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Extension of Semistar Operations

Extension des operations semistar

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ABSTRACT. Let $R \subset T$ be an extension of integral domains and $*$ be a semistar operation stable of finite type on R. We define a semistar operation $*_1$ on T in the following way: for each nonzero T-submodule E of the quotient field K_1 of T, let $E^{*_1} = \cup\{E :_{K_1} JT \mid J \in \mathcal{F}^*\}$, where K_1 denotes the quotient field of T and $\mathcal{F}^*$ the localizing system associated to $*$. In this paper we investigate the basic properties of $*_1$. Moreover, we show that the map φ which associates to a semistar operation $*$ stable and of finite type on R, the semistar operation $*_1$ is continuous. Furthermore, we give sufficient conditions for φ to be a homeomorphism.
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1. Introduction

All rings considererd in this paper are commutative with identity. Let R be an integral domain with quotient field K. Let $\overline{F}(R)$ be the set of all nonzero R-submodules of K and $f(R)$ be the set of all nonzero finitely generated R-submodules of K.

In 1994, Okabe and Matsuda introduced the notion of semistar operation [6], in order to generalize the concept of star operation. A semistar operation on R is a map $* : \overline{F}(R) \to \overline{F}(R); E \longmapsto E^*$, such that for all $x \in K\backslash\{0\}$ and for all $E, F \in \overline{F}(R)$, the following properties hold:

(1) $(xE)^* = xE^*$.

(2) If $E \subset F$ then $E^* \subset F^*$.

(3) $E \subset E^*$ and $E^{**} = (E^*)^* = E^*$.

The set of all semistar operations on R is denoted by $SStar(R)$.

The map $e : \overline{F}(R) \to \overline{F}(R); E \longmapsto E^e = K$ is a semistar operation on R called the trivial semistar operation on R. Note that if $*$ is a semistar operation on R, then $* = e$ if and only if $R^* = K$. The identity semistar operation on R is defined by $\overline{F}(R) \to \overline{F}(R); E \longmapsto E$ and will be denoted by d_R.

Any ring T such that, $R \subset T \subset K$ will be called an overring of R. If T is an overring of R, we denote by $*_{\{T\}}$ the semistar operation on R defined by setting $E^{*_{\{T\}}} = ET$, for any $E \in \overline{F}(R)$.

Recall that if $*$ is a semistar operation on R then for all $E, F \in \overline{F}(R)$, we have $(EF)^* = (E^*F)^* = (EF^*)^* = (E^*F^*)^*$.

The set $SStar(R)$ of all semistar operations on R is endowed with a natural partial order, defined by $*_1 \leq *_2$ if and only if $E^{*_1} \subset E^{*_2}$ for any $E \in \overline{F}(R)$. Or, equivalently, if $(E^{*_1})^{*_2} = (E^{*_2})^{*_1} = E^{*_2}$ for each $E \in \overline{F}(R)$.

Let $*$ be a semistar operation on R. For any $E \in \overline{F}(R)$, set $E^{*_f} = \cup\{F^* \mid F \in f(R) \text{ and } F \subset E\}$, then $*_f$ is a semistar operation on R called the semistar operation of finite type associated to $*$. For any semsitar operation on R, we have $*_f \leq *$. A semistar operation $*$ is said to be of finite type if $* = *_f$. Note that $(*_f)_f = *_f$. The set of all semistar operations of finite type on R is denoted by $SStar_f(R)$.

A semistar operation on R is stable if $(E \cap F)^* = E^* \cap F^*$ for each $E, F \in \overline{F}(R)$. The trivial semistar operation and the identity semistar operations are stable. However, if T is an overring of R, the semistar operation $*_{\{T\}}$ is stable if and only if T is flat over R.

A localizing system $\mathcal{F}$ on R is a subset of ideals of R such that:

- if $I \in \mathcal{F}$ and J is an ideal of R such that $I \subset J$, then $J \in \mathcal{F}$.
- if $I \in \mathcal{F}$ and J is an ideal of R such that, for each $i \in I$, $(J :_R iR) \in \mathcal{F}$, then $J \in \mathcal{F}$.

Note that if $\mathcal{F}$ is a localizing system, and $I, J \in \mathcal{F}$ then $I \cap J \in \mathcal{F}$ and $IJ \in \mathcal{F}$.

A localizing system $\mathcal{F}$ on R is of finite type if for each $I \in \mathcal{F}$ there exists a nonzero finitely generated ideal $J \in \mathcal{F}$ with $J \subset I$. The relation between localizing systems and stable semistar operations has been investigated by Fontana and Huckaba in [3]. Recall that if $*$ is a semistar operation on R, then the set $\mathcal{F}^* = \{I \text{ ideal of } R \mid I^* = R^*\}$ is a localizing system (called the localizing system associated to $*$). Moreover, if $*$ is of finite type then $\mathcal{F}^*$ is of finite type. Conversely, if $\mathcal{F}$ is a localizing system on R then the map $*_{\mathcal{F}} : \overline{F}(R) \to \overline{F}(R); E \longmapsto E^{*_{\mathcal{F}}} = \cup\{E :_K J \mid J \in \mathcal{F}\}$ is a stable semistar operation on R. Moreover, if $\mathcal{F}$ is of finite type then $*_{\mathcal{F}}$ is of finite type. If $*$ is a semistar operation, the map $\widetilde{*} = *_{\mathcal{F}^{*_f}}$ is a semistar operation stable and of finite type. Moreover $* = \widetilde{*}$ if and only if $*$ is stable and of finite type.

Let $*$ be a semistar operation on R and I be a nonzero ideal of R, we say that I is a quasi-$*$-ideal if $I = I^* \cap R$. A quasi-$*$-prime ideal is a quasi-$*$-ideal which is also a prime ideal. A quasi-$*$-maximal ideal is an ideal which is a maximal element in the set of quasi-$*$-prime ideals. We denote by $QSpec^*(R)$ the set of quasi-$*$-prime ideals of R and $M(*)$ the set of quasi-$*$-maximal ideals of R, then we have $M(*) \subset QSpec^*(R)$. If $*$ is a non trivial semistar operation of finite type then each proper quasi-$*$-ideal is contained in a quasi-$*$-maximal ideal [3].

Let $*$ be a semistar operation on R and $I \in \overline{F}(R)$, we denote by $I^{-1} = \{x \in K \mid xI \subset R\}$ and we say that I is $*$-invertible if $(II^{-1})^* = R^*$.

In [2], the authors endowed the set $SStar(R)$ with the Zariski topology for which a subbasis of open sets is the collection of all sets of the form $V_F^R = \{* \in SStar(R) \mid 1 \in F^*\}$, as F ranges among the nonzero R-submodules of K. Note that this topology is naturally linked to the order $\leq$, in the following way: if $U \subset SStar(R)$ is an open neighborhood of $*$ and $*' \geq *$, then $*' \in U$. So for any $* \in SStar(R)$, $Ad(\{*\}) = \{*' \in SStar(R) : *' \leq *\}$ where $Ad(Y)$ denotes the closure of $Y \subset SStar(R)$.

Following [2], let $A \subset B$ be an extension of integral domains, and let k be the quotient field of A. For any semistar operation $* \in SStar(B)$, the authors define a semistar operation $\sigma(*) \in SStar(A)$ by setting $F^{\sigma(*)} = (FB)^* \cap k$ for every nonzero A-submodule F of k. They show that the map $\sigma : SStar(B) \to SStar(A)$ is continuous and if $* \in SStar(B)$ is of finite type, then so is $\sigma(*)$. Moreover, if B has the same quotient field as A then $\sigma(*)$ is of finite type if and only if $*$ is of finite type and σ is a topological embedding.

Let $R \subset T$ be an extension of commutative rings, K (respectively K_1) the quotient field of R (respectiveley of T). In this paper, we work on an ascent type property which relate semistar operations on R with semistar operations on T. Let $*$ be a semistar operation stable of finite type on R. We extend the semistar operation $*$ to a semistar operation on T by defining a new semistar operation $*_1$ on T in the following way: for any nonzero T-submodule E of the quotient field $K_1$of T, let $E^{*_1} = \cup\{E :_{K_1} JT \mid J \in \mathcal{F}^*\}$. Then $*_1$ is a stable semistar operation of finite type. In the second section of this paper we give the main properties of the semistar $*_1$. In particular we determine the localizing system associated to $*_1$. More precisely we show that $\mathcal{F}^{*_1} = \{I$ nonzero ideal of T such that $I \cap R \neq (0)$ and $(I \cap R)^* = R^*\}$. If the Lying over (LO) property holds in the pair R, T, that is for any prime ideal P in R there exists a prime ideal Q in T such that $Q \cap R = P$, and if $*$ is non trivial then the quasi-$*$-maximal ideals of R are exactely of the form $M \cap R$ where M is a quasi-$*_1$-maximal ideal of T. Moreover under additional hypothesis, we show that the $*$-Noetherian property is inherited from T to R and vice versa.

Let $\widetilde{SStar}(R)$ (repsectively $\widetilde{SStar}(T)$) denote the set of all stable semistar operations of finite type on R (respectively on T).

We denote by φ the map $\widetilde{SStar}(R) \to \widetilde{SStar}(T); * \to *_1$. The third section of this paper was motivated by the fact that if T is an overring of R then the map σ is a topological embedding. It is natural to try in that case to construct the inverse function of σ. First, we prove that if $\widetilde{SStar}(R)$ (respectively $\widetilde{SStar}(T)$) is endowed with the Zariski topology, then the map φ is continuous and $\varphi(\widetilde{SStar}(R)) = U$, where $U = \{\star$ semistar operation stable and of finite type on T such that for any $I \in \mathcal{F}^\star$, $(I \cap R)T \in \mathcal{F}^\star\}$. In addition, for any semistar operation $\star$ of finite type on T, $\varphi \circ \sigma(\star) \leq \star$. On the other hand, let $\overline{\sigma}(\star) = *_{\mathcal{F}^{\sigma(\star)}}$, for any $\star \in \widetilde{SStar}(T)$. We prove that for any semistar operation $\star$ of U, $\varphi(\overline{\sigma}(\star)) = \star$. In the case where the pair $R \subset T$ is LO we deduce that φ is a homeomorphism from $\widetilde{SStar}(R)$ into U.

In the following, $\dim R$ denotes the Krull dimension of R, $qf(R)$ its quotient field and $spec(R)$ its prime spectrum.

2. Extension of semistar operations

Recall that if $*$ is a semistar operation stable and of finite type on an integral domain R and if $\mathcal{F}^*$ denotes the localizing system associated to $*$, that is $\mathcal{F}^* = \{I$ ideal of $R \mid I^* = R^*\}$, then for any $E \in \overline{F}(R)$ we have

$$E^* = \cup\{E :_K J \mid J \in \mathcal{F}^*\}. \tag{1}$$

Proposition 1. *Let $R \subset T$ be an extension of integral domains which are not fields, $K = qf(R)$ and $K_1 = qf(T)$. Let $*$ be a semistar operation of finite type on R, then the map*

$$*_1 : \overline{F}(T) \to \overline{F}(T); E \longmapsto E^{*_1} = \cup\{E :_{K_1} JT \mid J \in \mathcal{F}^*\}$$

is a semistar operation stable of finite type on T.

Proof. Let $E \in \overline{F}(T)$ then $E^{*_1} \in \overline{F}(T)$. In fact, if $x \in E$ then for any $J \in \mathcal{F}^*$, $xJT \subset xT \subset E$, so $x \in E^{*_1}$ which implies that $E \subset E^{*_1}$ and $E^{*_1} \neq (0)$.

Let $x, y \in E^{*_1}$, we show that $x+y \in E^{*_1}$. There exist $J_1, J_2 \in \mathcal{F}^*$ such that $xJ_1T \subset E$ and $yJ_2T \subset E$. As $J_1 \cap J_2 \in \mathcal{F}^*$ and $(x+y)(J_1 \cap J_2)T \subset E$ then $x+y \in E^{*_1}$.

Let $x \in E^{*_1}$ and $a \in T$, we show that $ax \in E^{*_1}$. There exists $J_1 \in \mathcal{F}^*$ such that $xJ_1T \subset E$, so $axJ_1T \subset xJ_1T \subset E$, which implies that $ax \in E^{*_1}$.

Now we prove that $*_1$ is a semistar operation on T.

•) Let $E \in \overline{F}(T)$ and $y \in K_1 \backslash \{0\}$, then $(yE)^{*_1} = yE^{*_1}$. In fact, let $x \in (yE)^{*_1}$. There exists $J \in \mathcal{F}^*$ such that $x \in yE : JT$. So $xJ \subset yE$ which is equivalent to $xy^{-1}J \subset E$. Thus $xy^{-1} \in E : JT \subset E^{*_1}$ and $x \in yE^{*_1}$. Conversely, let $x \in yE^{*_1}$, then $xy^{-1} \in E^{*_1}$. So there exists $J \in \mathcal{F}^*$ such that $xy^{-1} \in E : JT$ which is equivalent to $xy^{-1}JT \subset E$. Thus $xJT \subset yE$ that is $x \in yE : JT \subset (yE)^{*_1}$.

•) Let $E, F \in \overline{F}(T)$ such that $E \subset F$. Then $E^{*_1} \subset F^{*_1}$. In fact, let $x \in E^{*_1}$, and $J \in \mathcal{F}^*$ such that $x \in E : JT \subset F : JT \subset F^{*_1}$.

•) We have already shown that $E \subset E^{*_1}$.

•) Let $E \in \overline{F}(T)$, Then $E^{*_1} = (E^{*_1})^{*_1}$. In fact, we have $E \subset E^{*_1}$ so $E^{*_1} \subset (E^{*_1})^{*_1}$. Conversely, let $x \in (E^{*_1})^{*_1}$ and $J \in \mathcal{F}^*$ such that $x \in E^{*_1} : JT$. Thus $xJ \subset E^{*_1}$. As $*$ is of finite type then we can assume that J is finitely generated. Set $J = (a_1, ..., a_n)$. For each $i \in \{1, ..., n\}$, $xa_i \in E^{*_1}$, so there exists $J_i \in \mathcal{F}^*$ such that $xa_iJ_i \subset E$. Let $I = J_1...J_n \in \mathcal{F}^*$, then for each $i \in \{1, ..., n\}$, $xa_iI \subset xa_iJ_i \subset E$, so $xJI \subset E$. As $JI \in \mathcal{F}^*$, then $x \in E^{*_1}$.

We prove that $*_1$ is stable. Let $E, F \in \overline{F}(T)$. Remark that $E \cap F \neq (0)$. It is clear that $(E \cap F)^{*_1} \subset E^{*_1} \cap F^{*_1}$. Conversely, let $x \in E^{*_1} \cap F^{*_1}$. There exist $I, J \in \mathcal{F}^*$ such that $xI \subset E$ and $xJ \subset F$. As $I \cap J \in \mathcal{F}^*$ and $x(I \cap J) \subset E \cap F$, then $x \in (E \cap F)^{*_1}$.

Finally, we show that $*_1$ is of finite type. Let $E \in \overline{F}(T)$, then $E^{(*_1)_f} \subset E^{*_1}$. Let $x \in E^{*_1}$. There exists $J \in \mathcal{F}^*$ such that $xJT \subset E$. As $*$ is of finite type then we can assume that J is finitely generated. Set $J = a_1R + ... + a_nR$. For each $i \in \{1, ..., n\}$, $xa_i \in E$. Let $F = xa_1T + ... + xa_nT \subset E$. We have $xJ \subset F$, so $xJT \subset F$ and $x \in F : JT$. Then, $x \in F^{*_1} \subset E^{(*_1)_f}$. Finally, $E^{*_1} = E^{(*_1)_f}$. ■

In the sequel, we are going to limit ourselves to semistar operations stable and of finite type on R. We denote by $SStar(R)$ (repsectively $SStar(T)$) the set of all semistar operation stable and of finite type on R (respectively on T). We denote also by φ the map from $SStar(R)$ to $SStar(T)$ which associates to a semistar operation $*$ stable and of finite type on R, the semistar operation $*_1$ defined in the previous proposition.

Remark 2. 1. *If $* \leq *'$ are two semistar operations stable and of finite type on R then $\varphi(*) \leq \varphi(*')$. This results from the fact that if $* \leq *'$ then $\mathcal{F}^* \subset \mathcal{F}^{*'}$. In particular, this implies that $\varphi(Ad(\{*\}) \subset Ad\{\varphi(*)\}$.*

2. *Let $*$ be a semistar operation stable of finite type on R. If $E \in \overline{F}(R)$ and $F \in \overline{F}(T)$ are such that $E \subset F$ then $E^* \subset F^{\varphi(*)}$. In fact, if $x \in E^*$ then there exists $J \in \mathcal{F}^*$ such that $xJ \subset E \subset F$, so $x \in F^{\varphi(*)}$.*

3. *Let $*$ be a semistar operation stable and of finite type on R. For any $E \in \overline{F}(R)$, $E^* = \cup\{E :_K J \mid J \in \mathcal{F}^*\}$ and $E^{\sigma(\varphi(*))} = \cup\{ET :_{K_1} JT \mid J \in \mathcal{F}^*\} \cap K$ so $E^* \subset E^{\sigma(\varphi(*))}$.*

We present next some notable examples.

Example 3. *1. If T is an overring of R ($R \subset T \subset K$) and $*$ is a semistar operation on R then the map $*_i : \overline{F}(T) \to \overline{F}(T), E \mapsto E^{*_i} := E^*$ is a semistar operation on T.*

Let $*$ be a semistar operation stable and of finite type on R and T be T be an overring of R then $*_1 = *_i$. In fact, let $E \in \overline{F}(T)$ and $x \in E^{*_1}$. There exists $J \in \mathcal{F}^*$ such that $xJ \subset E$, so $xJ^* \subset E^* = E^{*_i}$. As $J^* = R^*$ then $x \in E^{*_i}$. Conversely, let $x \in E^{*_i} = E^*$. There exists $J \in \mathcal{F}^*$ such that $xJ \subset E$, so $xJT \subset E$ and then $x \in E^{*_1}$.

2. *Recall from [8] that if A is an integral domain, $\mathcal{F}$ is a localizing system of A and X is an indeterminate on A then $\mathcal{F}[X] = \{I \text{ ideal of } A[X] \mid JA[X] \subset I, \text{for some } J \in \mathcal{F}\}$ is a localizing system of $A[X]$ and we have $\mathcal{F}[X] = \{I \text{ ideal of } A[X] \mid I \cap A \in \mathcal{F}\}$. Moreover, if $\mathcal{F}$ is a finitely generated localizing system of A then $\mathcal{F}[X]$ is a finitely generated localizing system of $A[X]$.*

   Consider the extension of integral domains $R = A \subset T = A[X]$. Let $K_1 = qf(A[X])$. Let $*$ be a semistar operation on A, $*_f$ the semistar operation of finite type associated to $*$, $\mathcal{F}^{*_f}$ the localizing system associated to $*_f$ and $\mathcal{F}^{*_f}[X]$ the localizing system definied above on $A[X]$. We denote by $\star$ the semsistar operation $*_{\mathcal{F}^{*_f}[X]}$, then $\star = *_1$. In fact, for any $E \in \overline{F}(A[X])$, $E^\star = \cup\{E :_{K_1} J \mid J \in \mathcal{F}^{*_f}[X]\} = \cup\{E :_{K_1} J \mid J \cap A \in \mathcal{F}^{*_f}\}$. We prove that $E^\star = E^{*_1}$. In fact, let $f \in E^\star$ then $f \in K_1$ and there exists $J \in \mathcal{F}^{*_f}[X]$ such that $fJ \subset E$. Thus $f(J \cap A) \subset E$ with $J \cap A \in \mathcal{F}^{*_f}$, which implies that $f \in E^{*_1}$. Conversely, let $f \in E^{*_1}$, then $f \in K_1$ and there exists $I \in \mathcal{F}^{*_f}$ such that $fI \subset E$. As $I \in \mathcal{F}^{*_f}$ then $I[X] \in \mathcal{F}^{*_f}[X]$ and we have $fI[X] \subset E$. In fact, for any $g = \sum_{finite} a_i X^i \in I[X]$, $fg = \sum_{finite} (a_i f) X^i \in E$ (E is an $A[X]$-submodule of K_1). Then $f \in E^\star$.

3. *Recall from [5] that if A is an integral domain, $\mathcal{F}$ is a localizing system of A and X is an indeterminate on A then $\mathcal{F}[[X]] = \{I \text{ ideal of } A[[X]] \mid \text{there exists } J \in \mathcal{F} \text{ such that } JA[[X]] \subset I\}$ is a localizing system of $A[[X]]$ and we have $\mathcal{F}[[X]] = \{I \text{ ideal of } A[[X]] \mid I \cap A \in \mathcal{F}\}$. Moreover, if $\mathcal{F}$ is a finitely generated localizing system of A then $\mathcal{F}[[X]]$ is a finitely generated localizing system of $A[[X]]$.*

   Consider the extension of integral domains $R = A \subset T = A[[X]]$. Let $K_1 = qf(A[[X]])$ and $*$ be a semistar operation on A, $*_f$ the semistar of finite type associated to $*$, $\mathcal{F}^{*_f}$ the localizing system associated to $*_f$ and $\mathcal{F}^{*_f}[[X]]$ the localizing system definied above on $A[[X]]$. Let $\star_1 = *_{\mathcal{F}^{*_f}[[X]]}$, then $\star_1 = *_1$. In fact, for any $E \in \overline{F}(A[[X]])$, $E^{\star_1} = \cup\{E :_{K_1} J \mid J \in \mathcal{F}^{*_f}[[X]]\} = \cup\{E :_{K_1} J \mid J \cap A \in \mathcal{F}^{*_f}\}$. We show that $E^\star = E^{*_1}$. Let $f \in E^\star$ then $f \in K_1$ and there exists $J \in \mathcal{F}^{*_f}[[X]]$ such that $fJ \subset E$. So $f(J \cap A) \subset E$. As $J \cap A \in \mathcal{F}^{*_f}$, then $f \in E^{*_1}$. Conversely, let $f \in E^{*_1}$, then $f \in K_1$ and there exists a finitely generated ideal $I \in \mathcal{F}^{*_f}$ such that $fI \subset E$. As $I \in \mathcal{F}^{*_f}$ then $I[[X]] \in \mathcal{F}^{*_f}[[X]]$ and we have $fI[[X]] \subset E$. In fact, $I[[X]] = I.A[[X]]$ and for any $g = \sum_{finite} a_i g_i \in I[[X]]$, $fg = \sum_{finite} (a_i f) g_i \in E$ (E is an $A[[X]]$-submodule of K_1). So $f \in E^\star$.

4. *Let $R \subset T$ be an extension of integral domains such that R is a Prüfer domain. Recall from [[7], Lemma 4.4], that if $*$ is a semistar operation of finite type on R, then there exists an overring D of R such that for any $E \in \overline{F}(R)$, $E^{\widetilde{*}} = ED$. In particular, $R^{\widetilde{*}} = RD = D$, which implies that for each $J \in \mathcal{F}^*$, $JD = D$.*

   Now let $*$ be a semistar operation stable and of finite type on R and $I \in \mathcal{F}^{*_1}$. We have $I^{*_1} = T^{*_1}$ and thus $1 \in I^{*_1}$. Therefore there exists $J \in \mathcal{F}^*$ such that $J \subset I$ so $JDT \subset ID \subset DT$. As

$JD = D$ then $DT \subset ID \subset DT$ and $ID = DT$. We prove that $E^{*_1} = ED$ for any $E \in \overline{F}(T)$. Let $y \in E^{*_1}$ and $J \in \mathcal{F}^*$ such that $yJ \subset E$. So $yJD \subset ED$. As $JD = D$ then $yD \subset ED$ and so $y \in ED$. Conversely, we have $(ER^*)^{*_1} = E^{*_1}$, for any $E \in \overline{F}(T)$. In fact, $E \subset ER^*$ so $E^{*_1} \subset (ER^*)^{*_1}$. Conversely, let $y \in ER^*$ then there exists $a_1, ..., a_n \in E$ and $r_1, ..., r_n \in \widetilde{R^*}$ such that $y = \sum\limits_{i=1}^{n} a_i r_i$. For any $i \in \{1, ..., n\}$, there exists $J_i \in \mathcal{F}^*$ such that $r_i J_i \subset R$. Let $J = \prod\limits_{i=1}^{n} J_i \in \mathcal{F}^*$ then $r_i J \subset R$ for any $i \in \{1, ..., n\}$. So, $yJ = \sum\limits_{i=1}^{n} a_i r_i J \subset ER \subset E$ and $yJT \subset ET = E$. Hence, $y \in E^{*_1}$. So $ER^* \subset E^{*_1}$ and $(ER^*)^{*_1} \subset E^{*_1}$ which implies the equality $(ER^*)^{*_1} = E^{*_1}$. We get then $ED = ER^* \subset (ER^*)^{*_1} = E^{*_1}$ and so, for any $E \in \overline{F}(T)$, $E^{*_1} = ED$.

Proposition 4. *Let $R \subset T$ be an extension of integral domains which are not fields, $K = qf(R)$ et $K_1 = qf(T)$. Let $*$ be a non trivial semistar operation stable of finite type on R and $*_1$ the semistar operation stable of finite type defined in Proposition 1 on T. The following properties hold:*

1. *Let $\mathcal{F}^{*_1} = \{I$ nonzero ideal of T such that $I^{*_1} = T^{*_1}\}$ then $\mathcal{F}^{*_1} = \{I$ nonzero ideal of T such that $I \cap R \neq (0)$ and $(I \cap R)^* = R^*\}$. Moreover, $\{JT \mid J \in \mathcal{F}^*\} \subset \mathcal{F}^{*_1}$.*
2. *For any $E \in \overline{F}(T)$, $E^{*_1} = \cap\{ER_P \mid P \in M(*)\}$.*
3. *For any $E \in \overline{F}(T)$ and any $P \in M(*)$, $ER_P \cap K = (E \cap K)R_P$.*
4. *Let $I \in \overline{F}(R)$, then $I^* \subset (IT)^{*_1}$.*
5. *If J is a quasi-$*_1$-ideal of T such that $J \cap R \neq (0)$ then $J \cap R$ is a quasi-$*$-ideal of R.*
6. *For any $I \in \overline{F}(R)$, $(IT)^{*_1} = (I^*T)^{*_1}$.*
7. *For any $I \in \overline{F}(T)$, $(IR^*)^{*_1} = I^{*_1}$.*
8. *If $* = d_R$ then $*_1 = d_T$.*
9. *Let I be an ideal of R and L be an ideal of T such that $L \cap R = I$ then $I^* = L^{*_1} \cap R^*$. In particular, if I is a quasi-$*$-ideal of R then $I = L^{*_1} \cap R$.*
10. *If $I \in \overline{F}(R)$ is $*$-invertible then IT is $*_1$-inversible.*

Proof.

1. Let $I \in \mathcal{F}^{*_1}$ then $I^{*_1} = T^{*_1}$. Thus there exists $J \in \mathcal{F}^*$ such that $JT \subset I$, so $J \subset I \cap R$. In particular, we get $I \cap R \neq (0)$. As $J^* = R^*$, then $(I \cap R)^* = R^*$. Conversely, let I be an ideal of T such that $I \cap R \neq (0)$ and $(I \cap R)^* = R^*$ that is $I \cap R \in \mathcal{F}^*$. As $I \cap R \subset I$ then $1 \in I : I \cap R$, so $1 \in I^{*_1}$. This implies that $I^{*_1} = T^{*_1}$.
2. Let $E \in \overline{F}(T)$ and $x \in E^{*_1}$. Hence, there exists $J \in \mathcal{F}^*$ such that $xJ \subset E$. As $J^* = R^*$, then for any $P \in M(*)$, we have $J \nsubseteq P$. So for any $P \in M(*)$, there exists $a_P \in J \backslash P$, so $xa_p \in E$ which implies that $x \in ER_P$. Then, $E^{*_1} \subset \cap\{ER_P \mid E \in M(*)\}$. Conversely, let $x \in \cap\{ER_P \mid E \in M(*)\}$. For any $P \in M(*)$, there exists $s_P \in R \backslash P$ such that $xs \in E$. Set $J = (s_P \mid P \in M(*))_R$ then $J^* = R^*$ and $xJ \subset E$. Therefore $x \in E^{*_1}$.

3. Let $E \in \overline{F}(T)$, $P \in M(*)$ and $x \in ER_P \cap K$. There exists $a \in E$ and $s \in R\backslash P$ such that $x = \frac{a}{s}$. As $x \in K$ then $xs \in K$ so $a \in K \cap E$. Therefore, $x \in (E \cap K)R_P$. Conversely, $(E \cap K) \subset E$, so $(E \cap K)R_P \subset ER_P$. Hence, $E \cap K \subset K$ and $(E \cap K)R_P \subset KR_P = K$. Consequently, $(E \cap K)R_P \subset ER_P \cap K$. And thus we get the equality.

4. Let $I \in \overline{F}(R)$ and $x \in I^*$. There exists $J \in \mathcal{F}^*$ such that $xJ \subset I$, so $xJT \subset IT$. Consequently $x \in (IT)^{*_1}$.

5. Let J be a quasi-$*_1$-ideal of T such that $J \cap R \neq (0)$ and $x \in (J \cap R)^* \cap R$. Then there exists $F \in \mathcal{F}^*$ such that $xF \subset J \cap R$. So $xFT \subset JT = J$ and thus $x \in J^{*_1} \cap T = J$. Consequently, $x \in J \cap R$.

6. Let $I \in \overline{F}(R)$. As $I \subset I^*$ then $IT \subset I^*T$, so $(IT)^{*_1} \subset (I^*T)^{*_1}$. Moreover by $4/$ $I^* \subset (IT)^{*_1}$, thus $I^*T \subset (IT)^{*_1}$ and $(I^*T)^{*_1} \subset (IT)^{*_1}$. So we obtain the equality.

7. Let $I \in \overline{F}(T)$. As $I \subset IR^*$ then $I^{*_1} \subset (IR^*)^{*_1}$. Conversely, let $y \in IR^*$ then there exist $a_1, ..., a_n \in I$ and $r_1, ..., r_n \in R^*$ such that $y = \sum_{i=1}^{n} a_i r_i$. For any $i \in \{1, ..., n\}$, there exists $J_i \in \mathcal{F}^*$ such that $r_i J_i \subset R$. Set $J = \prod_{i=1}^{n} J_i \in \mathcal{F}^*$ and $r_i J \subset R$ for any $i \in \{1, ..., n\}$. Consequently, $yJ = \sum_{i=1}^{n} a_i r_i J \subset IR \subset I$ and $yJT \subset IT = I$. Thus $y \in I^{*_1}$.

8. If $* = d_R$ then $\mathcal{F}^* = \{R\}$. Let $E \in \overline{F}(T)$ and $x \in E^{*_1}$. There exists $J \in \mathcal{F}^{*_f}$ such that $xJ \subset E$. But $J = R$ so $xR \subset E$ and then $x \in E$. Thus $*_1 = d_T$.

9. Let I be an ideal of R and L be an ideal of T such that $L \cap R = I$ then $I^* \subset L^{*_1} \cap R^*$. Conversely, let $x \in L^{*_1} \cap R^{*_1}$ so there exist $J_1, J_2 \in \mathcal{F}^*$ such that $xJ_1 \subset L$ and $xJ_2 \subset R$. As $J_1 J_2 \in \mathcal{F}^*$ and $xJ_1 J_2 \subset L \cap R = I$ then $x \in I^*$.

   In particular, if I is a quasi-$*$-ideal of R then $I = I^* \cap R = L^{*_1} \cap R^* \cap R = L^{*_1} \cap R$.

10. Let $I \in \overline{F}(R)$ be an ideal $*$-invertible so $(II^{-1})^* = R^*$. We prove that IT is $*_1$-invertible.

    Note that $((IT)^{-1})^{*_1} = (I^{-1}T)^{*_1}$. In fact, $I^{-1}T \subset (IT)^{-1}$: let $f \in I^{-1}$ then $fI \subset R$. Thus $fIT \subset T$ and $f \in (IT)^{-1}$. Therefore $I^{-1} \subset (IT)^{-1}$, which implies tha $I^{-1}T \subset (IT)^{-1}$. Consequently, $(I^{-1}T)^{*_1} \subset ((IT)^{-1})^{*_1}$. Conversely, let $f \in (IT)^{-1}$ then $fIT \subset T$. Thus $fII^{-1} \subset I^{-1}T$ which implies that $f(II^{-1}T)^{*_1} \subset (I^{-1}T)^{*_1}$. But if $J \in \overline{F}(R)$, then $(JT)^{*_1} = (J^*T)^{*_1}$, by 6 of Proposition 4. So $f((II^{-1})^*T)^{*_1} \subset (I^{-1}T)^{*_1}$ or $f(R^*T)^{*_1} \subset (I^{-1}T)^{*_1}$ as I is $*$-invertible. As a consequence $f \in (I^{-1}T)^{*_1}$. Thus $((IT)^{-1})^{*_1} \subset (I^{-1}T)^{*_1}$ and then we get the equality $((IT)^{-1})^{*_1} = (I^{-1}T)^{*_1}$.

    We show that IT is $*_1$-invertible. First, we have $(IT(IT)^{-1})^{*_1} = (IT((IT)^{-1})^{*_1})^{*_1}$. By the first part of this proof, $((IT)^{-1})^{*_1} = (I^{-1}T)^{*_1}$ thus $(IT(IT)^{-1})^{*_1} = (ITI^{-1}T)^{*_1} = (II^{-1}T)^{*_1}$. But $(II^{-1}T)^{*_1} = ((II^{-1})^*T)^{*_1}$. Therefore $(IT(IT)^{-1})^{*_1} = (R^*T)^{*_1} = (RT)^{*_1} = T^{*_1}$. Consequently IT is $*_1$-invertible.

■

Remark 5. *One may ask the following question: if $*$ is a non trivial semistar operation stable and of finite type, is the semistar operation $*_1$ necessary non trivial?*

If we suppose that $T \cap K = R$ and $*$ is non trivial then $*_1$ is also non trivial. In fact, if $T^{*_1} = K_1$ then $R^* = K$: let $x \in K \subset K_1$, then there exists $J \in F^*$ such that $xJ \subset T$, thus $xJ \subset T \cap K = R$, so $x \in R^*$.

Remark that if for any $b \in R$, $bT \cap R = bR$ then $T \cap K = R$.

This is, in particular, the case of the examples of the polynomial ring and the formal power series ring given in Example 3 (2 and 3).

The same proof is also valuable if we suppose that $T \cap K \subset R^*$.

Generally, the result is false as shown in the following example: let A be an integral domain with $\dim A \geq 1$ and $R = A[X]$ so $\dim R[X] \geq 2$. Let $p \in spec(A)\backslash\{0\}$ and $P = p[X] \in spec(R)$. Consider the semistar operation stable and of finite type $*$ defined on R by, $E^* = ER_P$, for any $E \in \overline{F}(R)$. As $R^* = R_P \notin \{R, qf(R)\}$ then $* \notin \{d_R, e_R\}$. Let $K = qf(A)$ and $T = K[[X]]$ then $R \subset T$ and T is a DVR. So by [[6], Theorem 48], for any non trivial semistar operation $\star$ on T, we have $\star_f = d_T$. In particular $*_1 \in \{e_T, d_T\}$. We prove that $*_1 = e_T$. Suppose on the contrary, that $*_1 = d_T$ then for any $J \in \mathcal{F}^*$, $JT = T$. As $XR \in \mathcal{F}^*$ then $XK[[X]] = K[[X]]$ which is impossible. Consequently, $*_1 = e_T$.

Proposition 6. *Let $*$ be a semistar operation stable and of finite type on R. Suppose that $R \subset T$ is LO and for any ideal I of T, $I = (I \cap R)T$. The following properties hold:*

1. *$* = e_K$ if and only if $*_1 = e_{K_1}$.*
2. *If $*$ is non trivial, then $M(*) = \{M \cap R \mid M \in M(*_1)\}$.*

Proof.

1. If $*_1 \neq e_{K_1}$ then $M(*_1) \neq \emptyset$. Let $M \in M(*_1)$ and $P = M \cap R$ then P is a quasi-$*_f$-prime ideal (P is nonzero as $M = (M \cap R)T = PT \neq (0)$). thus, $* \neq e_K$.

   Conversely, suppose that $* \neq e_K$ then $M(*) \neq \emptyset$. Let $P \in M(*)$ then $P = PT \cap R$. In fact, it is clear that $P \subset PT \cap R$. We show that $PT \cap R \subset P$. As $R \subset T$ is LO, then there exists $Q \in spec(T)$ such that $P = Q \cap R$. Thus $PT \subset Q$ and $PT \cap R \subset Q \cap R = P$.

   If $(PT)^{*_1} = T^{*_1}$ then $1 \in (PT)^{*_1}$ so there exists $J \in \mathcal{F}^*$ such that $J \subset PT$. Consequently, $J \subset PT \cap R = P$ which is impossible, so $(PT)^{*_1} \subsetneq T^{*_1}$ and $*_1$ is non trivial.

2. Let $M \in M(*_1)$ and $P = M \cap R$, then P is a quasi-$*$-prime ideal of R. As $R \subset T$ is LO then $PT \cap R = P$.

   If $P \notin M(*)$ then there exists $Q \in M(*)$ such that $P \subsetneq Q$. Thus $PT = M \subsetneq QT$. If $(QT)^{*_1} = T^{*_1}$ then there exists $J \in F^*$ such that $J \subset QT$ which implies that $J \subset QT \cap R = Q$ and this is impossible. So $(QT)^{*_1} \cap T$ is a proper quasi-$*_1$-ideal of T and $M \subsetneq QT$ which is absurd as $M \in M(*_1)$. Thus $\{M \cap R \mid M \in M(*_1)\} \subset M(*)$.

   Conversely, let $P \in M(*)$, then $P = PT \cap R$. If $(PT)^{*_1} = T^{*_1}$ then there exists $J \in \mathcal{F}^*$ such that $J \subset PT$ which implies that $J \subset PT \cap R = P$. This is impossible as $P \in M(*)$. So $(PT)^{*_1} \neq T^{*_1}$.

Hence, there exists $M \in M(*_1)$ such that $PT \subset M$. Consequently, $PT \cap R = P \subset M \cap R$ so $P = M \cap R$. Therefore $M(*) = \{M \cap R \mid M \in M(*_1)\}$.

■

Recall from [1] that if R is an integral domain and $*$ is a semistar operation on R then we say that R is a $*$-Noetherian domain if R satisfies the ascending chain condition on quasi-$*$-ideals.

Proposition 7. *Let $*$ be a semistar operation stable of finite type on R. The following properties hold:*

1. *Suppose that for any ideal I of T, $I = (I \cap R)T$. If R is $*$-Noetherian then T is $*_1$-Noetherian.*
2. *Suppose that for any ideal I of R, $I = IT \cap R$. If T is $*_1$-Noetherian then R is $*$-Noetherian.*

Proof.

1. Suppose that R is $*$-Noetherian. Let $(H_n)_{n\in\mathbb{N}}$ be a chain of quasi-$*_1$-ideals of T then $(H_n \cap R)_{n\in\mathbb{N}}$ is a chain of nonzero quasi-$*$-ideals of R. So there exists $k \in \mathbb{N}$ such that for any $n \geq k$, $H_n \cap R = H_k \cap R$. Consequently $H_n = (H_n \cap R)T = (H_k \cap R)T = H_k$.
2. Suppose that T is $*_1$-Noetherian. Let $(I_n)_{n\in\mathbb{N}}$ be a chain of quasi-$*$-ideals of R then for any $n \in \mathbb{N}$, $I_n = I_nT \cap R$. Thus $I_n = (I_nT)^{*_1} \cap R$ by 9 of Proposition 4. For $n \in \mathbb{N}$, let $H_n = (I_nT)^{*_1} \cap T$, then $(H_n)_{n\in\mathbb{N}}$ is a chain of quasi-$*_1$-ideals of T. So there exists $k \in \mathbb{N}$ such that for any $n \geq k$, $H_n = H_k$, which implies that $H_n \cap R = H_k \cap R$. Therefore for any $n \geq k$, $I_n = I_k$.

■

Corollary 8. *Let R be an integral domain and $*$ be a semistar stable and of finite type on R. Let $T = R[X]$. If T is $*_1$-Noetherian then R is $*$-Noetherian.*

Corollary 9. *Let R be an integral domain and $*$ be a semistar stable and of finite type on R. Let $T = R[[X]]$. If T is $*_1$-Noetherian then R is $*$-Noetherian.*

3. Basic properties of the map φ

Recall that the Zariski topology on $SStar(R)$ is the topology which has as a subbasis of open sets the collection of all sets of the form $V_F^R = \{* \in SStar(R) \mid 1 \in F^*\}$, as F ranges among the nonzero R-submodules of K. The Zariski topology on $\widetilde{SStar}(R)$ is just the subspace topolgy induced by the Zariski topology on $SStar(R)$. Therefore $\{V_F^R \cap \widetilde{SStar}(R) \mid F \in \overline{F}(R)\}$ is a subbasis of the Zariski topology on $\widetilde{SStar}(R)$. Endowed with this topology, the space $SStar(R)$ is compact. Note that the Zariski topology of $\widetilde{SStar}(R)$ is determined by the finitely generated integral ideals of R that is the collection of sets of the form $V_F^R \cap \widetilde{SStar}(R) = \{* \in \widetilde{SStar}(R) \mid 1 \in F^*\}$, where F ranges among the finitely generated integral ideals of R is a subbasis. This follows from the definition of semistar operation stable and of finite type. Moreover, if F is an integral ideal of R then $\{* \in \widetilde{SStar}(R) \mid 1 \in F^*\} = \{* \in \widetilde{SStar}(R) \mid F^* = R^*\}$.

Since $1 \in K$, it follows that $e \in V_F^R \cap \widetilde{SStar(R)}$ for any $F \in \overline{F}(R)$. We deduce that $\widetilde{SStar(R)}$ is a dense subspace of $SStar(R)$.

Moreover as $SStar(R)$ is a T_0 space then $\widetilde{SStar(R)}$ is also a T_0 space.

In this section, we study the behaviour of the map φ.

Proposition 10. *The map φ is continuous from $\widetilde{SStar(R)}$ in $\widetilde{SStar(T)}$.*

Proof. Let $L \in \overline{F}(T)$, then $\varphi^{-1}(V_L^T \cap \widetilde{SStar(R)}) = \{* \in \widetilde{SStar(R)} \mid \varphi(*) \in V_L^T \cap \widetilde{SStar(T)})\} = \{* \in \widetilde{SStar(R)} \mid 1 \in L^{*_1}\} = \{* \in \widetilde{SStar(R)} \mid \exists\ J \in \mathcal{F}^* \text{ such that } J \subset L\}$. Note that if $\varphi^{-1}(V_L^T \cap \widetilde{SStar(T)})$ is non empty, $* \in \varphi^{-1}(V_L^T \cap \widetilde{SStar(T)})$ and $J \in \mathcal{F}^*$ are such that $J \subset L$ then $J \subset L \cap R$. Consequently, as $J \neq (0)$ then $L \cap R \in \overline{F}(R)$, so $1 \in V_{L\cap R}^R \cap \widetilde{SStar(R)}$. Therefore $\varphi^{-1}(V_L^T \cap \widetilde{SStar(T)}) \subset V_{L\cap R}^R \cap \widetilde{SStar(R)}$. Conversely, if $* \in V_{L\cap R}^R \cap \widetilde{SStar(R)}$ then $1 \in (L \cap R)^*$. So there exists $J \in \mathcal{F}^*$ such that $J \subset L \cap R \subset L$. Hence, $* \in \varphi^{-1}(V_L^T \cap \widetilde{SStar(T)})$. Consequently $\varphi^{-1}(V_L^T \cap \widetilde{SStar(T)}) = V_{L\cap R}^R \cap \widetilde{SStar(R)}$. So the map φ is continuous. ■

Proposition 11. *Suppose that $R \subset T$ is LO. The following properties hold:*

1. *Let $*$ be a semistar stable of finite type on R and J be an ideal of R such that $(JT)^{*_1} = T^{*_1}$, then $J^* = R^*$.*
2. *The map φ is injective.*

Proof.

1. Note first, that if $*$ is the trivial semistar operation on R then the result is clear. Assume that $*$ is non trivial. Let J be an ideal of R such that $(JT)^{*_1} = T^{*_1}$, we show that $J^* = R^*$. If $J^* \subsetneq R^*$ then there exists a quasi $*$-prime ideal P of R such that $J \subset P$. As $R \subset T$ is LO, there exists $Q \in spec(T)$ such that $Q \cap R = P$. Therefore, $J \subset P \subset Q$, so $JT \subset Q \subset T$, which implies that $Q^{*_1} = T^{*_1}$. Thus, $Q \cap R = P \in \mathcal{F}^*$ that is $P^* = R^*$, which is impossible.
2. Let $*$ and $*'$ be two semistar operations stable and of finite type on R such that $*_1 = *'_1$. To show that $* = *'$, we prove that $\mathcal{F}^* = \mathcal{F}^{*'}$. Let $J \in F^*$ then $JT \in F^{*_1}$. So $(JT)^{*_1} = T^{*_1}$ but $(JT)^{*_1} = (JT)^{*'_1}$. By 1, $J^{*'} = R^{*'}$ that is $J \in F^{*'}$. In a symmetric way we show that $F^{*'} \subset F^*$.

■

Remark 12. *Let $R \subset T$ be an extension of integral domains which are not fields and assume that $R \subset T$ is LO. Let $P \in spec(R)$ and $* = *_{\{R_P\}}$ then $M(*) = \{P\}$. Note that by 2 of Proposition 4 for each $E \in \overline{F}(T)$, $E^{*_1} = ER_P$. Let $Q \in spec(T)$ such that $Q \cap R = P$ then $*_1 \leq *_{\{T_Q\}}$. In fact, let $x \in E^{*_1}$ then there exists $s \in R \backslash P$ such that $xs \in E$. It is clear that $s \notin Q$ so $x \in ET_Q$ and then $E^{*_1} \subset ET_Q$. For example if $T = R[X]$ and $Q = P[X]$ then $*_1 \neq *_{\{T_Q\}}$. In fact $\mathcal{F}^{*_1} = \{I$:ideal of $R[X]$ such that $I \cap R \neq (0)$ and $I \cap R \nsubseteq P\}$ and $\mathcal{F}^{*\{R[X]_Q\}} = \{I$:ideal of $R[X]$ such that $I \nsubseteq P[X]\}$. So $XR[X] \in \mathcal{F}^{*\{R[X]_Q\}}$ but $XR[X] \notin \mathcal{F}^{*_1}$ which implies that $*_1 \neq *_{\{T_Q\}}$.*

Proposition 13. *Let $R \subset T$ be an extension of integral domains which are not fields and $U = \{\star$ semistar operation stable and of finite type on T such that for any $I \in \mathcal{F}^\star$, $(I \cap R)T \in \mathcal{F}^\star\}$. Then $\varphi(\widetilde{SStar}(R)) = U$.*

Proof. Let $\star \in \varphi(\widetilde{SStar}(R))$ then there exists $* \in \widetilde{SStar}(R)$ such that $\star = \varphi(*)$. Therefore $\star$ is stable and of finite type and by 1 of Proposition 4, if $I \in \mathcal{F}^\star$, $I \cap R \neq (0)$ and $I \cap R \in \mathcal{F}^*$. Consequently, $((I \cap R)T)^\star = ((I \cap R)^*T)^\star = (R^*T)^\star = (RT)^\star = T^\star$. So $\varphi(\widetilde{SStar}(R)) \subset U$.

Conversely, let $\star \in U$. We have to show that there exists a semistar operation $*$ stable and of finite type on R such that $*_1 = \star$. For this purpose we have to find $\mathcal{F}^*$.

let $\mathcal{F} = \{I$ nonzero ideal of R such that $(IT)^\star = T^\star\}$, then $\mathcal{F}$ is a localizing system. In fact:

•) Let $I \in \mathcal{F}$ and J be an ideal of R such that $I \subset J$ then $IT \subset JT \subset T$, so $(IT)^\star = T^\star \subset (JT)^\star \subset T^\star$. Therefore $(JT)^\star = T^\star$ and $J \in \mathcal{F}$.

•) Let $I \in \mathcal{F}$ and J be an ideal of R such that $(J :_R iR) \in \mathcal{F}, \forall i \in I$.

We show that $J \in \mathcal{F}$ that is $(JT)^\star = T^\star$.

For $i \in I$, $(J :_R iR)T \subset JT :_T iT$. So $T^\star = ((J :_R iR)T)^\star \subset (JT :_T iT)^\star \subset T^\star$, then $JT :_T iT \in \mathcal{F}^\star$, for any $i \in I$. Hence, $JT :_T xT \in \mathcal{F}^\star$, for any $x \in IT$ (let $x = \sum\limits_{finite} i_k t_k \in IT$, then $JT :_T xT = \{y \in T \mid y \sum\limits_{finite} i_k t_k T \subset JT\} = \{y \in T \mid y i_k T \subset JT, \text{for any } k\} = \bigcap\limits_{finite} JT :_T i_k T \in F^\star$ as $\mathcal{F}^\star$ is a localizing system. But $IT \in \mathcal{F}^\star$, then $JT \in \mathcal{F}^\star$ that is $J \in \mathcal{F}$.

Moreover, $\mathcal{F}$ is of finite type. In fact, let $I \in \mathcal{F}$, then $IT \in \mathcal{F}^\star$., As $\star$ is of finite type then $\mathcal{F}^\star$ is of finite type. Consequently, there exists a finitely generated ideal J of T such that $J \subset IT$ and $J \in \mathcal{F}^\star$. Set $J = (t_1, ..., t_n)$ where $t_i \in IT$, thus $t_i = \sum\limits_{finite} a_{ij} t_{ij}$, with $a_{ij} \in I$ and $t_{ij} \in T$. Let $I_1 =< a_{ij} >_R$ then $I_1 \subset I$ and $(I_1T)^\star = T^\star$.

Consider $* = *_\mathcal{F}$. Note, by [[4], Lemma $1.1(a)$] that $\mathcal{F}^* = \mathcal{F}$. We prove that $*_1 = \star$. For this purpose we show that $\mathcal{F}^{*_1} = \mathcal{F}^\star$. We have $\mathcal{F}^{*_1} = \{I$ nonzero ideal of T such that $I^{*_1} = T^{*_1}\} = \{I$ nonzero ideal of T such that $I \cap R \neq (0)$ and $(I \cap R)^* = R^*\} = \{I$ nonzero ideal of T such that $((I \cap R)T)^\star = T^\star\} = \{I$ ideal of T such that $I^\star = T^\star\}$, the last equality follows from the fact that $\star \in U$, then $\mathcal{F}^{*_1} = \mathcal{F}^\star$. The proof is now complete. ■

In the sequel, we are going to investigate the relation between the two maps φ and σ. Note that if T is an overring of R then for every $* \in \widetilde{SStar}(R)$ and every $E \in \overline{F}(R)$, $E^{\sigma \circ \varphi(*)} = (ET)^*$. This is a consequence of Proposition 3.2 of [2] and Example 3. Hence, $E^* \subset E^{\sigma \circ \varphi(*)}$. Moreover for every $E \in \overline{F}(T)$, $E^{\sigma \circ \varphi(*)} = E^*$.

Proposition 14. *Let $R \subset T$ be an extension of integral domains which are not fields, $K = qf(R)$ and $K_1 = qf(T)$.*

1. *Let $\star \in SStar(T)$ then $\mathcal{F}^{\sigma(\star)} = \{I$ ideal of R such that $(IT)^\star = T^\star\}$.*
2. *For any semistar operation $\star$ of finite type on T, $\varphi \circ \sigma(\star) \leq \star$.*

3. *Let $U = \{\star$ semistar operation stable and of finite type on T such that for any $I \in \mathcal{F}^\star$, $(I \cap R)T \in \mathcal{F}^\star\}$. Then for any semistar operation $\star \in U$, $\star = \varphi \circ \sigma(\star)$.*

4. *Assume that $R \subset T$ is LO. For any $\star \in \widetilde{SStar}(T)$, let $\overline{\sigma}(\star) = *_{\mathcal{F}^{\sigma(\star)}}$. Then for any $* \in \widetilde{SStar}(R)$, $\overline{\sigma}(\varphi(*)) = *$.*

5. *For any semistar operation $\star$ of U, $\varphi(\overline{\sigma}(\star)) = \star$.*

6. *The map $\overline{\sigma} : U \to SStar(R)$ is continuous.*

Proof.

1. Let $\star \in SStar(T)$ and $I \in \mathcal{F}^{\sigma(\star)}$ then $I^{\sigma(\star)} = R^{\sigma(\star)}$ that is $(IT)^\star \cap K = (RT)^\star \cap K = T^\star \cap K$ which implies that $(IT)^\star = T^\star$.

   So $\mathcal{F}^{\sigma(\star)} = \{I$ ideal of R such that $(IT)^\star = T^\star\}$.

2. Let $\star$ be a semistar operation of finite type on T and $E \in \overline{F}(T)$. Let $f \in E^{\varphi(\sigma(\star))}$, then there exists $J \in \mathcal{F}^{\sigma(\star)}$ such that $fJ \subset E$ so $fJT \subset ET = E$. As $J \in \mathcal{F}^{\sigma(\star)}$ then $(JT)^\star = T^\star$, so $JT \in \mathcal{F}^\star$. Thus $f(JT)^\star = fT^\star \subset E^\star$ therefore $f \in E^\star$.

3. Let $\star \in U$. We prove that $\star \leq \varphi \circ \sigma(\star)$. Let $E \in \overline{F}(T)$ and $f \in E^\star$ then there exists $I \in \mathcal{F}^\star$ such that $fI \subset E$, so $f(I\cap R) \subset E$. As $(I\cap R)T \in \mathcal{F}^\star$ then by 1, $I\cap R \in \mathcal{F}^{\sigma(\star)}$ and $(I\cap R)T \in \mathcal{F}^{\varphi(\sigma(\star))}$. Consequently, $f \in E^{\varphi(\sigma(\star))}$. By 2, we get $\star = \varphi \circ \sigma(\star)$.

4. Let $* \in \widetilde{SStar}(R)$, $E \in \overline{F}(R)$ and $x \in E^{\overline{\sigma}(\varphi(*))}$. Then there exists $I \in \mathcal{F}^{\sigma(\varphi(*))}$ such that $xI \subset E$. As $I \in \mathcal{F}^{\sigma(\varphi(*))}$ then by 1, $(IT)^{\varphi(*)} = T^{\varphi(*)}$ that is $(IT)^{*_1} = T^{*_1}$. By hypothesis, the extension $R \subset T$ is LO so by 1 of Proposition 9, $I^* = R^*$. Thus $x \in E^*$, so $\overline{\sigma}(\varphi(*)) \leq *$.

   Conversely, let $x \in E^*$ then there exists $J \in \mathcal{F}^*$ such that $xJ \subset E$. Moreover $(JT)^{\varphi(*)} = (JT)^{*_1} = T^{*_1}$ so $J \in \mathcal{F}^{\sigma(\varphi(*))}$ and $x \in E^{\overline{\sigma}(\varphi(*))}$. Therefore $* \leq \overline{\sigma}(\varphi(*))$ and then $* = \overline{\sigma}(\varphi(*))$.

5. Let $\star \in U$, $E \in \overline{F}(T)$ and $f \in E^\star$ then there exists $J \in \mathcal{F}^\star$ such that $fJ \subset E$. As $J \in \mathcal{F}^\star$ and $\star \in U$ then $(J \cap R)T \in \mathcal{F}^\star$ so $((J \cap R)T)^\star = T^\star$. Thus $J \cap R \in \mathcal{F}^{\sigma(\star)} = \mathcal{F}^{\overline{\sigma}(\star)}$ and $f(J \cap R)T \subset E$ which implies that $f \in E^{\varphi(\overline{\sigma}(\star)}$. Consequently, $\star \leq \varphi(\overline{\sigma}(\star))$. Conversely, let $f \in E^{\varphi(\overline{\sigma}(\star))}$ then there exists $J \in \mathcal{F}^{\overline{\sigma}(\star)}$ such that $fJ \subset E$. As $J \in \mathcal{F}^{\overline{\sigma}(\star)}$ then $(JT)^\star = T^\star$ so $f \in E^\star$ and thus $E^{\varphi(\overline{\sigma}(\star)} \subset E^\star$ and $\varphi(\overline{\sigma}(\star)) \leq \star$.

6. Let $F \in \overline{F}(R)$, then $\overline{\sigma}^{-1}(V_F^R \cap \widetilde{SStar}(R)) = \{\star \in U \mid 1 \in \mathcal{F}^{\overline{\sigma}(\star)}\}$. If $\star \in U$ is such that $1 \in \mathcal{F}^{\overline{\sigma}(\star)}$ then there exists an ideal I of R such that $(IT)^\star = T^\star$ and $I \subset F$ so $I \subset F \cap R$. Thus $IT \subset (F \cap R)T \subset T$ so $1 \in (IT)^\star = T^\star \subset ((F \cap R)T)^\star$ and $\star \in V^T_{(F\cap R)T} \cap U$. Conversely, let $\star \in V^T_{(F\cap R)T} \cap U$ then $1 \in ((F \cap R)T)^\star$. Therefore, $F \cap R \in \mathcal{F}^{\sigma(\star)}$ so $1 \in (F \cap R)^{\overline{\sigma}(\star)}$ and $1 \in \mathcal{F}^{\overline{\sigma}(\star)}$. Consequently, $\overline{\sigma}^{-1}(V_F^R \cap \widetilde{SStar}(R)) = V^T_{(F\cap R)T} \cap U$ which implies that the map $\overline{\sigma} : U \to SStar(R)$ is continuous.

■

Corollary 15. *Let $R \subset T$ be an extension of integral domains which are not fields and assume that $R \subset T$ is LO. Let $U = \{\star$ semistar operation stable and of finite type on T such that for any $I \in \mathcal{F}^\star$, $(I \cap R)T \in \mathcal{F}^\star\}$. Then the map φ is an homeomorphism from $\widetilde{SStar}(R)$ into U.*

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