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Zak Transform For Groupoids with Abelian Isotropy

Transformée de Zack pour les groupoides à groupe d'isotropie abélien

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ABSTRACT. Let $\mathcal{G}$ be a second countable locally compact Hausdorff groupoid with abelian isotropy groups and $\mathcal{L}$ be a lattice bundle in the isotropy subgroupoid $\mathcal{G}'$ of $\mathcal{G}$. In this paper, we define a Zak transform on $\mathcal{G}$ relatively to $\mathcal{L}$ and study some of its properties. Moreover, we use Zak transform to obtain some characterizations of the cyclic subbundle of the left regular representation of $\mathcal{G}$ restricted to $\mathcal{L}$.
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1 introduction

The Zak transform was first introduced by Gelfand [6] in 1950 on $\mathbb{R}$ for a problem in differential equations. Let $f \in L^2(\mathbb{R})$, the Zak transform of f is the function $Zf : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$Zf(x, y) = \sum_{k=-\infty}^{+\infty} f(x + k)e^{2\pi iyk}$$

It was extended after by Weil [20] in 1964 to locally compact abelian groups with respect to arbitrary closed subgroups. It was rediscovered in 1967 by Zak [23] in quantum mechanic representation. But Zak proved explicitly that its transform was actually a special case of the transform introduced by Weil on locally compact abelian groups. The Zak transform is a particular useful tool in condensed matter physics, signal processing, time-frequency analysis, and harmonic analysis in general. The Zak transform was developed by many authors [8, 11, 12, 14] on locally compact abelian groups. We also have now some generalizations of the notion to locally compact non abelian groups [1, 2, 13]. Groupoids are a generalization of groups. If the notion of group is the mathematical tool to describe the symetries, that of groupoid permits to describe local symetries. The theory of groupoids is becoming more used in many areas of mathematics [3, 4, 16, 17, 19] and theorical physics where group theory is useful. It has many applications for instance in mathematical foundations of classical and quantum mechanics as N. Landsman work [15] shows. In this paper, our purpose is to generalize the notion of Zak transform to groupoids. We actually extend to groupoids some results of [9, 22] using abelian group representations. In fact, we consider a groupoid $\mathcal{G}$ with abelian isotropy group and define the Zak transform by the restriction to a lattice bundle $\mathcal{L}$ of the regular representation of $\mathcal{G}$ on the measurable fields of square integrable functions. All our groupoids are locally compact, second countable, and Hausdorff. The paper is organized as follows. In the second section, we give definitions and notations useful for the understanding of the paper. The purpose of the third section is to introduce first all the necessary materials to define the Zak Transform in the context of groupoids. Then we obtain a groupoid version of elementary properties of Zak transform such as the fact that the Zak transform intertwines the regular representation and the modulation representation. In section 4, we study the properties of cyclic invariant subdundles of the left regular representation of the groupoid generalizing some results of [9, 22]. Cyclic invariant subbundles generalize the notion of cyclic invariant subspaces also known as principal shift invariant subspaces.

2 Notations and preliminaries

We use the notations and setup of this section in the rest of the paper without mentioning. We shall use definition of a locally compact groupoid and the definition of a Haar system on groupoid giving by J. Renault in [18]. Let $\mathcal{G}$ be a second countable locally compact Hausdorff groupoid with a left Haar system $\{\lambda^u, u \in \mathcal{G}^{(0)}\}$. For $x \in \mathcal{G}$, $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$ are respectively the range and the domain of x. For $u, v \in \mathcal{G}^{(0)}$, let us put $\mathcal{G}^u = r^{-1}(u)$, $\mathcal{G}_v = d^{-1}(v)$, $\mathcal{G}_v^u = \mathcal{G}^u \cap \mathcal{G}_v$ and for each unit element u, $\mathcal{G}(u) = \{x \in \mathcal{G} : r(x) = d(x) = u\}$ is the isotropy group at u. The set $\mathcal{G}' = \{x \in \mathcal{G} : r(x) = d(x)\}$ is called the isotropy subgroupoid of $\mathcal{G}$. The groupoid $\mathcal{G}$ is a groupoid with abelian isotropy if for any $u \in \mathcal{G}^{(0)}$, $\mathcal{G}(u)$ is an abelian group. Let μ be a measure on $\mathcal{G}^{(0)}$ and $\nu = \int \lambda^u d\mu(u)$ be the induced measure by μ on $\mathcal{G}$. For $u \in \mathcal{G}^{(0)}$, λ_u will denote the image of λ^u by the inverse map and $\{\lambda_u, u \in \mathcal{G}^{(0)}\}$ is a right Haar system on $\mathcal{G}$. Let $[\mu]$ be the saturation of μ. $[\mu]$ is quasi-invariant for the Haar system $\{\lambda^u, u \in \mathcal{G}^{(0)}\}$. There is a decomposition(see [21]) of ν with respect to the range map to get measures $\nu^u \in \mathcal{G}^u$ such that $\nu = \int \nu^u d[\mu]$. We denote the image of ν^u under inversion as ν_u. $\mathcal{C}_c(\mathcal{G})$ will denote the space of complex-valued continuous functions on $\mathcal{G}$ with compact support endowed with the inductive limit topology and $L^1(\mathcal{G}, \nu)$ the space of ν-integrable functions on $\mathcal{G}$. Let $\mathcal{H}$=$(H_u)_{u \in G^{(0)}}$ be a Hilbert bundle over $\mathcal{G}^{(0)}$ and $\mathcal{U}(\mathcal{H})$ the unitary groupoid of the bundle $\mathcal{H}$. $(\pi, \mathcal{H})$ is a unitary continuous representation of $\mathcal{G}$ if π is a groupoid morphism of $\mathcal{G}$ into the unitary groupoid $\mathcal{U}(\mathcal{H})$ such that for all square integrable sections ξ and η of $\mathcal{H}$, the map $x \mapsto < \pi(x)\xi(d(x)), \eta(r(x)) >$ is continuous. A closed nonzero subbundle $\mathcal{N}$=$(N_u)_{u \in \mathcal{G}^{(0)}}$ of $\mathcal{H}$ (i.e. N_u is a nonzero closed subspace of H_u for each $u \in \mathcal{G}^{(0)}$) is invariant under π if $\pi(x)N_{d(x)} \subset N_{r(x)}$, for each $x \in \mathcal{G}$. If π admits a non trivial closed invariant subbundle $\mathcal{N}$, it is called reducible. Otherwise it is called irreducible. If ξ is a section of $\mathcal{H}$, the subbundle $\mathcal{N}_\xi$ whose leaf at $u \in \mathcal{G}^{(0)}$ is the closed linear span of the set $\{\pi(x)\xi(d(x)) : x \in \mathcal{G}^u\}$ is called the cyclic subbundle generated by ξ. We say that ξ is cyclic if $(\mathcal{N}_\xi)_u$ is dense in H_u, for each $u \in \mathcal{G}^{(0)}$. We designate by $\widehat{\mathcal{G}}$ the set of equivalence classes of unitary irreducible representations of $\mathcal{G}$. It is called the unitary dual of $\mathcal{G}$. Let us recall that if $\mathcal{G}$ is an abelian group then any unitary irreducible representation of $\mathcal{G}$ is one dimensional and so its unitary dual is the set of characters of $\mathcal{G}$. For $f \in L^1(\mathcal{G})$, the Fourier transform is defined by: $\widehat{f}(\omega) = \int_{\mathcal{G}} f(x)\omega(x)dx$ where $\omega \in \widehat{\mathcal{G}}$ and dx is a left Haar measure on $\mathcal{G}$. A sequence $\{e_j : j \in J\}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis if there exists constants $0 < A \leq B < \infty$ such that for any $\{a_j : j \in J\} \in \ell^2(J)$

$$A \sum_{j \in J} |a_j|^2 \leq || \sum_{j \in J} a_j e_j || \leq B \sum_{j \in J} |a_j|^2$$

A sequence $\{e_j : j \in J\}$ in a Hilbert space $\mathcal{H}$ is a frame if there exists constants $0 < A \leq B < \infty$ such that for any $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum | < f, e_j > |^2 \leq A\|f\|^2$$

3 Zak Transform

Let $\mathcal{G}$ be a second countable locally compact Hausdorff groupoid with abelian isotropy. Let $\{\lambda^u, u \in \mathcal{G}^{(0)}\}$ be a left Haar system on $\mathcal{G}$. Let $\mathcal{G}'$ be the isotropy subgroupoid of $\mathcal{G}$ with a Haar system $\{\beta_u, u \in \mathcal{G}^{(0)}\}$. Then $\mathcal{G}'$ is a second countable locally compact Hausdorff subgroupoid which is closed in $\mathcal{G}$(see [7], page 12, Proposition I.33). We will assume that $\mathcal{G}'$ is a group bundle over $\mathcal{G}^{(0)}$, with bundle map $p = d = r$, whose fibres are the isotropy subgroups $\mathcal{G}(u)$. Each isotropy subgroup $\mathcal{G}(u)$ is second countable, so according to Weil's structural Theorem [20], there is a topological discrete subgroup $\mathcal{L}(u)$ of $\mathcal{G}(u)$ such that $T_{\mathcal{L}(u)} = \mathcal{G}(u)/\mathcal{L}(u)$ is compact in the quotient topology. $\mathcal{L}(u)$ is called a lattice. We will assume that $\mathcal{L} = \cup_{u\in\mathcal{G}^{(0)}}\mathcal{L}(u)$ is a group bundle over $\mathcal{G}^{(0)}$ with bundle map $p_{\mathcal{L}} : \mathcal{L} \to \mathcal{G}^{(0)}$ such that $p_{\mathcal{L}}(l) = u$ if $l \in \mathcal{L}(u)$. We will call $\mathcal{L}$ the lattice bundle. We will assume also that $\mathcal{G}'/\mathcal{L}$ is a group bundle. By considering the counting measure on each fibre $\mathcal{L}(u)$, we have a Haar system on $\mathcal{L}$ and it follows that $p_{\mathcal{L}}$ is open. For $u \in \mathcal{G}^{(0)}$, let $\widehat{\mathcal{G}(u)}$ be the dual group of $\mathcal{G}(u)$. We can then define the dual lattice for each $\mathcal{L}(u)$ by:

$$\mathcal{L}(u)^{\perp} = \{\omega \in \widehat{\mathcal{G}(u)} : \forall k \in \mathcal{L}(u), \omega(k) = 1\}$$

and set $\mathcal{L}^{\perp} = \cup_{u\in\mathcal{G}^{(0)}}\mathcal{L}(u)^{\perp}$. We know by Goelhe in [7] that $\widehat{\mathcal{G}'} = \cup_{u\in\mathcal{G}^{(0)}}\widehat{\mathcal{G}(u)}$ is a group bundle called the dual group bundle, where the bundle map $\widehat{p} : \widehat{\mathcal{G}'} \to \mathcal{G}^{(0)}$ is such that $\widehat{p}(\omega) = u$ if $\omega \in \widehat{\mathcal{G}(u)}$. Goelhe, always in [7], proves that $\widehat{\mathcal{G}'}$ is second countable, Hausdorff and locally compact. Since each $\mathcal{L}(u)^{\perp}$ is closed in $\widehat{\mathcal{G}(u)}$ then $\mathcal{L}^{\perp}$ is a bundle where the bundle map is the restriction $\widehat{q} = \widehat{p}|_{\mathcal{L}^{\perp}}$. We will call it the dual lattice bundle. We have $\widehat{\mathcal{G}(u)}/\mathcal{L}(u)^{\perp}$ isomorphic to $\widehat{\mathcal{L}(u)}$(see [10]). Let us denote by $\{\widehat{\beta}^u\}$ the dual Haar system on $\widehat{\mathcal{L}}$ (see definition 2.50. in [7]). We can take $\widehat{\beta}^u$ normalized since each $\widehat{\mathcal{L}(u)}$ is compact. Let us consider the measurable field of Hilbert spaces $\{L^2(\mathcal{G}^u, \nu^u), u \in \mathcal{G}^{(0)}\}$ (resp.$\{L^2(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u), u \in \mathcal{G}^{(0)}\}$). We denote by $L^2(\mathcal{G}, \nu) = \int^{\oplus} L^2(\mathcal{G}^u, \nu^u)d[\mu](u)$ (resp. $L^2(\widehat{\mathcal{L}}, \widehat{\beta}) = \int^{\oplus} L^2(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u)$) the space of square integrable sections for the field $\{L^2(\mathcal{G}^u, \nu^u), u \in \mathcal{G}^{(0)}\}$ (resp.$\{L^2(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u), u \in \mathcal{G}^{(0)}\}$). Let δ be the modular function coming from $[\mu]$. For $\gamma \in \mathcal{G}$,

$$\delta(\gamma) = \frac{d(\gamma.\nu^{d(\gamma)})}{d(\nu^{r(\gamma)})}$$

In particular, if $\gamma \in \mathcal{G}'$ then $\delta(\gamma) = 1$. We denote by D the left regular representation of $\mathcal{G}$ on $L^2(\mathcal{G}, \nu)$. For $\gamma \in \mathcal{G}$, $x \in \mathcal{G}^{r(\gamma)}$ and $f \in L^2(\mathcal{G}^{d(\gamma)}, \nu^{d(\gamma)})$, we have

$$D(\gamma)f(x) = \delta(\gamma)^{\frac{1}{2}}f(\gamma^{-1}x)$$

For $l \in \mathcal{L}$ and $\phi \in L^2(\widehat{\mathcal{L}(p_{\mathcal{L}}(l))}, \widehat{\beta}^{p_{\mathcal{L}}(l)})$ we define an operator $M(l)\phi$ by

$$M(l)\phi(\omega) = \omega(l)\phi(\omega)$$

where $\omega \in \widehat{\mathcal{L}}(\widehat{p}(l))$. The map M defines a unitary representation of $\mathcal{L}$ on $L^2(\widehat{\mathcal{L}}, \widehat{\beta})$ which is called the modulation representation. For any $u \in \mathcal{G}^{(0)}$, since $\widehat{\mathcal{L}(u)}$ is compact, we have(see [5]) an orthonormal set $\{e_l; l \in \mathcal{L}(u)\}$ in $L^2(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u)$ defined by $e_l(\omega) = \omega(l)$. Now, we are ready to give our definition of Zak transform.

Definition 3.1. For $\psi \in L^2(\mathcal{G}, \nu)$, the Zak transform of ψ is a section $Z\psi$ of the bundle of functions $\{\mathcal{G}^u \to L^2(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u); u \in \mathcal{G}^{(0)}\}$ defined by $Z\psi(u) := Z_u\psi(u)$ where

$$Z_u\psi(u)(x, \omega) = \sum_{l\in\mathcal{L}(u)} (D(l)\psi(u))(x)\omega(l^{-1})$$

The following results extend some well-known properties to groupoids setting.

Theorem 3.2. *For $\psi \in L^2(\mathcal{G}, \nu)$, $u \in \mathcal{G}^{(0)}$,*

$$Z_u\psi(u)(kx, \omega\theta) = \omega(k^{-1})Z_u\psi(u)(x, \omega),$$
$$\forall (x, k) \in \mathcal{G}^u \times \mathcal{L}(u), (\omega, \theta) \in \widehat{\mathcal{L}(u)} \times \mathcal{L}(u)^{\perp}$$

Proof.

$$\begin{aligned} Z_u\psi(u)(kx, \omega\theta) &= \sum_{l\in\mathcal{L}(u)} (D(l)\psi(u))(kx)\omega(l^{-1})\theta(l^{-1}) \\ &= \sum_{l\in\mathcal{L}(u)} \psi(u)(l^{-1}kx)\omega(l^{-1}) \\ &= \sum_{l\in\mathcal{L}(u)} \psi(u)(l^{-1}x)\omega(l^{-1}k^{-1}) \\ &=\omega(k^{-1}) \sum_{l\in\mathcal{L}(u)} \psi(u)(l^{-1}x)\omega(l^{-1}) \\ &=\omega(k^{-1})Z_u\psi(u)(x, \omega) \end{aligned}$$

□

Theorem 3.3. *The Zak transform Z intertwines the restriction of the representation D to $\mathcal{L}$ and the modulation representation M of $\mathcal{L}$ on the image under Z of $L^2(\mathcal{G}, \nu)$.*

Proof. For $l \in \mathcal{L}$ and $\psi \in L^2(\mathcal{G}, \nu)$, we have

$$\begin{aligned} Z_{p_{\mathcal{L}}(l)}(D(l)\psi(p_{\mathcal{L}}(l))(x, \omega) &= \sum_{k\in\mathcal{L}(p(l))} D(k)(D(l)\psi(p_{\mathcal{L}}(l)))(x)\omega(k^{-1}) \\ &= \sum_{k\in\mathcal{L}(p(l))} (D(l)\psi(p_{\mathcal{L}}(l)))(k^{-1}x)\omega(k^{-1}) \\ &= \sum_{k\in\mathcal{L}(p(l))} \psi(p_{\mathcal{L}}(l))(l^{-1}k^{-1}x)\omega(k^{-1}) \\ &= \omega(l) \sum_{k\in\mathcal{L}(p(l))} \psi(p_{\mathcal{L}}(l))(k^{-1}x)\omega(k^{-1}) \\ &= \omega(l) \sum_{k\in\mathcal{L}(p(l))} (D(k)\psi(p_{\mathcal{L}}(l)))(x)\omega(k^{-1}) \\ &= \omega(l)(Z_{p_{\mathcal{L}}(l)}\psi(p_{\mathcal{L}}(l)))(x, \omega) \\ &= M(l)(Z_{p_{\mathcal{L}}(l)}\psi(p_{\mathcal{L}}(l)))(x, \omega) \end{aligned}$$

□

Now, we denote by $\mathcal{M}$ the Hilbert space of sections Φ on $\mathcal{G}^{(0)}$ of the bundle of functions $\{F(u) : \mathcal{G}^u \times L^2(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u) \to \mathbb{C}; u \in \mathcal{G}^{(0)}\}$ which intertwines the resrtictions to $\mathcal{L}$ of representations D and M and $||\Phi||^2_{\mathcal{M}} = \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}^u} \int_{\widehat{\mathcal{L}(u)}} |\Phi(u)(x, \omega)|^2 d\widehat{\beta}^u(\omega) d\nu^u(x) d[\mu](u) < \infty$.

Theorem 3.4. *The Zak transform is an isometry from $L^2(\mathcal{G},\nu)$ onto the Hilbert space $\mathcal{M}$.*

Proof. For $\psi \in L^2(\mathcal{G},\nu)$ and an orbit-cross section set $\mathcal{C} \subset \mathcal{G}$ for the natural action of $\mathcal{L}$ on $\mathcal{G}$. $\mathcal{C}^u = \mathcal{C} \cap \mathcal{G}^u$.

$$\begin{aligned}
||Z\psi||^2_{\mathcal{M}} &= \int\limits_{\mathcal{G}^{(0)}} \int\limits_{\mathcal{C}^u} \int\limits_{\widehat{\mathcal{L}}(u)} |Z_u\Psi(u)(x,\omega)|^2 d\widehat{\beta}^u(\omega) d\nu^u(x) d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} \int\limits_{\mathcal{C}^u} \int\limits_{\widehat{\mathcal{L}}(u)} |\sum_{l\in\mathcal{L}(u)} \psi(u)(l^{-1}x)\omega(l^{-1})|^2 d\widehat{\beta}^u(\omega) d\nu^u(x) d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} \int\limits_{\mathcal{C}^u} \sum_{l\in\mathcal{L}(u)} |\psi(u)(l^{-1}x)|^2 d\nu^u(x) d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} \sum_{l\in\mathcal{L}(u)} \int\limits_{l.\mathcal{C}^u} |\psi(u)(x)|^2 d\nu^u(x) d[\mu](u) = ||\psi||^2_{L^2(\mathcal{G},\nu)}
\end{aligned}$$

□

We achieve this section by an application to the groupoid action $\mathcal{G} = \mathbb{R} \propto S^1$. $\mathbb{R}$ is the set of real numbers and S^1 the unit circle. Let us consider the action of $\mathbb{R}$ on S^1 defined by: $t.s = e^{2\pi i t}s$ for $t \in \mathbb{R}, s \in S^1$. So $\mathcal{G}$ is defined by:

$$d(t,s) = s; r(t,s) = t.s; (t, t'.s')(t', s') = (t+t', s'); (t,s)^{-1} = (-t, t.s)$$

$\mathcal{G}^{(0)}$ is identified with S^1. We equip $\mathcal{G}$ with the Haar system $\nu = \{\lambda \times \delta_s, s \in S^1\}$, where λ is the Lebesgue measure on $\mathbb{R}$ and δ_s the point mass at s. For each $s \in S^1$, $\mathcal{G}(s)$ is the stabilizer subgroup of s which is here $\mathbb{Z}$, thus we can take $\mathcal{L}(s) = \mathbb{Z}$ as lattice in each $\mathcal{G}(s)$. The unitary dual is then $\widehat{\mathcal{L}(u)} = \widehat{\mathbb{Z}}$ which is isomorphic to $\mathbb{T}$, the one dimensional torus and consequently to the unit circle S^1. For any $s \in S^1$, the corresponding character is defined by $\omega_s(k) = e^{2\pi i k.s}$ for $k \in \mathbb{Z}$ and $k.s$ designates the scalar product in $\mathbb{C}$. Now for any $s \in S^1$, we have

$$Z_s\psi(s)(t, t^{-1}s, s') = \sum_{k\in\mathbb{Z}} (D(k)\psi(s))(t, t^{-1}s)\omega_{s'}(-k) = \sum_{k\in\mathbb{Z}} \psi(s)(t-k, t^{-1}s)e^{-2\pi i k.s'}$$

4 Properties of the D-cyclic subbundle $\mathcal{H}_\psi$

For $\psi \in L^2(\mathcal{G},\nu) \setminus \{0\}$, we denote by $\mathcal{H}_\psi$ the cyclic subbundle generated by ψ whose leaf at $u \in \mathcal{G}^{(0)}$ is the closed linear span of the set $\{D(l)\psi(d(l)) : l \in \mathcal{L}(u)\}$. Let $L^1(\widehat{\mathcal{L}}, \widehat{\beta})$ be the space of integrable sections of the bundle $\{L^1(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u), u \in \mathcal{G}^{(0)}\}$ where $L^1(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u)$ is the space of the class of integrable functions on $\widehat{\mathcal{L}(u)}$. Let $\phi, \psi \in L^2(\mathcal{G},\nu)$, we define the following bracket at $u \in \mathcal{G}^{(0)}$ by $[\phi,\psi](u) = [\phi(u), \psi(u)]$ where

$$[\phi(u), \psi(u)](\omega) = \int\limits_{\mathcal{C}^u} Z_u\phi(u)(x,\omega)\overline{Z_u\psi(u)(x,\omega)} d\nu^u(x)$$

It is clear that $[\phi,\psi] \in L^1(\widehat{\mathcal{L}}, \widehat{\beta})$.

Remark 4.1. The map $(\phi, \psi) \mapsto [\phi, \psi]$ is bounded, sesquilinear, Hermitian symmetric from $L^2(\mathcal{G}, \nu) \times L^2(\mathcal{G}, \nu)$ to $L^1(\widehat{\mathcal{L}}, \widehat{\beta})$ and it is positive semi-definite that is $[\psi, \psi] \geq 0$, in other words for each $u \in \mathcal{G}^{(0)}, \omega \in L^1(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u), [\psi, \psi](u)(\omega) \geq 0$.

We denote by $\mathcal{S}_{\psi(u)}$ the set defined by $\mathcal{S}_{\psi(u)} = \{\omega \in L^1(\widehat{\mathcal{L}(u)}, \widehat{\beta}^u) : [\psi, \psi](u)(\omega) \neq 0\}$ and by $\mathcal{S}_\psi$ the bundle $\{\mathcal{S}_{\psi(u)}, u \in \mathcal{G}^{(0)}\}$. The following results generalize in groupoids setting some well-known results.

Theorem 4.2. *For $\phi, \psi \in L^2(\mathcal{G}, \nu) \setminus \{0\}$ and $l \in \mathcal{L}$, we have*

(i) $[D(l)\phi(d(l)), \psi(d(l))] = [\phi(d(l)), D(l^{-1})\psi(d(l))] = e_l[\phi(d(l)), \psi(d(l))]$;

(ii)

$$< D(l)\phi(d(l)), \psi(d(l)) >_{L^2} = \int_{\widehat{\mathcal{L}}(d(l))} \omega(l)[\phi(d(l)), \psi(d(l))](\omega) d\widehat{\beta}^{d(l)}(\omega)$$

where L^2 here designates $L^2(\mathcal{G}(d(l)), \nu^{d(l)})$;

(iii) $\mathcal{H}_\phi$ *is orthogonal to* $\mathcal{H}_\psi$ *iff* $[\phi, \psi] = 0$ *a.e.;*

(iv) The restriction to $\mathcal{H}_\psi$ of the map $\mathcal{J}_\psi : \phi \to \frac{[\phi,\psi]}{[\psi,\psi]^{\frac{1}{2}}} \chi_{\mathcal{S}_\psi}$ is unitary onto $L^2(\mathcal{S}_\psi, \widehat{\beta})$.

Proof.

(i) Using the Theorem 2.2, we have

$$\begin{aligned}
[D(l)\phi(d(l)), \psi(d(l))](\omega) &= \int_{\mathcal{C}^{d(l)}} Z_{d(l)} D(l)\phi(d(l))(x, \omega) \overline{Z_{d(l)}\psi(d(l))(x, \omega)} d\nu^{d(l)}(x) \\
&= \int_{\mathcal{C}^{d(l)}} M(l) Z_{d(l)}\phi(d(l))(x, \omega) \overline{Z_{d(l)}\psi(d(l))(x, \omega)} d\nu^{d(l)}(x) \\
&= \int_{\mathcal{C}^{d(l)}} \omega(l) Z_{d(l)}\phi(d(l))(x, \omega) \overline{Z_{d(l)}\psi(d(l))(x, \omega)} d\nu^{d(l)}(x) \\
&= \omega(l)[\phi(d(l)), \psi(d(l))](\omega) \\
&= \int_{\mathcal{C}^{d(l)}} Z_{d(l)}\phi(d(l))(x, \omega) \overline{\omega(l^{-1}) Z_{d(l)}\psi(d(l))(x, \omega)} d\nu^{d(l)}(x) \\
&= \int_{\mathcal{C}^{d(l)}} Z_{d(l)}\phi(d(l))(x, \omega) \overline{M(l^{-1}) Z_{d(l)}\psi(d(l))(x, \omega)} d\nu^{d(l)}(x) \\
&= \int_{\mathcal{C}^{d(l)}} Z_{d(l)}\phi(d(l))(x, \omega) \overline{Z_{d(l)} D(l^{-1})\psi(d(l))(x, \omega)} d\nu^{d(l)}(x) \\
&= [\phi(d(l)), D(l^{-1})\psi(d(l))](\omega)
\end{aligned}$$

(ii) Using (i) and the fact that the Zak transform is an isometry, we have

$$< D(l)\phi(d(l)), \psi(d(l)) >_{L^2} = < Z_{d(l)}D(l)\phi(d(l)), Z_{d(l)}\psi(d(l)) >$$

$$= \int\limits_{\mathcal{C}^{d(l)}} \int\limits_{L^2(\widehat{\mathcal{G}^{d(l)}},\beta^{d(l)})} Z_{d(l)}D(l)\phi(d(l))(x,\omega)\times \overline{Z_{d(l)}\psi(d(l))}(x,\omega)d\beta^{d(l)}(\omega)d\nu^{d(l)}(x)$$

$$= \int\limits_{L^2(\widehat{\mathcal{G}^{d(l)}},\beta^{d(l)})} [D(l)\phi(d(l)), \psi(d(l))](\omega)d\beta^{d(l)}(\omega)$$

$$= \int\limits_{L^2(\widehat{\mathcal{G}^{d(l)}},\beta^{d(l)})} \omega(l)[\phi(d(l)), \psi(d(l))](\omega)d\beta^{d(l)}(\omega)$$

(iii) Using (ii), We have for $k, l \in \widehat{\mathcal{L}}(u)$

$$< D(l)\phi(u), D(k)\psi(u) > = < D(k^{-1}l)\phi(u), \psi(u) >$$

$$= \int\limits_{\widehat{\mathcal{L}}(u)} \omega(k^{-1}l)[\phi(u), \psi(u)](\omega)d\widehat{\beta}^u(\omega)$$

So, $< D(l)\phi(u), D(k)\psi(u) > = 0$ if and only if $\int_{\widehat{\mathcal{L}}(u)} \omega(k^{-1}l)[\phi(u), \psi(u)](\omega)d\widehat{\beta}^u(\omega) = 0$ and by the unicity of the Fourier transform on $\widehat{\mathcal{L}}(u)$, $[\phi(u), \psi(u)] = 0$ for a.e.

(iv) Let us first show that the bracket $[,]$ verifies that $|[\phi, \psi]| \leq [\phi, \phi]^{\frac{1}{2}}[\psi, \psi]^{\frac{1}{2}}$. In fact, for any $u \in \mathcal{G}^{(0)}$ and $\omega \in \widehat{\mathcal{L}}(u)$

$$|[\phi(u), \psi(u)](\omega)| = |\int\limits_{\mathcal{C}^u} Z_u\phi(u)(x,\omega)\overline{Z_u\psi(u)(x,\omega)}d\nu^u(x)|$$

$$= |< Z_u\phi(u)(.,\omega), Z_u\psi(u)(.,\omega) >_{L^2(\mathcal{G}^u,\nu^u)} |$$

$$\leq \|Z_u\phi(u)(.,\omega)\|\|Z_u\psi(u)(.,\omega)\|$$

$$= (\int\limits_{\mathcal{C}^u} Z_u\phi(u)(x,\omega)\overline{Z_u\phi(u)(x,\omega)}d\nu^u(x))^{\frac{1}{2}}\times (\int\limits_{\mathcal{C}^u} Z_u\psi(u)(x,\omega)\overline{Z_u\psi(u)(x,\omega)}d\nu^u(x))^{\frac{1}{2}}$$

$$= [\phi(u), \phi(u)]^{\frac{1}{2}}(\omega)[\psi(u), \psi(u)]^{\frac{1}{2}}(\omega)$$

where the inequality of the third line is due to the Cauchy-Schwartz inegality in $L^2(\mathcal{G}^u, \nu^u)$. Let us notice that this property of the bracket shows that $[\phi(u), \psi(u)](\omega) = 0$ when ω is off $\mathcal{S}_\psi$. Now for $\phi \in L^2(\mathcal{G}, \nu) \setminus \{0\}$, we have

$$\int\limits_{\mathcal{G}^{(0)}} \int\limits_{\widehat{\mathcal{L}}(u)} |\mathcal{J}_\psi(\phi)(u)(\omega)|^2 d\widehat{\beta}^u(\omega)d[\mu](u) = \int\limits_{\mathcal{G}^{(0)}} \int\limits_{\widehat{\mathcal{L}}(u)} |\frac{[\phi(u), \psi(u)]^2(\omega)}{[\psi(u), \psi(u)](\omega)}\chi_{\mathcal{S}_\psi}(\omega)|d\widehat{\beta}^u(\omega)d[\mu](u)$$

$$
\begin{aligned}
&\leq \int\limits_{\mathcal{G}^{(0)}} \int\limits_{\widehat{\mathcal{L}}(u)} \frac{[\phi(u),\phi(u)](\omega)[\psi(u),\psi(u)](\omega)}{[\psi(u),\psi(u)](\omega)} \times \\
&\quad \chi_{\mathcal{S}_{\psi(u)}}(\omega) d\widehat{\beta}^u(\omega) d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} \int\limits_{\mathcal{S}_{\psi(u)}} [\phi(u),\phi(u)](\omega) d\widehat{\beta}^u(\omega) d[\mu](u) \\
&= \|Z\phi\|^2_{\mathcal{M}} = \|\phi\|^2_{L^2(\mathcal{G},\nu)}
\end{aligned}
$$

Thus, we have shown that $\mathcal{J}_\psi$ maps $L^2(\mathcal{G},\nu)$ into $L^2(\widehat{\mathcal{L}},\widehat{\beta})$. It is clear that $\mathcal{J}_\psi$ is linear. We have thanks to (i)

$$
\frac{[D(l)\psi(d(l)),\psi(d(l))](\omega)}{[\psi(d(l)),\psi(d(l))]^{\frac{1}{2}}(\omega)} = \frac{e_l(\omega)[\psi(d(l)),\psi(d(l))](\omega)}{[\psi(d(l)),\psi(d(l))]^{\frac{1}{2}}(\omega)} = e_l(\omega)[\psi(d(l)),\psi(d(l))]^{\frac{1}{2}}(\omega)
$$

which shows that $\mathcal{J}_\psi(\mathcal{H}_\psi) \subset L^2(\mathcal{S}_\psi,\widehat{\beta})$. Also for $u \in \mathcal{G}^{(0)}$ and $\phi_1(u) = \sum_{l\in\mathcal{L}(u)} a_l D(l)\psi(d(l))$, $\phi_2(u) = \sum_{k\in\mathcal{L}(u)} b_k D(k)\psi(d(k))$, the sum are finite.

$$
\begin{aligned}
< \mathcal{J}_\psi(\phi_1), \mathcal{J}_\psi(\phi_2) > &= \int\limits_{\mathcal{G}^{(0)}} < \mathcal{J}_\psi(\phi_1)(u), \mathcal{J}_\psi(\phi_2)(u) > d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} \int\limits_{\widehat{\mathcal{L}}(u)} \mathcal{J}_\psi(\phi_1)(u)(\omega)\overline{\mathcal{J}_\psi(\phi_2)(u)(\omega)} d\widehat{\beta}^u(\omega) d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} \int\limits_{\widehat{\mathcal{L}}(u)} \sum a_l \overline{b_k} e_l(\omega)\overline{e_k(\omega)}[\psi(u),\psi(u)](\omega) d\widehat{\beta}^u(\omega) d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} \sum a_l \overline{b_k} (\int\limits_{\widehat{\mathcal{L}}(u)} e_{k^{-1}l}(\omega)[\psi(u),\psi(u)](\omega) d\widehat{\beta}^u(\omega)) d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} \sum a_l \overline{b_k} < D(k^{-1}l)\psi(u),\psi(u) > d[\mu](u) \\
&= \int\limits_{\mathcal{G}^{(0)}} < \phi_1(u), \phi_2(u) > d[\mu](u) = < \phi_1, \phi_1 >
\end{aligned}
$$

Since this finite sums are dense in $(\mathcal{H}_\psi)_u$, we have proved the unitarity. □

Remark 4.3. $\mathcal{J}_\psi$ intertwines the left regular representation D on $\mathcal{H}_\psi$ with the modulation representation of $\mathcal{L}$ on $L^2(\mathcal{S}_\psi,\widehat{\beta})$. In fact,

$$
\begin{aligned}
\mathcal{J}_\psi(D(l)\psi)(d(l))(\omega) &= e_l(\omega)[\psi(d(l)),\psi(d(l))]^{\frac{1}{2}}(\omega)\chi_{\mathcal{S}_\psi}(\omega) \\
&= e_l(\omega)\mathcal{J}_\psi(\psi(d(l)))(d(l))(\omega) = M(l)\mathcal{J}_\psi(\psi)(d(l))(\omega)
\end{aligned}
$$

Corollary 4.4. *For each $\psi \in L^2(\mathcal{G},\nu)\setminus\{0\}$ and $u \in \mathcal{G}^{(0)}$, the set $\mathcal{A}_{\psi_u} = \{D(l)\psi(d(l)) : l \in \mathcal{L}(u)\}$ spanning $(\mathcal{H}_\psi)_u$ is*

(i) an orthonormal basis if and only if $[\psi,\psi] = 1$ a.e.

(ii) a Riesz basis for $(\mathcal{H}_\psi)_u$ if and only if $||[\psi(u),\psi(u)]||_\infty$ and $\frac{1}{||[\psi(u),\psi(u)]||_\infty}$ are finite.

(iii) a frame if and only if there exists non-negative constants A and B such that $A\chi_{\mathcal{S}_{\psi(u)}} \leq [\psi(u),\psi(u)] \leq B\chi_{\mathcal{S}_{\psi(u)}}$ a.e.

Proof.

(i) For $l \in \mathcal{L}(u)$, we have thanks to Theorem 3.2 (ii)

$$< D(l)\psi(u),\psi(u) >= \int_{\widehat{\mathcal{L}}(u)} e_l(\omega)[\psi(u),\psi(u)](\omega)d\widehat{\beta}^u(\omega)$$

If $\mathcal{A}_{\psi_u}$ is orthonormal then $< D(l)\psi(u),\psi(u) >=< D(l)\psi(u),D(u)\psi(u) >= \delta_{lu}$ where δ_{lu} is the symbol of Kronecker. So the integral above is null for all $l \neq u$ and equal 1 for $l = u$. It follows, looking at the integral for various l as Fourier coefficients of $[\psi(u),\psi(u)] \in L^1(\widehat{\mathcal{L}}(u),\widehat{\beta}^u)$, that $[\psi(u),\psi(u)] = 1$ for a.e $\omega \in \widehat{\mathcal{L}}(u)$. Arbitrariness of $u \in \mathcal{G}^{(0)}$ implies that $[\psi,\psi] = 1 \forall u \in \mathcal{G}^{(0)}$ and a.e. for $\omega \in \widehat{\mathcal{L}}(u)$. Conversely if $[\psi,\psi] = 1\forall u \in \mathcal{G}^{(0)}$ and a.e. for $\omega \in \widehat{\mathcal{L}}(u)$ then for any $u \in \mathcal{G}^{(0)}$, we have $< D(l)\psi(d(l)),D(k)\psi(d(k)) >= \int_{\widehat{\mathcal{L}}(u)} e_{k^{-1}l}(\omega)d\widehat{\beta}^u(\omega) =< e_l,e_k >= \delta_{lk}$, since the set $\{e_l : l \in \mathcal{L}(u)\}$ is orthonormal.

(ii) Let us suppose that $\mathcal{A}_{\psi_u}$ is a Riesz basis, then there exists constants $0 < A \leq B < \infty$ such that for any $\{a_l : l \in \mathcal{L}(u)\} \in \ell^2(\mathcal{L}(u))$

$$A\sum_{l\in\mathcal{L}(u)}|a_l|^2 \leq ||\sum_{l\in\mathcal{L}(u)} a_l D(l)\psi(u)\|^2_{L^2(\mathcal{G}^u,\nu^u)} \leq B\sum_{l\in\mathcal{L}(u)}|a_l|^2$$

Since $\mathcal{J}_\psi$ is unitary and

$$\mathcal{J}_\psi(D(l)\psi)(d(l))(\omega) = e_l(\omega)[\psi(d(l)),\psi(d(l))]^{\frac{1}{2}}(\omega)\chi_{\mathcal{S}_{\psi(d(l))}}(\omega)(1)$$

we have

$$A\sum_{l\in\mathcal{L}(u)}|a_l|^2 \leq \|\sum_{l\in\mathcal{L}(u)} a_l e_l[\psi(u),\psi(u)]^{\frac{1}{2}}\chi_{\mathcal{S}_{\psi(u)}}\|^2_{L^2(\mathcal{L}(u),\beta^u)} \leq B\sum_{l\in\mathcal{L}(u)}|a_l|^2$$

So we deduce, since the set $\{e_l : l \in \mathcal{L}(u)\}$ is orthonormal, that

$$A \leq \int_{\widehat{\mathcal{L}}(u)} [\psi(u),\psi(u)](\omega)d\widehat{\beta}^u(\omega) \leq B$$

and consequently $||[\psi(u),\psi(u)]||_\infty$ and $\frac{1}{||[\psi(u),\psi(u)]||_\infty}$ are finite. Conversely if $||[\psi(u),\psi(u)]||_\infty$ and $\frac{1}{||[\psi(u),\psi(u)]||_\infty}$ are finite then there exists some positive constants A and B such that for any $\omega \in \widehat{\mathcal{L}}(u)$

$$A \leq [\psi(u),\psi(u)](\omega) \leq B$$

and it follows that

$$A \leq \int_{\widehat{\mathcal{L}}(u)} [\psi(u),\psi(u)](\omega)d\widehat{\beta}^u(\omega) \leq B$$

Now multiplying each members by $\sum_{l\in\mathcal{L}(u)}|a_l|^2|e_l(\omega)|^2$ and using relation (1) and the unitarity of $\mathcal{J}_\psi$, we have the result.

(iii) Let us suppose that $\mathcal{A}_{\psi_u}$ is a frame with frame bounds $0 < A \leq B < \infty$ such that for any $\phi \in (\mathcal{H}_\psi)_u$

$$A\|\phi\|^2 \leq \sum | < \phi, D(l)\psi(d(l)) > |^2 \leq A\|\phi\|^2$$

Since $\mathcal{J}_\psi$ is unitary, we have

$$A\|\mathcal{J}_\psi(\phi)\|^2 \leq \sum | < \mathcal{J}_\psi(\phi), \mathcal{J}_\psi(D(l)\psi(d(l)) > |^2 \leq A\|\mathcal{J}_\psi(\phi)\|^2$$

In particular, for $\phi = D(k)\psi(d(k))$, we have using relation (1)

$$\begin{aligned} A \int_{\mathcal{S}_{\psi(u)}} [\psi(u), \psi(u)](\omega)d\widehat{\beta}^u(\omega) \leq & (\int_{\mathcal{S}_{\psi(u)}} [\psi(u), \psi(u)](\omega)d\widehat{\beta}^u(\omega))^2 \\ \leq & B \int_{\mathcal{S}_{\psi(u)}} [\psi(u), \psi(u)](\omega)d\widehat{\beta}^u(\omega) \end{aligned}$$

The map $[\psi(u), \psi(u)]$ is continuous and $[\psi(u), \psi(u)] > 0$ on $\mathcal{S}_{\psi(u)}$, so

$$A \leq \int_{\mathcal{S}_{\psi(u)}} [\psi(u), \psi(u)](\omega)d\widehat{\beta}^u(\omega) \leq B$$

and it follows that $A\chi_{\mathcal{S}_{\psi(u)}} \leq [\psi(u), \psi(u)] \leq B\chi_{\mathcal{S}_{\psi(u)}}$. Conversely let us suppose that $A\chi_{\mathcal{S}_{\psi(u)}} \leq [\psi(u), \psi(u)] \leq B\chi_{\mathcal{S}_{\psi(u)}}$ and let $\phi \in (\mathcal{H}_\psi)_u$. Multiplying the inequality by $|\mathcal{J}_\psi(\phi)|^2$ an passing to integral, we obtain

$$A\|\mathcal{J}_\psi(\phi)\|^2 \leq \int_{\widehat{\mathcal{L}}(u)} |\mathcal{J}_\psi(\phi)(\omega)[\psi(u)(\omega), \psi(u)(\omega)]^{\frac{1}{2}}\chi_{\mathcal{S}_{\psi(u)}}(\omega)|^2 d\widehat{\beta}^u(\omega) \leq B\|\mathcal{J}_\psi(\phi)\|^2$$

Now using the Plancherel theorem, we have

$$\begin{aligned} \int_{\widehat{\mathcal{L}}(u)} |\mathcal{J}_\psi(\phi)(\omega)[\psi(u)(\omega), \psi(u)(\omega)]^{\frac{1}{2}}\chi_{\mathcal{S}_{\psi(u)}}(\omega)|^2 d\widehat{\beta}^u(\omega) = & \sum_{l \in \mathcal{L}(u)} \int_{\widehat{\mathcal{L}}(u)} |\mathcal{J}_\psi(\phi)(\omega) \times \\ & [\psi(u)(\omega), \psi(u)(\omega)]^{\frac{1}{2}} e_l(\omega)\chi_{\mathcal{S}_{\psi(u)}}(\omega)|^2 d\widehat{\beta}^u(\omega) \\ & = \sum_{l \in \mathcal{L}(u)} < \mathcal{J}_\psi(\phi), [\psi(u), \psi(u)]^{\frac{1}{2}} e_l \chi_{\mathcal{S}_{\psi(u)}} > \end{aligned}$$

Using then the relation (1) and the unitarity of $\mathcal{J}_\psi(\phi)$, we have

$$\int_{\widehat{\mathcal{L}}(u)} |\mathcal{J}_\psi(\phi)(\omega)[\psi(u)(\omega), \psi(u)(\omega)]^{\frac{1}{2}}\chi_{\mathcal{S}_{\psi(u)}}(\omega)|^2 d\widehat{\beta}^u(\omega) = \sum_{l \in \mathcal{L}(u)} < \phi, D(l)\psi(d(l)) >$$

this completes the proof. $\square$

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