Some remarks on surjections of unit groups
Remarques sur les surjections des groupes unitaires

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ABSTRACT. The main purpose of this paper is to study unital ring homomorphisms of associative rings \( \varphi : R \rightarrow S \) satisfying one of the following conditions: (a) the unit-preserving property: \( \varphi(R^\times) = S^\times \) and (b) the inverse unit-preserving property: \( \varphi^{-1}(S^\times) = R^\times \). We establish the relationship between these two conditions. Several characterizations of such conditions are settled. An application to the index of unit groups of rings \( R \subset S \) having a nonzero common ideal is given.

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1. introduction

Throughout this paper we assume that all rings are associative with identity and (as usual) that all ring homomorphisms are unital. By an ideal of a ring \( R \), we mean a two-sided ideal. If \( R \) is a ring, we let \( R^\times \) denote its group of units, \( \text{Spec}(R) \) (resp., \( \text{Max}(R) \)) its set of prime (resp., maximal) ideals. In [2], Chen has studied surjective ring homomorphisms \( \varphi : R \rightarrow S \) of commutative rings such that \( \varphi(R^\times) = S^\times \). So several important results are obtained. We will call a ring homomorphism \( \varphi \), satisfying the above condition, a homomorphism with the unit-preserving property. Motivated by the work of Chen, we will say that a ring homomorphism \( \varphi : R \rightarrow S \) of arbitrary rings satisfies the inverse unit-preserving property if \( \varphi^{-1}(S^\times) = R^\times \). In Section 2, we characterize surjective ring homomorphisms satisfying the inverse-unit preserving property (see Corollary 2.4). We show in Proposition 2.7 that any lying over surjective ring homomorphism of commutative rings satisfies the inverse unit-preserving property. Theorem 2.9 states that if \( R \) is a von Neumann regular ring, then a ring homomorphism \( \varphi : R \rightarrow S \) of commutative rings satisfies the inverse unit-preserving property if and only if \( \varphi \) is injective. In Section 3, we extend Chen’s definition of ring homomorphisms satisfying the unit-preserving property for arbitrary rings. We establish in Proposition 3.4 a relationship between the unit-preserving and the inverse unit-preserving properties. As a consequence, we recover [2, Corollary 2.1] for arbitrary rings (see Corollary 3.5). We demonstrate in Theorem 3.7 that if \( \varphi : R \rightarrow S \) is a surjective ring homomorphism with kernel \( I \) such that \( R/\text{Ann}_R(I) \) is left Artinian, then \( \varphi \) satisfies the unit-preserving property. In Theorem 4.3, we establish that for rings \( R \subset S \) sharing an ideal \( I \), if the canonical surjection \( \pi : S \rightarrow S/I \) satisfies the unit-preserving property, then \([S^\times : R^\times] = [(S/I)^\times : (R/I)^\times]\).

Any unexplained terminology is standard as in [3], [4] and [5].

2. Inverse unit-preserving property

For any ring homomorphism \( \varphi : R \rightarrow S \), we always have \( R^\times \subseteq \varphi^{-1}(S^\times) \). This inclusion relation may be strict, in general, even if \( \varphi \) is surjective or injective. To see this, let us consider, for instance, the
canonical surjection \( \pi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \), where \( p \) is a prime number. Clearly, \( \pi^{-1}((\mathbb{Z}/p\mathbb{Z})^\times) = \mathbb{Z} \setminus p\mathbb{Z} \supseteq \mathbb{Z}^\times = \{ \pm 1 \} \). We provide another example of an injective ring homomorphism with the above strict inclusion relation. Take, for instance, the canonical injection \( \varphi : \mathbb{Z} \to \mathbb{Q} \). Clearly, \( \varphi^{-1}(\mathbb{Q}^\times) = \mathbb{Z} \setminus \{0\} \supseteq \mathbb{Z}^\times \). These facts encourage us to introduce the following definition.

**Definition 2.1.** Let \( R \) and \( S \) be arbitrary rings. We say that a ring homomorphism \( \varphi : R \to S \) satisfies the inverse unit-preserving property if \( \varphi^{-1}(S^\times) = R^\times \); or equivalently, if \( \varphi^{-1}(S^\times) \subseteq R^\times \).

**Remark 2.2.** (1) Of course, if \( \varphi : R \to S \) is a ring isomorphism, then \( \varphi \) satisfies the inverse unit-preserving property.

(2) Let \( R \) and \( S \) be rings and let \( \varphi : R \to S \) be a ring homomorphism satisfying the inverse unit-preserving property. Then \( S = \{0\} \) if and only if \( R = \{0\} \). Indeed, assume first that \( S = \{0\} \). As \( \varphi \) satisfies the inverse unit-preserving property, then \( R = R^\times \). Thus, \( 0 \in R^\times \) and so \( R = \{0\} \). Conversely, suppose \( R = \{0\} \). Since \( \varphi^{-1}(S^\times) = R^\times = \{0\} \), then \( \varphi(0) = 0 \in S^\times \). Hence, \( S = \{0\} \). This is why we will assume in the sequel that if \( \varphi : R \to S \) is a ring homomorphism satisfying the inverse unit-preserving property, then \( R \) and \( S \) can be chosen to be nonzero.

In what follows, we will investigate some properties of ring homomorphisms satisfying the inverse unit-preserving property. But, first recall from [5] that the Jacobson radical of a ring \( R \), denoted by \( J(R) \), is the intersection of maximal left ideals of \( R \). It follows from [5, Section 4] that \( x \in J(R) \) if and only if \( 1 - yxz \in R^\times \) for all \( y, z \in R \).

**Proposition 2.3.** Let \( \varphi : R \to S \) be a ring homomorphism satisfying the inverse unit-preserving property. Then \( \text{Ker}(\varphi) \subseteq J(R) \).

**Proof.** Let \( x \in \text{Ker}(\varphi) \) and \( y, z \in R \). According to the above comments, we need to show that \( 1 - yxz \in R^\times \). As \( \varphi \) is a ring homomorphism, we have \( \varphi(1 - yxz) = \varphi(1) - \varphi(y)\varphi(x)\varphi(z) = \varphi(1) = 1 \). Thus, \( 1 - yxz \in \varphi^{-1}(S^\times) \). But, as \( \varphi \) satisfies the inverse unit-preserving property, then \( \varphi^{-1}(S^\times) = R^\times \). Hence, we infer that \( 1 - yxz \in R^\times \), as desired. This completes the proof. \( \Box \)

As a consequence, we get the following characterization.

**Corollary 2.4.** Let \( \varphi : R \to S \) be a surjective ring homomorphism. Then the following statements are equivalent:

1. \( \varphi \) satisfies the inverse unit-preserving property.

2. \( \text{Ker}(\varphi) \subseteq J(R) \).

**Proof.** The implication (1)\( \Rightarrow \)\( (2) \) follows from Proposition 2.3 without the “surjective” hypothesis. Let us prove that \( (2) \) implies \( (1) \). Assume \( (2) \). Let \( x \in \varphi^{-1}(S^\times) \). Then \( \varphi(x) \in S^\times \). So there exists \( y \in S \) such that \( y\varphi(x) = \varphi(x)y = 1 \). But, as \( \varphi \) is surjective, then \( y = \varphi(r) \) for some \( r \in R \). It follows that \( \varphi(r)\varphi(x) = \varphi(x)\varphi(r) = 1 \); or equivalently, \( \varphi(rx) = \varphi(xr) = 1 \). Therefore, \( rx - 1, xr - 1 \in \text{Ker}(\varphi) \). Hence \( rx, xr \in 1 + \text{Ker}(\varphi) \subseteq 1 + J(R) \subseteq R^\times \). So \( x \) has both a right inverse, namely \( r(xr)^{-1} \), and a left inverse, namely \( (rx)^{-1}r \). It follows that \( x \in R^\times \). The proof is complete. \( \Box \)
Recall from [4] that a ring homomorphism $\varphi : R \to S$ satisfies the lying-over property (briefly, LO), if for each $P \in \text{Spec}(R)$, there exists $Q \in \text{Spec}(S)$ such that $\varphi^{-1}(Q) = P$. As an example of such ring homomorphism, one can take for instance, $\varphi$ to be the inclusion map of an integral ring extension $R \subseteq S$.

**Proposition 2.5.** Let $\varphi : R \to S$ be an homomorphism of commutative rings satisfying LO. Then $\varphi$ satisfies the inverse unit-preserving property.

**Proof.** By using Remark 2.2 (2), we can assume that $R \neq \{0\}$ and so $R \setminus R^\times \neq \emptyset$. Now, suppose that the assertion fails. Then there exists $r \in R \setminus R^\times$ such that $y := \varphi(r) \in S^\times$. An application of Krull’s theorem ensures the existence of a maximal ideal $M$ of $R$ such that $r \in M$. As $\varphi$ satisfies LO, there exists $Q \in \text{Spec}(S)$ such that $\varphi^{-1}(Q) = M$. Hence $y = \varphi(r) \in \varphi(M) = \varphi(\varphi^{-1}(Q)) \subseteq Q$. Since $y \in S^\times$, this gives the desired contradiction completing the proof. \(\square\)

**Proposition 2.6.** Let $\varphi : R \to S$ be an homomorphism of commutative rings such that the induced map $\varphi_M : R_M \to S_M$ satisfies the inverse unit-preserving property for all $M \in \text{Max}(R)$. Then $\varphi$ satisfies the inverse unit-preserving property.

**Proof.** Without loss of generality, we can assume that $R \neq \{0\}$ accordingly to Remark 2.2 (2). We need to show that if $y \in S^\times$ and $y = \varphi(x)$ for some $x \in R$, then $x \in R^\times$. Deny. Then we can choose $M \in \text{Max}(R)$ such that $x \in M$. As $\varphi_M(x/1) = \varphi(x)/1 = y/1 \in S^\times_M$ and $\varphi_M : R_M \to S_M$ satisfies the inverse unit-preserving property, we get $x/1 \in R^\times_M$, which is a contradiction since $x/1 \in MR_M$. \(\square\)

Recall that a ring $R$ is called von Neumann regular if for every element $a \in R$ there exists an element $x$ in $R$ with $a = axa$. It is well known that for every von Neumann regular ring $R$, we have $J(R) = 0$.

**Theorem 2.7.** Let $\varphi : R \to S$ be an homomorphism of commutative rings. Assume that $R$ is a von Neumann regular ring, then the following statements are equivalent:

1. $\varphi$ satisfies the inverse unit-preserving property.
2. $\varphi$ is injective.

**Proof.** As $\varphi$ satisfies the inverse unit-preserving property, then Proposition 2.3 guarantees that $\text{Ker}(\varphi) \subseteq J(R)$. Since $R$ is a von Neumann regular ring, then $J(R) = 0$. It follows that $\text{Ker}(\varphi) = 0$. Hence, $\varphi$ is injective. This proves that (1) implies (2). Conversely, assume (2). According to Remark 2.2 (2), we can assume that $R \neq \{0\}$. By Proposition 2.6, it is enough to prove that the induced map $\varphi_M : R_M \to S_M$ satisfies the inverse unit-preserving property for all $M \in \text{Max}(R)$. Fix any such $M$. Since $R_M$ is a flat $R$-module, the monomorphism $\varphi$ induces a monomorphism $\varphi_M : R_M \to S \otimes_R R_M \cong S_M$. As $R$ is von Neumann regular, $K := R_M$ is a field and $B := S_M$ is a nonzero $K$-algebra. Since $B \neq \{0\}$, we get $0 \notin B^\times$ and so, since $K$ is a field, it follows that $K \to B$ satisfies the inverse unit-preserving property. This proves assertion (1). The proof is complete. \(\square\)

### 3. Some remarks on Unit-preserving property

In the nice paper [2], Chen has introduced the following definition.
Definition 3.1. (1) Let $R$ and $S$ be commutative rings. We say that a surjective ring homomorphism $\varphi : R \to S$ has ($\ast$) if the induced map $\varphi^\times : R^\times \to S^\times$ is surjective.

(2) We say that the ring $R$ has ($\ast$) if every surjective ring homomorphism $\varphi : R \to S$ (for any ring $S$) has ($\ast$).

(3) An ideal $I$ of a commutative ring $R$ is said to have ($\ast$) if the canonical surjection $R \to R/I$ has ($\ast$).

Next, we extend Chen’s definition for arbitrary rings.

Definition 3.2. (1) Let $R$ and $S$ be rings (not necessarily commutative). We say that a surjective ring homomorphism $\varphi : R \to S$ satisfies the unit-preserving property if $\varphi^\times(R^\times) = S^\times$; or equivalently, if $S^\times \subseteq \varphi(R^\times)$.

(2) We say that the ring $R$ satisfies the unit-preserving property if every surjective ring homomorphism $\varphi : R \to S$ (for any ring $S$) satisfies the unit-preserving property.

(3) An ideal $I$ of an arbitrary ring $R$ is said to satisfy the unit-preserving property if the canonical surjection $R \to R/I$ satisfies the unit-preserving property.

Remark 3.3. It is worth noticing that we can limit ourselves in Definition 3.2 to only nonzero rings $S$. In fact, let $R$ be a ring and let $\varphi : R \to \{0\}$ be the only possible surjective ring homomorphism from $R$ to $\{0\}$. Then $\varphi(R^\times) = \{0\}$. So, $\varphi$ satisfies the unit-preserving property.

We start our investigation with the following straightforward result.

Proposition 3.4. Let $\varphi : R \to S$ be a surjective ring homomorphism satisfying the inverse unit-preserving property, then $\varphi$ satisfies the unit-preserving property.

It is worth mentioning that the converse to Proposition 3.4 does not hold in general. To see this, it is enough to consider the mapping $\varphi : R \to \{0\}$ for some nonzero ring $R$. Then $\varphi$ satisfies the unit-preserving property by virtue of Remark 3.3, however $\varphi$ does not satisfy the inverse unit-preserving property according to Remark 2.2 (2).

As a consequence of Corollary 2.4 and Proposition 3.4, we recover [2, Corollary 2.1] for arbitrary rings.

Corollary 3.5. Let $R$ be a ring and let $I$ be an ideal of $R$. If $I \subseteq J(R)$, then $I$ satisfies the unit-preserving property.

Proposition 3.6. Let $R$ and $S$ be commutative rings and let $\varphi : R \to S$ be a surjective ring homomorphism satisfying LO. Then $\varphi$ satisfies the unit-preserving property.

Proof. Combine Propositions 2.7 and 3.4. □

Recall that if $R$ is a ring and $A$ is a subset of $R$, then the (left) annihilator of $A$, denoted $\text{Ann}_R(A)$, is the set of all elements $r$ in $R$ such that, for all $a$ in $A$, $ra = 0$. In set notation, $\text{Ann}_R(A) := \{r \in R \mid \forall a \in A, ra = 0\}$. Clearly, $\text{Ann}_R(A)$ is a left ideal of $R$. If moreover, $A$ is a left ideal of $R$, then $\text{Ann}_R(A)$ is an ideal of $R$.
Theorem 3.7. Let $R$ and $S$ be rings and let $\varphi : R \to S$ be a surjective ring homomorphism with kernel $I$. If $R/\text{Ann}_R(I)$ is left Artinian, then $\varphi$ satisfies the unit-preserving property.

Proof. Let $\text{End}(I)$ be the endomorphism ring of the additive group $(I, +)$ and let $\rho : R \to \text{End}(I)$ be the ring homomorphism defined by $\rho(r) = L_r$, where $L_r$ is the left multiplication by $r$. Set $J := \text{Ker}(\rho)$. Clearly, $J = \text{Ann}_R(I)$ is an ideal of $R$. Set $E := \text{Im}(\rho)$. By the first ring isomorphism theorem, we have $R/J \cong E$. It follows from [1, Lemm 3.5] that the combined map $R \to S \times E$ induces a ring isomorphism $\phi : R/(I \cap J) \to S \times R/(I+J)$. We claim that the map $(R/(I \cap J))^\times \to S^\times$ is surjective. Indeed, let $u \in S^\times$. Write $v$ for the image of $u$ in $(R/(I + J))^\times$. Since $E$ is left Artinian, we can choose by virtue of [1, Lemma 3.4] an element $w \in E^\times$ mapping to $v \in (R/(I + J))^\times$. Thus, $(u, w) \in S^\times \times (R/(I + J))^\times$. Hence, $\phi^{-1}(u, w)$ is a unit of $R/(I \cap J)$ that maps to $u \in S^\times$. This proves our claim. Since $(I \cap J)(I \cap J) \subseteq JJ = 0$, we infer that for any $x \in I \cap J$ the element $1 + x$ has inverse $1 - x$ and therefore belongs to $R^\times$. This yields that $I \cap J \subseteq J(R)$, so by Corollary 3.5 the map $R^\times \to (R/(I \cap J))^\times$ is surjective. It follows that the composed map $R^\times \to S^\times$ is also surjective. $\Box$

4. The case of rings sharing a nonzero ideal

We start with the following straightforward result. We include a proof for the sake of completeness.

Lemma 4.1. Let $R \subset S$ be an extension of rings having $I$ as a common ideal. Then $I \subseteq J(R)$ if and only if $I \subseteq J(S)$.

Proof. For the “Only if” part, assume that $I \subseteq J(R)$. Our task is to show that $I \subseteq J(S)$. For, let $x \in I$ and $y, z \in S$. As $yxz \in I \subseteq J(R)$, then $1 - yxz = 1 - 1(yxz)1 \in R^\times$. In particular, $1 - yxz \in S^\times$. This proves that $x \in J(S)$. Thus $I \subseteq J(S)$, as desired.

For the “If” part, suppose that $I \subseteq J(S)$ and let $x \in I$ and $y, z \in R$. As $yxz \in I \subseteq J(S)$, then $1 - yxz \in S^\times$. Thus, $(1 - yxz)\alpha = 1$ for some $\alpha \in S$. This implies $\alpha = (yxz)\alpha \in IS \subseteq I \subseteq R$. Therefore, $1 - yxz \in R^\times$. This proves that $x \in J(R)$. Thus $I \subseteq J(R)$. This completes the proof. $\Box$

Corollary 4.2. Let $R \subset S$ be an extension of rings having $I$ as a common ideal. Then the canonical surjection $\pi : S \to S/I$ satisfies the inverse unit-preserving property if and only if so does the restriction $\pi|_R : R \to R/I$.

Proof. Combine Corollary 2.4 and Lemma 4.1. $\Box$

Theorem 4.3. Let $R \subset S$ be an extension of rings having $I$ as a common ideal. Then the following hold true:

1. $[S^\times : R^\times] \leq [(S/I)^\times : (R/I)^\times]$.

2. If moreover the canonical surjection $\pi : S \to S/I$ satisfies the unit-preserving property, then $[S^\times : R^\times] = [(S/I)^\times : (R/I)^\times]$.
Proof.

1. The canonical surjection $\pi : S \to S/I$ induces a group homomorphism $\pi^\times : S^\times \to (S/I)^\times$. We claim that $\pi^\times^{-1}((R/I)^\times) = R^\times$. Indeed, the inclusion relation $R^\times \subseteq \pi^\times^{-1}((R/I)^\times)$ is clear. Conversely, let $x \in \pi^\times^{-1}((R/I)^\times)$. Then $\pi(x) = x + I \in (R/I)^\times$. Hence, there exists $r \in R$ such that $(x + I)(r + I) = (r + I)(x + I) = 1 + I$. This implies that $xr = 1 + a$ for some element $a \in I$. But $x \in S^\times$, so $sx = xs = 1$ for some $s \in S$. Therefore, $sxr = s + sa$. Hence, $s = r - sa \in R + aS \subseteq R + I = R$. This shows that $x$ is invertible in $R$ with inverse $s$. This completes the proof of our claim. Now, since $\pi^\times^{-1}((R/I)^\times) = R^\times$, the homomorphism $\pi^\times$ reduces to an embedding of sets $\pi^\times : S^\times/R^\times \to (S/I)^\times/(R/I)^\times$. It follows that $[S^\times : R^\times] \leq [(S/I)^\times : (R/I)^\times]$.

2. As the canonical surjection $\pi : S \to S/I$ satisfies the unit-preserving property, then the group homomorphism $\pi^\times : S^\times \to (S/I)^\times$ is surjective. Thus, $\pi^\times$ is also surjective. This yields the equality of unit indexes. □

We close the paper with the following corollary.

**Corollary 4.4.** Let $R \subset S$ be an extension of rings having $I$ as a common ideal. If $S/\text{Ann}_S(I)$ is left Artinian, then $[S^\times : R^\times] = [(S/I)^\times : (R/I)^\times]$.

**Proof.** This follows readily from Theorems 3.7 and 4.3 since the canonical surjection $\pi : S \to S/I$ satisfies the unit-preserving property in case $S/\text{Ann}_S(I)$ is left Artinian. □

**Bibliography**


