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Multiplicative Jordan type higher Derivations of Unital Rings with non trivial Idempotents

Dérivations supérieures multiplicatives de type Jordan des anneaux unitaires avec idempotants non-triviaux

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ABSTRACT. Suppose $\mathcal{R}$ is a non-zero unital associative ring with a nontrivial idempotent "e". In this paper, we prove that under some mild conditions every multiplicative jordan n-higher derivations on $\mathcal{R}$ is additive. Moreover, at the end of the paper, we have presented some applications of multiplicative Jordan n-higher derivations on triangular rings, nest algebra, upper triangular block matrix algebra, prime rings, von Neumann algebras.
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1 Introduction

The question of to what extent the multiplicative structure of rings and algebra determines its additive structure has been considered by many researchers over the past decades. In particular, they have investigated conditions under which bijective mappings between algebras preserving the multiplicative structure necessarily preserve the additive structure as well. The most fundamental result in this direction is due to W. S. Martindale III [12], who proved that every bijective multiplicative mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. Let $\mathcal{R}$ be an associative ring. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a *derivation* if $d(st) = d(s)t + sd(t)$ for all $s, t \in \mathcal{R}$. For any $s, t \in \mathcal{R}$, we denote a "new product" of s and t by $\{s, t\} = st + ts$, this new product is usually known as *Jordan product*. Such kind of product based on Jordan bracket naturally appears in relation with the so-called Jordan derivations. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a *Jordan derivation* if $d(\{s, t\}) = \{d(s), t\} + \{s, d(t)\}$ holds for all $s, t \in \mathcal{R}$. It is to be remarked that if the additivity is dropped in the above definition, then d is said to be a *multiplicative Jordan derivation*.

The concept of derivation and Jordan derivation has been extended to higher derivation, Jordan higher derivations and Jordan triple higher derivation of rings. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{D} = \{d_m\}_{m\in\mathbb{N}}$ be a family of additive mappings $d_m : \mathcal{R} \to \mathcal{R}$ such that $d_0 = id_{\mathcal{R}}$, the identity map on $\mathcal{R}$. Then $\mathcal{D}$ is called a higher derivation on $\mathcal{R}$ if for every $m \in \mathbb{N}$, $d_m(st) = \sum_{i+j=m} d_i(s)d_j(t)$ for all $s, t \in \mathcal{R}$. A family $\mathcal{D} = \{d_m\}_{m\in\mathbb{N}}$ of mappings $d_m : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) is said to be:

(i) a *multiplicative Jordan higher derivation* on $\mathcal{R}$ if for every $m \in \mathbb{N}$,
$d_m(\{s, t\}) = \sum_{i+j=m} \{d_i(s), d_j(t)\}$ for all $s, t \in \mathcal{R}$.

Given the consideration of Jordan derivations and Jordan triple derivations, Lin [7, 8] further developed them in a more general way. Suppose that $n \geq 2$ is a fixed positive integer. Let us consider a sequence of polynomials:

$$\begin{aligned}
p_1(Y_1) &= Y_1, \\
p_2(Y_1, Y_2) &= \{p_1(Y_1), Y_2\} \overset{\text{set}}{=} \{Y_1, Y_2\}, \\
p_3(Y_1, Y_2, Y_3) &= \{p_2(Y_1, Y_2), Y_3\} = \{\{Y_1, Y_2\}, Y_3\}, \\
p_4(Y_1, Y_2, Y_3, Y_4) &= \{p_3(Y_1, Y_2, Y_3), Y_4\} = \{\{\{Y_1, Y_2\}, Y_3\}, Y_4\}, \\
&\vdots \qquad\qquad \vdots \;, \\
p_n(Y_1, Y_2, \cdots, Y_n) &= \{p_{n-1}(Y_1, Y_2, \cdots, Y_{n-1}), Y_n\}
\end{aligned}$$

A *multiplicative Jordan n-derivation* is a mapping $d : \mathcal{R} \longrightarrow \mathcal{R}$ satisfying the condition

$$d(p_n(s_1, s_2, \cdots, s_n)) = \sum_{k=1}^{n} p_n\big(s_1, \cdots, s_{k-1}, d(s_k), s_{k+1}, \cdots, s_n\big) \tag{1.1}$$

for all $s_1, s_2, \cdots, s_n \in \mathcal{R}$. This notion makes the best use of the definition of Jordan-type derivation and that of Lie-type derivation, see [7, 8]. By the definition, it is clear that every Jordan derivation is Jordan 2-derivation and each Jordan triple derivation is an Jordan 3-derivation. One can easily check that each multiplicative Jordan derivation on $\mathcal{R}$ is a multiplicative Jordan triple derivation. But, we don't know whether the converse statement is true. Jordan 2-derivations, Jordan 3-derivations and Jordan n-derivations are collectively referred to as *Jordan-type derivations*.

Let $p_n(Y_1, Y_2, \cdots, Y_n)$ be the polynomial defined by n indeterminates $Y_1, \cdots, Y_n$ and their Jordan multiple products. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{D} = \{d_m\}_{m \in \mathbb{N}}$ be a family of mappings $d_m : \mathcal{R} \to \mathcal{R}$ such that $d_0 = id_{\mathcal{R}}$, the identity mapping on $\mathcal{R}$. Then $\mathcal{D}$ is called a *multiplicative Jordan n-higher derivation* if $\mathcal{D}$ satisfies the condition

$$d_m(p_n(s_1, s_2, \cdots, s_n)) = \sum_{i_1+i_2+\cdots+i_n=m} p_n(d_{i_1}(s_1), d_{i_2}(s_2), \cdots, d_{i_n}(s_n)) \tag{1.2}$$

for all $s_1, s_2, \cdots, s_n \in \mathcal{R}$ and for each $m \in \mathbb{N}$. It is to be noted that $\{d_m\}$ is called a *multiplicative Jordan higher derivation* whenever $n = 2$, and is called a *multiplicative Jordan triple higher derivation* whenever $n = 3$. Jordan higher derivations, Jordan triple higher derivations and Jordan n higher derivations are collectively referred to as *Jordan type higher derivations*.

Jordan type derivations in different backgrounds are extensively studied by several authors. Herstein [5] first proved that every additive Jordan derivation from a prime ring of characteristic not 2 into itself is a derivation. Li and Lu [10] proved that every additive Jordan derivation on reflexive algebras is an additive derivation. Brešar [2], proved that every Jordan triple derivation from a 2-torsion free semiprime ring into itself is a derivation. Recently, Lee and Quynh [6] gave a complete characterization of additive Jordan triple derivations of arbitrary semiprime rings, which generalized the results in Ref. [2]. Lu [11] proved that every multiplicative Jordan derivation on a 2-torsion free unital prime ring containing a nontrivial idempotent element is an additive derivation. Li and Fang [9] discussed the additivity of multiplicative Jordan (triple) derivations on nest algebras. Benkovic and Sirvonik [1] proposed a kind of algebras. Assume that $\mathcal{R}$ is a non-zero unital associative algebra with a non trivial idempotent e, and write $f = 1 - e$. Then $\mathcal{R}$ can be represented as $\mathcal{R} = e\mathcal{R}e + e\mathcal{R}f + f\mathcal{R}e + f\mathcal{R}f$, where $e\mathcal{R}e$ and $f\mathcal{R}f$ are subalgebras with units e and f , respectively, $e\mathcal{R}f$ is an $(e\mathcal{R}e, f\mathcal{R}f)$-bimodule and $f\mathcal{R}e$ is an $(f\mathcal{R}f, e\mathcal{R}e)$-bimodule. Furthermore, assume that $\mathcal{R}$ satisfies the following general conditions:

$$\left\{ \begin{array}{l} exe.e\mathcal{R}f = 0 = f\mathcal{R}e.exe \implies exe = 0 \\ \\ e\mathcal{R}f.fxf = 0 = fxf.f\mathcal{R}e \implies fxf = 0. \end{array} \right\} \tag{1.3}$$

for all $x \in \mathcal{R}$. Throughout this paper, we denote $e\mathcal{R}e$ by $\mathcal{R}_{11}$, $e\mathcal{R}f$ by $\mathcal{R}_{12}$, $f\mathcal{R}e$ by $\mathcal{R}_{21}$ and $f\mathcal{R}f$ by $\mathcal{R}_{22}$.

Zhang [21] independently studied 1-Jordan derivations on factor von Neumann algebras. Until recently, Xiaofei Qi et.al, [16] gave the characterization of Jordan n-derivations of unital rings containing idempotents. On the other hand, many mathematicians have made outstanding contribution for higher derivations on some classical algebras (see [3],[13], [14],[15], [18],[19],[20]).

Motivated by the afore-mentioned works, we investigate multiplicative Jordan n-higher derivation on non-zero unital associative rings. In fact, we prove that every multiplicative Jordan n-higher derivation $\mathcal{D} = \{d_m\}_{m\in\mathbb{N}}$ from $\mathcal{R}$ to itself is an additive higher derivation.

2 Main Theorem

In this section, we discuss the additivity of multiplicative Jordan n-higher derivation on unital rings.

Theorem 2.1. *Let $\mathcal{R}$ be a non-zero unital associative ring with a nontrivial idempotent e satisfying (1.3). Assume that the $Ch(\mathcal{R}) \neq 2$, where $Ch(\mathcal{R})$ denotes the characteristic of $\mathcal{R}$. Then for all $n \geq 2$, every multiplicative Jordan n-higher derivation on $\mathcal{R}$ is additive.*

Proof of Theorem 2.1. Assume $\psi_m : \mathcal{R} \to \mathcal{R}$ is a multiplicative jordan n-higher derivation. let $z_0 = e\psi_m(e)f - f\psi_m(e)e$. Now define two maps h_m and ψ'_m as follows ; $h_m : \mathcal{R} \to \mathcal{R}$ and $\psi'_m : \mathcal{R} \to \mathcal{R}$ such that $h_m(z) = [z, z_o]$, for all $z \in \mathcal{R}$ and $\psi'_m = \psi_m - h_m$. Clearly ψ'_m is also a jordan n-higher derivation and satisfies $e\psi'_m(e)f = f\psi'_m(e)e = 0$. So, without loss of generality we will assume that $e\psi_m(e)e = f\psi_m(e)f = 0$ in the following claims.

Claim 2.1. $\psi_m(0) = 0$

Proof. To Prove the claim we use the principle of mathematical induction. For $m = 1$, result is true by [16]. Now suppose that it also holds for all $k < m$ i.e, $\psi_k(0) = 0$, we will prove that it is also true for all $m \in \mathbb{N}$. Since, $0 = p_n(0, 0, 0, \cdots, 0)$ and by our assumption, this implies

$$\begin{aligned} \psi_m(0) &= \psi_m(p_n(0, 0, 0, \cdots, 0)) \\ &= \sum_{i=1}^{n} p_n(0, 0, 0, \cdots, \psi_m(0)_i, \cdots 0) + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(0), \psi_{i_2}(0), \cdots, \psi_{i_n}(0)) \\ &= 0. \end{aligned}$$

□

Claim 2.2. $\psi_m(e) \in \mathcal{R}_{11}$ *and* $\psi_m(f) \in \mathcal{R}_{22}$.

Proof. we will prove this with the help of induction,for $m = 1$ it is true, now suppose it is also true for all $k < m$.We will show it holds for all $m \in \mathbb{N}$. since $0 = p_n(e, f, f, \cdots, f)$ by induction hypothesis and using Claim 2.1 ,we get

$$\begin{aligned}
0 &= \psi_m(p_n(e, f, f, \cdots, f)) \\
&= p_n(\psi_m(e), f, f, \cdots, f) + p_n(e, \psi_m(f), f, \cdots, f) + p_n(e, f, \psi_m(f), \cdots, f) \\
&+ \cdots + p_n(e, f, f, \cdots, \psi_m(f)) + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(e), \psi_{i_2}(f), \cdots, \psi_{i_n}(f)) \\
&= p_n(\psi_m(e), f, f, \cdots, f) + p_n(e, \psi_m(f), f, \cdots, f) \\
&\quad = 2^{n-1} f\psi_m(e)f + e\psi_m(f)f + f\psi_m(f)e. \qquad (2.1)
\end{aligned}$$

and

$$\begin{aligned}
0 &= \psi_m(p_n(f, e, e, \cdots, e)) \\
&= p_n(\psi_m(f), e, e, \cdots, e) + p_n(f, \psi_m(e), e, \cdots, e) + p_n(f, e, \psi_m(e), \cdots, e) \\
&\quad + \cdots + p_n(f, e, e, \cdots, \psi_m(e)) + \sum_{\substack{i_1+i_2,\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(f), \psi_{i_2}(e), \cdots, \psi_{i_n}(e)) \\
&= p_n(\psi_m(f), e, e, \cdots, e) + p_n(f, \psi_m(e), e, \cdots, e) \\
&\quad = 2^{n-1} e\psi_m(f)e + f\psi_m(f)e + e\psi_m(e)f. \qquad (2.2)
\end{aligned}$$

from equations (2.2) and (2.3), we get $e\psi_m(e)f = f\psi_m(e)e = 0$, $f\psi_m(e)e = e\psi_m(e)f = 0$ and $f\psi_m(e)f = e\psi_m(f)e = 0$, as $ch(\mathcal{R}) \neq 2$. Therefore, $\psi_m(e) = e\psi_m(e)e \in \mathcal{R}_{11}$ and $\psi_m(f) = f\psi_m f \in \mathcal{R}_{22}$. □

Claim 2.3. $\psi_m(\mathcal{R}_{11}) \subseteq \mathcal{R}_{11}$ *and* $\psi_m(\mathcal{R}_{22}) \subseteq \mathcal{R}_{22}$.

Proof. The result is true for $m = 1$ by [16]. Suppose it also holds for all $k < m$.i.e, $\psi_k(\mathcal{R}_{11}) \subseteq \mathcal{R}_{11}$ and $\psi_k(\mathcal{R}_{22}) \subseteq \mathcal{R}_{22}$, for all $k < m$. We show it also holds for each $m \in \mathbb{N}$. Since, $0 = p_n(s_{11}, f, f, \cdots, f)$ by above two claims and induction hypothesis, we have

$$\begin{aligned}
0 &= \psi_m(p_n(s_{11}, f, f, \cdots, f)) \\
&= p_n(\psi_m(s_{11}), f, f, \cdots, f) + p_n(s_{11}, \psi_m(f), f, \cdots, f) + p_n(s_{11}, f, \psi_m(f), \cdots, f) \\
&+ \cdots + p_n(s_{11}, f, f, \cdots, \psi_m(f)) + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11}), \psi_{i_2}(f), \cdots, \psi_{i_n}(f))i \\
&= p_n(\psi_m(s_{11}), f, f, \cdots, f) \\
&= \psi_m(s_{11})f + f\psi_m(s_{11}) + \sum_{i=1}^{n-2} C_{n-1}^i f\psi_m(s_{11})f \\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11}), \psi_{i_2}(f), \cdots, \psi_{i_n}(f))
\end{aligned}$$

$$= \psi_m(s_{11})f + f\psi_m(s_{11}) + \sum_{i=1}^{n-2} C_{n-1}^i f\psi_m(s_{11})f. \tag{2.3}$$

Premultiplying by e and postmultiplying by f to equation (2.4) gives $e\psi_m(s_{11})f = 0$ and by premultiplying f and postmultiplying e gives, $f\psi_m(s_{11})f = 0$. Also by premultiplying and postmultiplying by f, it gives

$$\begin{aligned}
0 &= f\psi_m(s_{11})f + f\psi_m(s_{11})f + \sum_{i=1}^{n-2} C_{n-1}^i f\psi_m(s_{11})f. \\
&= \sum_{i=0}^{n-1} C_{n-1}^i f\psi_m(s_{11})f.
\end{aligned}$$

Since $ch(\mathcal{R}) \neq 2$, we get $\psi_m(s_{11})f = 0$. Therefore, $\psi_m(s_{11}) = e\psi_m(s_{11})e \in \mathcal{R}$. Which implies $\psi_m(\mathcal{R}_{11}) \subseteq \mathcal{R}_{11}$.

Similarly we can show that $\psi_m(\mathcal{R}_{22}) \subseteq \mathcal{R}_{22}$, which proves our claim. □

Claim 2.4. $\psi_m(\mathcal{R}_{12}) \in \mathcal{R}_{12} + \mathcal{R}_{21}$ *and* $\psi_m(\mathcal{R}_{21}) \in \mathcal{R}_{12} + \mathcal{R}_{21}$

Proof. To prove our claim, we use induction hypothesis on m, for $m = 1$ it is true by [16]. Suppose it also holds for all $k < m$, i.e, $\psi_k(\mathcal{R}_{12}) \in \mathcal{R}_{12} + \mathcal{R}_{21}$ and $\psi_k(\mathcal{R}_{21}) \in \mathcal{R}_{12} + \mathcal{R}_{21}$. We show it is also true for all $m \in \mathbb{N}$. For any $s_{12} \in \mathcal{R}_{12}$. Since, $s_{12} = p_n(s_{12}, f, f, \cdots, f)$. This implies

$$\begin{aligned}
\psi_m(s_{12}) &= \psi_m(p_n(s_{12}, f, f, \cdots, f)) \\
&= p_n(\psi_m(s_{12}), f, f, \cdots, f) + p_n(s_{12}, \psi_m(f), f, \cdots, f) + \cdots \\
&+ p_n(s_{12}, f, f, \cdots, \psi_m(f)) + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{12}), \psi_{i_2}(f), \cdots, \psi_{i_n}(f)) \\
&= \psi_m(s_{12lt})f + \sum_{i=1}^{n-2} C_{i-1}^i f\psi_m s_{12} f + f\psi_m s_{12} + (n-1)s_{12}\psi_m f.
\end{aligned} \tag{2.4}$$

Now multiply by e from the left and the right in equation (2.5), we get $e\psi_m(s_{12})e = 0$. Again since, $s_{12} = p_n(s_{12}, e, e, \cdots, e)$. Therefore from the relation $\psi_m(s_{12}) = \psi_m(p_n(s_{12}, f, f, \cdots, f))$. One can get $f\psi_m(s_{12})f = 0$. hence $\psi_m(s_{12}) = e\psi_m(s_{12})f + f\psi_m(s_{12})e \in R_{12} + R_{21}$. Similarly, we can show $\psi_m(s_{21}) = e\psi_m(s_{12})f + f\psi_m(s_{12})e \in R_{12} + R_{21}$. □

Claim 2.5. *For any* $s_{ii} \in \mathcal{R}_{ii}$,$s_{ij} \in \mathcal{R}_{ij}$ *and* $s_{ji} \in \mathcal{R}_{ji}$, *we have,*
$\psi_m(s_{ii} + s_{ij}) = \psi_m(s_{ii}) + \psi_m(s_{ij})$ *and* $\psi_m(s_{ii} + s_{ji}) = \psi_m(s_{ii}) + \psi_m(s_{ji})$, $1 \leq i \neq j \leq 2$.

Proof. We will prove only one part, second follows same arguments. For $m = 1$ it is true by [16], now assume that it holds for all $k < m$, i.e,$\psi_k(s_{ii}+s_{ij}) = \psi_k(s_{ii})+\psi_k(s_{ij})$, $1 \leq i \neq j \leq 2$. We show it is also true for all $m \in \mathbb{N}$. Take any $s_{11} \in \mathcal{R}_{11}$ and $s_{12} \in \mathcal{R}_{12}$ and write, $z = \psi_m(s_{11}+s_{12})-\psi_m(s_{11})-\psi_m(s_{12})$.

Now

$$\begin{aligned}
&\psi_m(p_n(s_{11}+s_{12},f,f,\cdots,f))\\
&\qquad = p_n(\psi_m(s_{11}+s_{12}),f,f,\cdots,f)+p_n(s_{11}+s_{12},\psi_m(f),f,\cdots,f)\\
&\qquad +\cdots+p_n(s_{11}+s_{12},f,f,\cdots,\psi_m(f))\\
&\qquad +\sum_{\substack{i_1+i_2,\cdots+i_n=m\\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11}+s_{12}),\psi_{i_2}(f),\psi_{i_3}(f),\cdots,\psi_{i_n}(f)),
\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}
&\psi_m(p_n(s_{11}+s_{12},f,f,\cdots,f))\\
&\qquad = \psi_m(p_n(s_{11},f,f,\cdots,f))+\psi_m(p_n(s_{12},f,f,\cdots,f))\\
&\qquad = p_n(\psi_m(s_{11}),f,f,\cdots,f)+p_n(s_{11},\psi_m(f),f,\cdots,f)+\\
&\qquad \cdots+p_n(s_{11},f,f,\cdots,\psi_m(f))\\
&\qquad +\sum_{\substack{i_1+i_2,\cdots+i_n=m\\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11}),\psi_{i_2}(f),\psi_{i_3}(f),\cdots,\psi_{i_n}(f))\\
&\qquad +p_n(\psi_m(s_{12}),f,f,\cdots,f)+p_n(s_{12},\psi_m(f),f,\cdots,f)\\
&\qquad +\cdots+p_n(s_{12},f,f,\cdots,\psi_m(f))\\
&\qquad +\sum_{\substack{i_1+i_2,\cdots+i_n=m\\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{12}),\psi_{i_2}(f),\psi_{i_3}(f),\cdots,\psi_{i_n}(f))\\
&\qquad = p_n(\psi_m(s_{11}),f,f,\cdots,f)+p_n(\psi_m(s_{12}),f,f,\cdots,f)\\
&\qquad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11}+s_{12}),\psi_{i_2}(f),\psi_{i_3}(f),\cdots,\psi_{i_n}(f)).
\end{aligned}\tag{2.6}$$

Comparing equations (2.6) and (2.7), we get $p_n(z,f,f,\cdots,f) = zf+fz+\sum_{i=1}^{n-2}C_{n-1}^{i}fzf = 0$. Which implies $ezf=fze=fzf=0$. Finally to prove our claim, we need to show $eze=0$. Now, since

$$\begin{aligned}
&\psi_m(p_n(s_{11}+s_{12},e-f,e-f,\cdots,e-f))\\
&\qquad = p_n(\psi_m(s_{11}+s_{12}),e-f,e-f,\cdots,e-f)\\
&\qquad +p_n(s_{11}+s_{12},\psi_m(e-f),e-f,\cdots,e-f)\\
&\qquad +\cdots+p_n(s_{11}+s_{12},e-f,e-f,\cdots,\psi_m(e-f))\\
&\qquad +\sum_{\substack{i_1+i_2,\cdots+i_n=m\\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11}+s_{12}),\psi_{i_2}(e-f),\psi_{i_3}(e-f),\cdots,\psi_{i_n}(e-f)).
\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}
&\psi_m(p_n(s_{11}+s_{12},e-f,e-f,\cdots,e-f))\\
&\quad = \psi_m(p_n(s_{11},e-f,e-f,\cdots,e-f))+\psi_m(p_n(s_{12},e-f,e-f,\cdots,f))\\
&\quad = p_n(\psi_m(s_{11}),e-f,e-f,\cdots,f)+p_n(s_{11},\psi_m(e-f),e-f,\cdots,e-f)\\
&\quad +\cdots+p_n(s_{11},e-f,e-f,\cdots,\psi_m(e-f))\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11}),\psi_{i_2}(e-f),\psi_{i_3}(e-f),\cdots,\psi_{i_n}(e-f))
\end{aligned}$$

$$
\begin{aligned}
&+ p_n(\psi_m(s_{12}), e-f, e-f, \cdots, e-f) + p_n(s_{12}, \psi_m(e-f), e-f, \cdots, e-f) \\
&+ \cdots + p_n(s_{12}, e-f, e-f, \cdots, \psi_m(e-f)) \\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{12}), \psi_{i_2}(e-f), \psi_{i_3}(e-f), \cdots, \psi_{i_n}(e-f)) \\
&= p_n(\psi_m(s_{11}), e-f, e-f, \cdots, e-f) + p_n(\psi_m(s_{12}), e-f, e-f, \cdots, e-f) \\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11}+s_{12}), \psi_{i_2}(e-f), \psi_{i_3}(e-f), \cdots, \psi_{i_n}(e-f)) \qquad (2.8)
\end{aligned}
$$

From equations (2.8) and (2.9), we get

$$
\begin{aligned}
p_n(z, e-f, e-f, \cdots, e-f) &= p_n(eze, e-f, e-f, \cdots, e-f) \\
&= 2^{n-1}eze \\
&= 0.
\end{aligned}
$$

Now, since $ch(\mathcal{R}) \neq 2$, it follows that $eze = 0$. Hence $z = 0$ i.e, $z = \psi_m(s_{11}+s_{12}) - \psi_m(s_{11}) - \psi_m(s_{12}) = 0$. Which implies $\psi_m(s_{11}+s_{12}) = \psi_m(s_{11}) + \psi_m(s_{12})$. For the other cases, the proofs are similar and we omit these here. □

Claim 2.6. *ψ_m is additive on $\mathcal{R}_{ij}$, $i \neq j \in 1, 2$*

Proof. For any $s_{12}, t_{12} \in \mathcal{R}_{12}$ and $s_{21}, t_{21} \in \mathcal{R}_{21}$, we have to show, $\psi_m(s_{12}+t_{12}) = \psi_m(s_{12}) + \psi_m(t_{12})$ and $\psi_m(s_{21}+t_{21}) = s_{21} + t_{21}$. We will prove first one and second can be proved similarly. Note that, $s_{12}+t_{12} = p_n(e+s_{12}, f+t_{12}, \cdots, f+t_{12})$. Then, $\psi_m(s_{12}+t_{12}) = \psi_m(p_n(e+s_{12}, f+t_{12}, \cdots, f+t_{12}))$ and, by claim 2.2 and claim 2.5, we have

$$
\begin{aligned}
&\psi_m(s_{12}+t_{12}) = \\
&\quad p_n(\psi_m(e+s_{12}), f+t_{12}, \cdots, f+t_{12}) + p_n(e+s_{12}, \psi_m(f+t_{12}), \cdots, f+t_{12}) \\
&\quad + \cdots + p_n(e+s_{12}, f+t_{12}, \cdots, \psi_m(f+t_{12})) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(e+s_{12}), \psi_{i_2}(f+t_{12}), \psi_{i_3}(f+t_{12}), \cdots, \psi_{i_n}(f+t_{12})) \\
&\quad = p_n(\psi_m(e), f+t_{12}, \cdots, f+t_{12}) + p_n(e, \psi_m(f+t_{12}), \cdots, f+t_{12}) + \cdots \\
&\quad + p_n(e, f+t_{12}, \cdots, \psi_m(f+t_{12})) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(e), \psi_{i_2}(f+t_{12}), \psi_{i_3}(f+t_{12}), \cdots, \psi_{i_n}(f+t_{12})) \\
&\quad + p_n(\psi_m(s_{12}), f+t_{12}, \cdots, f+t_{12}) + p_n(s_{12}, \psi_m(f+t_{12}), \cdots, f+t_{12}) + \cdots \\
&\quad + p_n(s_{12}, f+t_{12}, \cdots, \psi_m(f+t_{12})) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{12}), \psi_{i_2}(f+t_{12}), \psi_{i_3}(f+t_{12}), \cdots, \psi_{i_n}(f+t_{12})) \\
&\quad = \psi_m(p_n(e, f+t_{12}, \cdots, f+t_{12})) + \psi_m(p_n(s_{12}, f+t_{12}, \cdots, f+t_{12})) \\
&\quad = \psi_m(s_{12}) + \psi_m(t_{12}) \qquad (2.9)
\end{aligned}
$$

i.e, $\psi_m(s_{12}+t_{12}) = \psi_m(s_{12}) + \psi_m(t_{12})$, Which proves the claim. The proof of the other case can be verified similarly. □

Claim 2.7. *ψ_m is additive on $\mathcal{R}_{ii}$, $i = 1, 2$ i.e, for $s_{11}, t_{11} \in \mathcal{R}_{11}$ and $s_{22}, t_{22} \in \mathcal{R}_{22}$ $\psi_m(s_{11} + t_{11}) = \psi_m(s_{11}) + \psi_m(t_{11})$ and $\psi_m(s_{22} + t_{22}) = s_{22} + t_{22}$.*

Proof. We will prove the first one and second will be proved similarly. For $s_{11}, t_{11} \in \mathcal{R}_{11}$, write $z = \psi_m(s_{11} + t_{11}) - \psi_m(s_{11}) - \psi_m(t_{11})$. By claim 2.3, $z = eze \in \mathcal{R}_{11}$. Now, for any $s_{12} \in \mathcal{R}_{12}$, we have,

$$\begin{aligned}
&\psi_m(p_n(s_{11} + t_{11}, s_{12} + f, s_{12} + f, \cdots, s_{12} + f)) \\
&\quad = p_n(\psi_m(s_{11} + t_{11}), s_{12} + f, s_{12} + f, \cdots, s_{12} + f) \\
&\quad + p_n(s_{11} + t_{11}, \psi_m(s_{12} + f), s_{12} + f, \cdots, s_{12} + f) + \dots \\
&\quad + p_n(s_{11} + t_{11}, \psi_m(s_{12} + f), s_{12} + f, \cdots, s_{12} + f) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11} + t_{11}), \psi_{i_2}(s_{12} + f), \cdots, \psi_{i_n}(s_{12} + f)). \qquad (2.10)
\end{aligned}$$

Also,

$$\begin{aligned}
&\psi_m(p_n(s_{11} + t_{11}, s_{12} + f, s_{12} + f, \cdots, s_{12} + f)) \\
&\quad = \psi_m(p_n(s_{11}, s_{12} + f, s_{12} + f, \cdots, s_{12} + f)) \\
&\quad + \psi_m(p_n(t_{11}, s_{12} + f, s_{12} + f, \cdots, s_{12} + f)) \\
&\quad = p_n(\psi_m(s_{11}), s_{12} + f, s_{12} + f, \cdots, s_{12} + f) \\
&\quad + p_n(s_{11}, \psi_m(s_{12} + f), s_{12} + f, \cdots, s_{12} + f) + \dots \\
&\quad + p_n(s_{11}, f + s_{12} + f, s_{12} + f, \cdots, \psi_m(s_{12} + f)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11}), \psi_{i_2}(s_{12} + f), , \cdots, \psi_{i_n}(s_{12} + f)) \\
&\quad + p_n(\psi_m(t_{11}), s_{12} + f, s_{12} + f, \cdots, s_{12} + f) \\
&\quad + p_n(t_{11}, \psi_m(s_{12} + f), s_{12} + f, \cdots, s_{12} + f) + \dots \\
&\quad + p_n(t_{11}, \psi_m(s_{12} + f), s_{12} + f, \cdots, s_{12} + f) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(t_{11}), \psi_{i_2}(s_{12} + f), , \cdots, \psi_{i_n}(s_{12} + f)) \\
&\quad = p_n(\psi_m(s_{11}) + \psi_m(t_{11}), s_{12} + f, s_{12} + f, \cdots, s_{12} + f) \\
&\quad + p_n(s_{11} + t_{11}, \psi_m(s_{12} + f), s_{12} + f, \cdots, s_{12} + f) + \dots \\
&\quad + p_n(s_{11} + t_{11}, \psi_m(s_{12} + f), s_{12} + f, \cdots, s_{12} + f) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11} + t_{11}), \psi_{i_2}(s_{12} + f), \cdots, \psi_{i_n}(s_{12} + f)). \qquad (2.11)
\end{aligned}$$

Therefore by comparing equations (2.11) and (2.12), we get, $p_n(eze, s_{12} + f, s_{12} + f, \cdots, s_{12} + f) = p_n(z, s_{12} + f, s_{12} + f, \cdots, s_{12} + f) = 0$, and so, $ezes_{12} = 0$ for all $s_{12} \in \mathcal{R}_{12}$. Similarly for any $s_{21} \in \mathcal{R}_{21}$, by calculating $\psi_m(p_n(s_{11} + t_{11}, s_{21}, e, \cdots, e))$ in two ways, we obtain,$s_{21}eze = 0$. Therefore from this equation and $ezes_{12} = 0$, and using equation (1.3), we obtain, $z = eze = 0$, i.e,$z = \psi_m(s_{11} + t_{11}) - \psi_m(s_{11}) - \psi_m(t_{11}) = 0$. which implies, $\psi_m(s_{11} + t_{11}) = \psi_m(s_{11}) + \psi_m(t_{11})$. similarly, we can prove,$\psi_m(s_{22} + t_{22}) = \psi_m(s_{22}) + \psi_m(t_{22})$. Hence our claim is proved. ☐

Claim 2.8. *For any $s_{11} \in \mathcal{R}_{11}$ and $s_{22} \in \mathcal{R}_{22}$, we have, $\psi_m(s_{11} + s_{22}) = \psi_m(s_{11}) + \psi_m(s_{22})$.*

Proof. Let $z = \psi_m(s_{11} + s_{22}) - \psi_m(s_{11}) - \psi_m(s_{22})$, so, to prove our claim we need to show $z = 0$. For any $s_{11} \in \mathcal{R}_{11}$ and $s_{22} \in \mathcal{R}_{22}$, we have

$$\begin{aligned}
&\psi_m(p_n(s_{11} + s_{22}, f, f, \cdots, f)) \\
&\qquad = p_n(\psi_m(s_{11} + s_{22}), f, f, \cdots, f) + p_n(s_{11} + s_{22}, \psi_m(f), f, \cdots, f) \\
&\qquad + \cdots + p_n(s_{11} + s_{12}, f, f, \cdots, \psi_m(f)) \\
&\qquad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11} + s_{22}), \psi_{i_2}(f), \psi_{i_3}(f), \cdots, \psi_{i_n}(f))
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\psi_m(p_n(s_{11} + s_{22}, f, f, \cdots, f)) \\
&\qquad = \psi_m(p_n\left(s_{11}, f, f, \cdots, f\right)) + \psi_m(p_n\left(s_{22}, f, f, \cdots, f\right)) \\
&\qquad = p_n(\psi_m(s_{11}), f, f, \cdots, f) + p_n(s_{11}, \psi_m(f), f, \cdots, f) \\
&\qquad + \cdots + p_n(s_{11}, f, f, \cdots, \psi_m(f)) + \\
&\qquad \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11}), \psi_{i_2}(f), \psi_{i_3}(f), \cdots, \psi_{i_n}(f)) \\
&\qquad + p_n(\psi_m(s_{22}), f, f, \cdots, f) + p_n(s_{22}, \psi_m(f), f, \cdots, f) \\
&\qquad + \cdots + p_n(s_{22}, f, f, \cdots, \psi_m(f)) + \\
&\qquad \sum_{\substack{i_1+i_2,\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{22}), \psi_{i_2}(f), \psi_{i_3}(f), \cdots, \psi_{i_n}(f)) \\
&\qquad = p_n(\psi_m(s_{11}), f, f, \cdots, f) + p_n(\psi_m(s_{22}), f, f, \cdots, f) + \\
&\qquad p_n(s_{11} + s_{22}, \psi_m(f), f, \cdots, f) + \cdots + p_n(s_{11} + s_{22}, f, f, \cdots, \psi_m(f)) \\
&\qquad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11} + s_{22}), \psi_{i_2}(f), \psi_{i_3}(f), \cdots, \psi_{i_n}(f))
\end{aligned} \tag{2.13}$$

From equations (2.13) and (2.14), we obtain

$$p_n(z, f, f, \cdots, f) = zf + fz + \sum_{i=1}^{n-2} C_{n-1}^{i} fzf = 0$$

, and since the $ch(\mathcal{R} \neq 2)$, therefore we have, $eze = fze = fzf = 0$. On the other hand by calculating, $\psi_m(p_n\left(s_{11} + s_{22}, e, e, \cdots, e\right))$ in two ways, one can get $p_n(z, e, e, \cdots, e) = 0$, which implies $eze = 0$. Hence, $z = 0$, i.e, $\psi_m(s_{11} + s_{22}) = \psi_m(s_{11}) + \psi_m(s_{22})$. Which proves our claim. ☐

Claim 2.9. *For any $s_{ij} \in \mathcal{R}_{ij}$, $(1 \leq i \neq j \leq 2)$, we have $\psi_m(s_{ii} + s_{ij} + s_{ji}) = \psi_m(s_{ii}) + \psi_m(s_{ij} + s_{ji})$. i.e, for any $s_{ij} \in \mathcal{R}_{ij};\ i, j = 1, 2$.*

Proof. Since, $\psi_m(s_{11} + s_{12} + s_{21}) = \psi_m(s_{11}) + \psi_m(s_{12} + s_{21})$ and $\psi_m(s_{22} + s_{12} + s_{21}) = \psi_m(s_{22}) + \psi_m(s_{12} + s_{21})$. Let $z = \psi_m(s_{11} + s_{12} + s_{21}) - \psi_m(s_{11}) - \psi_m(s_{12} + s_{21})$. Then as in claim 2.8, by calculating $\psi_m(p_n\left(s_{11} + s_{12} + s_{21}, f, f, \cdots, f\right))$ in two ways one can obtain, $p_n(z, f, f, \cdots, f) = 0$, implies that $esf = fse = fsf = 0$. On the other hand, by calculating $\psi_m(p_n\left(s_{11} + s_{12} + s_{21}, e - f, e - f, \cdots, e - f\right))$ in two ways we obtain,$p_n(z, e-f, e-f, \cdots, e-f) =$

0. This implies that $eze = z = 0$, i.e, $\psi_m(s_{11}+s_{12}+s_{21}) = \psi_m(s_{11}) + \psi_m(s_{12}+s_{21})$. Hence, we are done.

Now by taking $g = \psi_m(s_{22}+s_{12}+s_{21}) - \psi_m(s_{22}) - \psi_m(s_{12}+s_{21})$ and calculating $\psi_m(p_n(g, e, e, \cdots, e))$ and $\psi_m(p_n(g, f-e, f-e, \cdots, f-e))$ in two ways, one can show that $g = 0$, i.e, $\psi_m(s_{22}+s_{12}+s_{21}) = \psi_m(s_{22}) + \psi_m(s_{12}+s_{21})$. Hence proves the claim. □

Claim 2.10. *For all* $s_{ij} \in \mathcal{R}_{ij}, (1 \leq i \neq j \leq 2)$*, we have* $\psi_m(s_{11}+s_{12}+s_{21}+s_{22}) = \psi_m(s_{11}) + \psi_m(s_{12}+s_{21}) + \psi_m(s_{22})$.

Proof. Write $z = \psi_m(s_{11}+s_{12}+s_{21}+s_{22}) - \psi_m(s_{11}) - \psi_m(s_{12}+s_{21}) - \psi_m(s_{22})$ and since $p_n(s_{22}, e, e, \cdots, e) = 0$.

By using claim 2.9, we have

$$\begin{aligned}
&\psi_m(p_n(s_{11}+s_{12}+s_{21}+s_{22}, e, e, \cdots, e)) \\
&\quad = p_n(\psi_m(s_{11}+s_{12}+s_{21}+s_{22}), e, e, \cdots, e) \\
&\quad + p_n(s_{11}+s_{12}+s_{21}+s_{22}, \psi_m(e), e, \cdots, e) + \cdots \\
&\quad + p_n(s_{11}+s_{12}+s_{21}+s_{22}, e, e, \cdots, \psi_m(e)) + \\
&\quad \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11}+s_{12}+s_{21}+s_{22}), \psi_{i_2}(e), \cdots, \psi_{i_n}(e)) \\
&\quad = p_n(\psi_m(s_{11}+s_{12}+s_{21}+s_{22}), e, e, \cdots, e) + \\
&\quad \sum_{i=2}^{n} p_n(s_{11}+s_{12}+s_{21}+s_{11}, e, e, \cdots, \psi_m(e)_i, \cdots, e) + \\
&\quad \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11}+s_{12}+s_{21}+s_{22}), \psi_{i_2}(e, \cdots, \psi_{i_n}(e))) \qquad (2.14)
\end{aligned}$$

Also,

$$\begin{aligned}
&\psi_m(p_n(s_{11}+s_{12}+s_{21}+s_{22}, e, e, \cdots, e)) \\
&\quad = \psi_m(p_n(s_{11}+s_{12}+s_{21}, e, e, \cdots, e)) + \psi_m(p_n(s_{22}, e, e, \cdots, e)) \\
&\quad = \psi_m(p_n(s_{11}, e, e, \cdots, e)) + \psi_m(p_n(s_{12}+s_{21}, e, e, \cdots, e)) \\
&\quad + \psi_m(p_n(s_{22}, e, e, \cdots, e)) \\
&\quad = p_n(\psi_m(s_{11}), e, e, \cdots, e) + \sum_{i=2}^{n} p_n(s_{11}, e, e, \cdots, \psi_m(e)_i, \cdots, e) + \\
&\quad p_n(\psi_m(s_{12}+s_{21}), e, e, \cdots, e) + \sum_{i=2}^{n} p_n(s_{12}+s_{21}, e, e, \cdots, \psi_m(e)_i, \cdots, e) \\
&\quad + p_n(\psi_m(s_{22}), e, e, \cdots, e) + \sum_{i=2}^{n} p_n(s_{22}, e, e, \cdots, \psi_m(e)_i, \cdots, e) + \\
&\quad \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leq i_1,i_2,\cdots,i_n\leq m-1}} p_n(\psi_{i_1}(s_{11}+s_{12}+s_{21}+s_{22}), \psi_{i_2}(e), \cdots, \psi_{i_n}(e)) \\
&= p_n(\psi_m(s_{11}), e, e, \cdots, e) + p_n(\psi_m(s_{12}+s_{21}), e, e, \cdots, e) +
\end{aligned}$$

$$
\begin{aligned}
&p_n(\psi_m(s_{22}), e, e, \cdots, e)+\\
&\sum_{i=2}^{n} p_n(s_{11}+s_{12}+s_{21}+s_{11}, e, e, \cdots, \psi_m(e)_i, \cdots, e)+\\
&\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(s_{11}+s_{12}+s_{21}+s_{22}), \psi_{i_2}(e), \cdots, \psi_{i_n}(e)).
\end{aligned} \tag{2.15}
$$

Therefore by comparing equations (0.15) and (0.16), we get, $p_n(z, e, e, \cdots, e) = 0$, Which gives $eze = ezf = fze = 0$. Similarly by calculating, $\psi_m(p_n(s_{11}+s_{12}+s_{21}+s_{22}, f, f, \cdots, f))$ in two ways gives $fzf = 0$. Which implies $z = 0$. i.e, $\psi_m(s_{11}+s_{12}+s_{21}+s_{22}) = \psi_m(s_{11})+\psi_m(s_{12}+s_{21})+\psi_m(s_{22})$. □

Claim 2.11. *For any* $s_{12} \in \mathcal{R}_{12}, s_{21} \in \mathcal{R}_{21})$*, we have* $\psi_m(s_{12}+s_{21}) = \psi_m(s_{12}) + \psi_m(s_{21})$.

Proof. To Prove the claim we use the principle of mathematical induction. For $m = 1$ it is true, now suppose that it holds for all $k < m$ i.e, $\psi_k(s_{12}+s_{21}) = \psi_k(s_{12}) + \psi_k(s_{21})$. we will prove that it is also true for all $m \in \mathbb{N}$. Take any $s_{12} \in \mathcal{R}_{12}$ and $s_{21} \in \mathcal{R}_{21})$. If $n \ge 3$, since $p_n(e+s_{12}, e+s_{21}, e, e, \cdots, e) = s_{12}+s_{21}+2^{n-2}s_{12}s_{21}$. By claim 2.9, one obtains

$$
\begin{aligned}
&\psi_m(s_{12}+s_{21}) + \psi_m(2^{n-2}s_{12}s_{21})\\
&= \psi_m(s_{12}+s_{21}+2^{n-2}s_{12}s_{21})\\
&= \psi_m(p_n(e+s_{12}, f+s_{21}, e, e, \cdots, e))\\
&= p_n(\psi_m(e+s_{12}), f+s_{21}, e, e, \cdots, e) + p_n(e+s_{12}, \psi_m(f+s_{21}), e, e, \cdots, e)\\
&+ \sum_{i=3}^{n} p_n(e+s_{12}, f+s_{21}, e, e, \cdots, \psi_m(e)_i, \cdots, e)\\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\le i_1,i_2,\cdots,i_n\le m-1}} p_n(\psi_{i_1}(e+s_{12}), \psi_{i_2}(f+s_{21}), \psi_{i_3}(e), \psi_{i_4}(e), \cdots, \psi_{i_n}(e))\\
&= \psi_m(p_n(e, f, e, e, \cdots, e)) + \psi_m(p_n(s_{12}, f, e, e, \cdots, e))\\
&+ \psi_m(p_n(e, s_{21}, e, e, \cdots, e)) + \psi_m(p_n(s_{12}, s_{21}, e, e, \cdots, e))\\
&= \psi_m(s_{12}) + \psi_m(s_{21}) + \psi_m(2^{n-2}s_{12}s_{21})
\end{aligned}
$$

i.e, $\psi_m(s_{12}+s_{21}) = \psi_m(s_{12}) + \psi_m(s_{21})$. Which proves the claim for $n \ge 3$.

Now if $n = 2$, by claim 2.10, one obtains

$$
\begin{aligned}
\psi_m(s_{12}+s_{21}) + \psi_m(s_{12}s_{21}) + \psi_m(s_{21}s_{12})
&= \psi_m(s_{12}+s_{21}+s_{12}s_{21}+s_{21}s_{12})\\
&= \psi_m(p_2(e+s_{12}, f+s_{21}))\\
&= p_2(\psi_m(e+s_{12}), f+s_{21}) + p_2(e+s_{12}, \psi_m(f+s_{21}))\\
&+ \sum_{\substack{i_1+i_2=m\\ 0\le i_1,i_2\le m-1}} p_2(\psi_{i_1}(e+s_{12}), \psi_{i_2}(f+s_{21}))\\
&= p_2(\psi_m(e)+\psi_m(s_{12}), f+s_{21}) + p_2(e+s_{12}, \psi_m(f)+\psi_m(s_{21}))\\
&+ \sum_{\substack{i_1+i_2=m\\ 0\le i_1,i_2\le m-1}} p_2(\psi_{i_1}(e)+\psi_{i_1}(s_{12}), \psi_{i_2}(f)+\psi_{i_2}(s_{21}))
\end{aligned}
$$

$$
\begin{aligned}
&= p_2(\psi_m(e), f) + p_2(\psi_m(e), s_{21}) + p_2(\psi_m(s_{12}), f) + p_2(\psi_m(s_{12}), s_{21}) \\
&+ p_2(e, \psi_m(f)) + p_2(e, \psi_m(s_{21})) + p_2(s_{12}, \psi_m(f)) + p_2(s_{12}, \psi_m(s_{21})) \\
&+ \sum_{\substack{i_1+i_2=m \\ 0\leq i_1, i_2 \leq m-1}} p_2(\psi_{i_1}(e), \psi_{i_2}(f)) + \sum_{\substack{i_1+i_2=m \\ 0\leq i_1, i_2 \leq m-1}} p_2(\psi_{i_1}(e), \psi_{i_2}(s_{21})) \\
&+ \sum_{\substack{i_1+i_2=m \\ 0\leq i_1, i_2 \leq m-1}} p_2(\psi_{i_1}(s_{12}), \psi_{i_2}(s_{21})) + \sum_{\substack{i_1+i_2=m \\ 0\leq i_1, i_2 \leq m-1}} p_2(\psi_{i_1}(s_{12}), \psi_{i_2}(f)) \\
&= \psi_m(p_2(e, f)) + \psi_m(p_2(e, s_{21})) + \psi_m(p_2(s_{12}, f)) + \psi_m(p_2(s_{12}, s_{21})) \\
&= \psi_m(s_{12}) + \psi_m(s_{21}) + \psi_m(s_{12}s_{21}) + \psi_m(s_{21}s_{12}),
\end{aligned}
$$

i.e, $\psi_m(s_{12} + s_{21}) = \psi_m(s_{12}) + \psi_m(s_{21})$. Which implies that result also holds for $n = 2$. Hence for all $m \in \mathbb{N}$,

$$\psi_m(s_{12} + s_{21}) = \psi_m(s_{12}) + \psi_m(s_{21}).$$

□

Claim 2.12. *$\psi_m(s + t) = \psi_m(s) + \psi_m(t)$ holds for all $s, t \in \mathcal{R}$.*

Proof. For any $s = s_{11} + s_{12} + s_{21} + s_{22},\ t = t_{11} + t_{12} + t_{21} + t_{22} \in \mathcal{R}$, by claim 2.10 and claim 2.11, we have

$$
\begin{aligned}
\psi_m(s + t) &= \psi_m(s_{11} + s_{12} + s_{21} + s_{22} + t_{11} + t_{12} + t_{21} + t_{22}) \\
&= \psi_m(s_{11} + t_{11}) + \psi_m(s_{12} + t_{12}) + \psi_m(s_{21} + t_{21}) + \psi_m(s_{22} + t_{22}) \\
&= \psi_m(s_{11}) + \psi_m(t_{11}) + \psi_m(s_{12}) + \psi_m(t_{12}) + \psi_m(s_{21}) + \psi_m(t_{21}) \\
&+ \psi_m(s_{22}) + \psi_m(t_{22}) \\
&= \psi_m(s_{11} + s_{12} + s_{21} + s_{22}) + \psi_m(t_{11} + t_{12} + t_{21} + t_{22}) \\
&= \psi_m(s) + \psi_m(t).
\end{aligned}
$$

Which completes the proof of Theorem. □

□

In the sequel, we present some applications of multiplicative Jordan n-higher derivations to some unital rings: triangular rings, nest algebra,upper triangular block matrix algebra, prime rings, von Neumann algebras.

2.1 Triangular rings

Suppose $\mathcal{H}$ and $\mathcal{K}$ be any unital rings over a commutative ring $\mathcal{R}$ and $\mathcal{S}$ be a faithful $(\mathcal{H}, \mathcal{K})$-bimodule. Then the $\mathcal{R}$-ring $\mathcal{T} = Tri(\mathcal{H}, \mathcal{S}, \mathcal{K}) = \left\{ \begin{pmatrix} h & s \\ o & k \end{pmatrix} : h \in \mathcal{H}, s \in \mathcal{S}, k \in \mathcal{K} \right\}$ under usual matrix operations is called a triangular ring and the idempotent element $\mathcal{E} = \begin{pmatrix} \mathcal{I}_{\mathcal{H}} & O \\ o & O \end{pmatrix}$ is called the standard idempotent of the triangular ring $\mathcal{T}$, where $\mathcal{I}$ and $\mathcal{I}_{\mathcal{H}}$ are units of $\mathcal{T}$ and $\mathcal{H}$ respectively. By definition, triangular rings are unital rings containing non-trivial idempotent $\mathcal{E}$ and satisfying (1.3). Then using Theorem 2.1, we have the following result:

Corollary 2.1. *Let $\mathcal{T} = Tri(\mathcal{H}, \mathcal{S}, \mathcal{K})$ be a triangular ring with characteristic not equal to 2. Then every multiplicative jordan n-higher derivation on $\mathcal{T}$ is additive.*

2.2 Nest algebra

A nest $\mathcal{N}$ on a banach space $\mathcal{X}$ is a chain of closed subspaces of $\mathcal{X}$, which is closed under the formation of arbitrary closed linear span and intersection, and which includes $\{0\}$ and $\mathcal{X}$. The nest algebra associated with the nest $\mathcal{N}$, denoted by $Alg\mathcal{N}$, is the weakly closed operator algebra consisting of all operators that leave $\mathcal{N}$ invariant. As an application of Theorem 2.1, we have

Corollary 2.2. *Let $\mathcal{N}$ be a nest on a banach space $\mathcal{X}$ over the real or complex field $\mathbb{F}$ and $Alg\mathcal{N}$ be the associated nest algebra. If there exist a non-trivial element in $\mathcal{N}$ that is complemented in $\mathcal{X}$, then every multiplicative Jordan n-higher derivation on $Alg\mathcal{N}$ is additive.*

2.3 Upper triangular block matrix algebras

Let $\mathcal{U}$ be an upper triangular block matrix algebra and $\mathcal{W}_m(\mathbb{F})$ the algebra of all $m \times m$ matrices over a field $\mathbb{F}$. Then $\mathcal{U} = \mathcal{U}(n_1, n_2, \cdots, n_l)$ is a subalgebra of $\mathcal{W}_m(\mathbb{F})$ of all $m \times m$ matrices of the form

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ 0 & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_{kk} \end{pmatrix},$$

where $n_1, n_2, \cdots, n_l$ are finite sequence of positive integers satisfying $n_1 + n_2 + \cdots + n_l = m$ and $B_{ij} \in \mathcal{W}_{n_i \times n_j}(\mathbb{F})$, the space of $n_i \times n_j$ matrices over $\mathbb{F}$. Since, every nest algebra on a finite dimensional space is isomorphic to an upper triangular block matrix algebra. Thus, by Theorem 2.1, we have the following result:

Corollary 2.3. *Let $\mathbb{F}$ be the real or complex field, and $m \geq 1$ be a positive integer. Assume that $\mathcal{U}$ is an upper triangular block matrix algebra. Then every multiplicative Jordan n-higher derivation on $\mathcal{U}$ is additive.*

2.4 Prime rings

A ring $\mathcal{R}$ is said to be a prime ring if for any $a, b \in \mathcal{R}$, $a\mathcal{R}b = \{0\}$ implies $a = 0$ or $b = 0$. Clearly every unital prime ring with a non-trivial idempotent satisfies the condition (1.3). By the application of Theorem 2.1, we have the following result:

Corollary 2.4. *Let $\mathcal{R}$ be a 2 torsion free unital prime ring with a non-trivial idempotent e. Then every multiplicative Jordan n-higher derivation on $\mathcal{R}$ is additive.*

2.5 von Neumannn Algebras

Let $\mathcal{V}$ a von Neumann algebra, then $\mathcal{V}$ is a $\mathbb{C}^*$-subalgebra of some $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. Clearly, if $\mathcal{V}$ has no central summands of type I_1, then $\mathcal{V}$ satisfies the corresponding assumption (1.3)[16]. Also note that $\mathcal{V}$ is semiprime, as any von Neumann algebra is semiprime. Hence by Theorem 2.1, we have the following corollary.

Corollary 2.5. *Let $\mathcal{V}$ a von Neumann algebra without central summands of type I_1. Then every multiplicative Jordan n-higher derivation on $\mathcal{V}$ is additive.*

finally we will

References

[1] D. Benkovič, N. Širovnik, : Jordan derivations of unital algebras with idempotents. Linear Algebra Appl. 437, 2271–2284 (2012)

[2] M. Brešar,: Jordan mappings of semiprime rings. J. Algebra 127, 218–228 (1989)

[3] M. Ferrero and C. Haetinger, *Higher derivations of semiprime rings*, Comm. Algebra, 30 (2002), 2321-2333.

[4] A. Fošner, F. Wei and Z.-K. Xiao, *Nonlinear Lie-type derivations of von Neumann algebras and related topics*, Colloq. Math., **132** (2013), 53-71.

[5] I. N. Herstein, : Jordan derivations of prime rings. Proc. Am. Math. Soc. 8, 1104–1110 (1957)

[6] T.-K. Lee, T.C. Quynh, : Centralizers and Jordan triple derivations of semiprime rings. Commun.Algebra 47, 236–251 (2019)

[7] W.-H. Lin, *Nonlinear ∗-Lie-type derivations on standard operator algebras*,Acta Math. Hungar., **154** (2018), 480-500.

[8] W.-H. Lin,*Nonlinear ∗-Lie-type derivations on von Neumann algebras*,Acta Math. Hungar., https://doi.org/10.1007/s10474-018-0803-1.

[9] C.-J. Li, X.-C. Fang, : Lie triple and Jordan derivable mappings on nest algebras. Linear Multilinear Algebra 61, 653–666 (2013)

[10] J. Li, F.-Y. Lu,: Additive Jordan derivations of reflexive algebras. J. Math. Anal. Appl. 329, 102–111(2007)

[11] F.-Y. Lu,: Jordan derivable maps of prime rings. Commun. Algebra 38, 4430–4440 (2010)

[12] W. S. Martindale III, *When are multiplicative mappings additive ?* , Proc. Amer. Math. Soc., **21** (1969), 695-698.

[13] C. R. Mires, *Lie homomorphisms of operator algebras*, Pacific J. Math., 38 (1971) 717-735.

[14] A. Nowicki, *Inner derivations of higher orders*, Tsukuba J. Math. 8(2) (1984), 219-225.

[15] X.-F. Qi and J.-C. Hou, *Lie higher derivations on nest algebras*, Commun. Math. Res. 26(2) (2010), 131-143.

[16] X. Qi, Z. Guo and T. Zhang, *Characterizing Jordan n-derivation of unital rings containing idempotents*, Bull. of the Iran. Math. Soc. 46, (2020), 1639-1658.

[17] P. Šemrl, *Jordan ∗-derivations of standard operator algebras*, Proc. Amer. Math. Soc., **120** (1994), 515-518.

[18] F. Wei and Z.-K. Xiao, *Higher derivations of triangular algebras and its generalizations*, Linear Algebra Appl. 435 (2011), 1034-1054.

[19] Z.-K. Xiao and F. Wei, *Nonlinear Lie higher derivations on triangular algebras*, Linear Multilinear Algebra 60(8) (2012), 979-994.

[20] F.-F. Zhao and C.-J. Li, *Nonlinear ∗-Jordan triple derivations on von Neumann algebras*, Math. Slovaca, **68** (2018), 163-170.

[21] F.-J. Zhang, *Nonlinear skew Jordan derivable maps on factor von Neumann algebras*, Linear Multilinear Algebra, **64** (2016), 2090-2103.

