D)]\backslash\{0\},\ i = 4, 5.$$

(ii) If $C \in \mathcal{F}_{+}^{\varepsilon}(X, Y)$ then,

$$\sigma_{e1,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e1,\varepsilon}(A) \cup \sigma_{e1,\varepsilon}(D)]\backslash\{0\}.$$

(iii) If $C \in \mathcal{F}_{-}^{\varepsilon}(X, Y)$ then,

$$\sigma_{e2,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{e2,\varepsilon}(A) \cup \sigma_{e2,\varepsilon}(D)]\backslash\{0\}.$$

Proof. (i) Let $\mathcal{P} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in \mathcal{L}(X \times Y)$ such that $\|\mathcal{P}\| < \varepsilon$. Then, for all $\lambda \in \mathbb{C}\backslash\{0\}$, we have

$$\lambda - L - \mathcal{P} = \begin{pmatrix} \lambda - A - P_1 & -B - P_2 \\ -C - P_3 & \lambda - D - P_4 \end{pmatrix}$$

$$= \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ -(C+P_3)(A+P_1) & -(C+P_3)(B+P_2) \end{pmatrix}$$
$$+ \begin{pmatrix} I & 0 \\ \frac{-(C+P_3)}{\lambda} & I \end{pmatrix}\begin{pmatrix} \lambda-A-P_1 & -B-P_2 \\ 0 & \lambda-D-P_4 \end{pmatrix}.$$

Suppose $\lambda \notin [\sigma_{e5,\varepsilon}(A) \cup \sigma_{e5,\varepsilon}(D)]\backslash\{0\}$, then by Lemma 6.6.1 in [11],

$$\begin{pmatrix} \lambda-A-P_1 & -B-P_2 \\ 0 & \lambda-D-P_4 \end{pmatrix} \in \Phi(X\times Y) \text{ and } i\begin{pmatrix} \lambda-A-P_1 & -B-P_2 \\ 0 & \lambda-D-P_4 \end{pmatrix} = 0.$$

Since $\begin{pmatrix} I & 0 \\ \frac{-(C+P_3)}{\lambda} & I \end{pmatrix}$ is invertible then $\begin{pmatrix} I & 0 \\ \frac{-(C+P_3)}{\lambda} & I \end{pmatrix}\begin{pmatrix} \lambda-A-P_1 & -B-P_2 \\ 0 & \lambda-D-P_4 \end{pmatrix}$ is a Fredholm matrix and

$$i\begin{pmatrix} I & 0 \\ \frac{-(C+P_3)}{\lambda} & I \end{pmatrix}\begin{pmatrix} \lambda-A-P_1 & -B-P_2 \\ 0 & \lambda-D-P_4 \end{pmatrix} = 0.$$

On the other hand, it follows from the hypothesis that $(C+P_3)(B+P_2) \in \mathcal{F}^b(Y)$ and $(C+P_3)(A+P_1) \in \mathcal{F}^b(X,Y)$ and so,

$$\begin{pmatrix} 0 & 0 \\ -(C+P_3)(A+P_1) & -(C+P_3)(B+P_2) \end{pmatrix} \in \mathcal{F}^b(X\times Y).$$

So, $\lambda - L - \mathcal{P} \in \Phi(X\times Y)$ and $i(\lambda - L - \mathcal{P}) = 0$. Thus, $\lambda \notin \sigma_{e5,\varepsilon}(L)\backslash\{0\}$. Hence,

$$\sigma_{ei,\varepsilon}(L)\backslash\{0\} \subset [\sigma_{ei,\varepsilon}(A) \cup \sigma_{ei,\varepsilon}(D)]\backslash\{0\},\ i = 4, 5.$$

(ii) The proof of (ii) and (iii) may be checked in the same way as the proof of (i). Q.E.D.

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