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Constrained Hitting Set and Steiner Tree in SC_k and $2K_2$-free Graphs

Ensemble de Hitting avec contraintes et arbres de Steiner dans les graphes libres de type SC_k et $2K_2$

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ABSTRACT. *Strictly Chordality-k graphs (SC_k)* are graphs which are either cycle-free or every induced cycle is of length exactly $k, k \geq 3$. Strictly chordality-3 and strictly chordality-4 graphs are well known chordal and chordal bipartite graphs, respectively. For $k \geq 5$, the study has been recently initiated in [1] and various structural and algorithmic results are reported. In this paper, we study SC_k graphs in the algorithmic front and the study concerns the class of graphs where $k \geq 5$. We show that recognizing vertex cycle ordering (VCO) for SC_k, $k \geq 5$ graphs, maximum independent set (MIS), minimum vertex cover, minimum dominating set, feedback vertex set (FVS), odd cycle transversal (OCT), even cycle transversal (ECT) and Steiner tree problem are linear time solvable on SC_k graphs, $k \geq 5$. We next consider $2K_2$-free graphs and discussed the algorithmic problems such as FVS, OCT, ECT and Steiner tree problem on the subclasses of $2K_2$-free graphs.

KEYWORDS. Strictly Chordality-k graphs, $2K_2$-free graphs, Feedback Vertex Set, Odd (Even) Cycle Transversal, Steiner tree.

1. Introduction

Strictly Chordality-k graphs (SC_k graphs) are graphs which are either cycle-free or every induced cycle is of length k. This graph class was introduced very recently by Dhanalakshmi et al. in [1] by generalizing Chordal and Chordal bipartite graphs in a larger dimension. SC_3 and SC_4 graphs are well known chordal graphs and chordal bipartite graphs, which are well studied as it helps to identify the gap between NP-Complete input instances and polynomial-time solvable input instances on many problems. Problems such as clique, independent set, coloring have polynomial-time algorithms restricted to $SC_3(SC_4)$ graphs. On a similar line, authors of [1] have explored $SC_{k\geq 5}$ in detail from both structural and algorithmic front. In [1], polynomial-time algorithms for problems such as testing, Hamiltonian cycle, coloring, tree-width, and minimum fill-in have been presented.

In this paper, we revisit SC_k graphs and study classical problems such as maximum independent set (MIS), dominating set, feedback vertex set (FVS), odd cycle transversal (OCT), even cycle transversal (ECT) and Steiner tree. In recent times, these problems are extensively studied in the context of parameterized complexity [3, 4]. Also, cycle hitting problems such as FVS, OCT, ECT have polynomial-time algorithms restricted to chordal and chordal bipartite graphs [2]. Further, independent set and vertex cover also have polynomial-time algorithms in chordal [5] and chordal bipartite graphs. Steiner tree, a generalization of classical minimum spanning tree problem, is known to be NP-Complete in chordal [6] and chordal bipartite graphs [7]. The problem of finding a minimum dominating set is NP-Complete in chordal [8] and chordal bipartite graphs [7].

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It is important to highlight that chordal (chordal bipartite) graphs have a special ordering, on vertices namely *perfect vertex elimination ordering* (*perfect edge elimination ordering*) and this ordering is crucially used in solving all of the above combinatorial problems. For $SC_{k\geq 5}$ graphs, a *vertex cycle ordering (VCO)* is proposed in [1]. It would be an interesting attempt to see whether VCO helps in solving the above mentioned combinatorial problems restricted to SC_k graphs. This is the first focus of this paper and also we focus on finding VCO in linear time.

The second focus of this paper is to study subclasses of $2K_2$-free graphs from minimal vertex separator (MVS) perspective and analyze the complexity of cycle hitting problems in $2K_2$-free graphs. $2K_2$-free graphs have received a good attention in the literature as it is a subclass of P_5-free graphs and a superclass of split graphs. Interestingly, Steiner tree [10] and Dominating set [10] are NP-Complete on $2K_2$-free graphs and other classical problems are polynomial-time solvable [11, 12, 13]. In this paper, we investigate the complexity of cycle hitting problems and Steiner tree on subclasses of $2K_2$-free graphs and present polynomial-time algorithms for all of them.

Organization of the paper: In Section 2, we introduce basic terminologies and theorems used in this paper. The algorithmic results on SC_k graphs; Vertex Cycle Ordering in linear time, maximum independent set, odd (even) cycle transversal, feedback vertex set, dominating set and Steiner tree are presented in Section 3. In Section 4, we present the structural and algorithmic results on the subclasses of $2K_2$-free graphs.

2. Preliminaries

2.1. *Graph Preliminaries*

We follow the notation as in [15, 16]. Let G be a simple, connected and undirected graph with the non-empty vertex set $V(G)$ and the edge set $E(G)$= $\{\{u,v\} \mid u,v \in V(G)$ and u is adjacent to v in G and $u \neq v\}$. The *neighborhood* of a vertex v of G, $N_G(v)$, is the set of vertices adjacent to v in G. The degree of the vertex v is $d_G(v) = |N_G(v)|$. Let $S \subset V(G)$, we define $N_G(S)$ as $\{u \in V(G) | \ \forall \ v \in S, \{u,v\} \in E(G)\}$. A *cycle* C on n-vertices is denoted as C_n, where $V(C) = \{x_1, x_2, \ldots, x_n\}$ and $E(C) = \{\{x_1,x_2\},\{x_2,x_3\},\ldots,\{x_{n-1},x_n\},\{x_n,x_1\}\}$. The graph G is said to be *connected* if every pair of vertices in G has a path and if the graph is not connected it can be divided into disjoint connected *components* $G_1, G_2, \ldots, G_k$, $k \geq 2$, where $V(G_i)$ denotes the set of vertices in the component G_i. The graph G is said to be *k-connected* (or *k-vertex connected*) if there does not exist a set of $k-1$ vertices whose removal disconnects the graph. The graph M is called a *subgraph* of G if $V(M) \subseteq V(G)$ and $E(M) \subseteq E(G)$. The subgraph M of a graph G is said to be *induced subgraph*, if for every pair of vertices u and v of M, $\{u,v\} \in E(M)$ if and only if $\{u,v\} \in E(G)$ and it is denoted by $[M]$. An *induced cycle* is a cycle that is an induced subgraph of G. The graph G is said to be *cycle free* if there is no induced cycle in G.

2.2. *Definitions and properties on SC_k graphs*

The definitions and theorems given in this section are from [1].

Theorem 1. *A graph G is an SC_k graph if and only if it can be constructed iteratively by any one of the following operations.*

(i) K_1 *is an* SC_k *graph.*

(ii) C_k *is an* SC_k *graph.*

(iii) If G *is an* SC_k *graph, then the graph* G'*, where,* $V(G') = V(G) \cup \{v\}$*,* $E(G') = E(G) \cup \{u, v\}$ *such that* $v \notin V(G)$ *and* u *is any vertex in* $V(G)$*, is also an* SC_k *graph.*

(iv) If G *is an* SC_k *graph, then the graph* G'*, where,* $V(G') = V(G) \cup \{v_1, v_2, \ldots, v_{k-1}\}$*,* $E(G') = E(G) \cup \{\{u, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-2}, v_{k-1}\}, \{v_{k-1}, u\}\}$ *such that* $\{v_1, v_2, \ldots, v_{k-1}\} \cap V(G) = \phi$ *and* u *is any vertex in* $V(G)$*, is also an* SC_k *graph.*

(v) If G *is an* SC_k *graph, then the graph* G'*, where,* $V(G') = V(G) \cup \{v_1, v_2, \ldots, v_{k-2}\}$*,* $E(G') = E(G) \cup \{\{u, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-2}, v\}\}$ *such that* $\{v_1, v_2, \ldots, v_{k-2}\} \cap V(G) = \phi$ *and* $\{u, v\}$ *is any edge in* $E(G)$*, is also an* SC_k *graph.*

(vi) If G *is an* SC_k *graph and* $k = 2m + 4, m \geq 1$*, then the graph* G'*, where,* $V(G') = V(G) \cup \{v_1, v_2, \ldots, v_{\frac{k}{2}-1}\}$*,* $E(G') = E(G) \cup \{\{u_1, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{\frac{k}{2}-1}, u_{\frac{k}{2}+1}\}\}$ *such that* $\{v_1, v_2, \ldots, v_{\frac{k}{2}-1}\} \cap V(G) = \phi$ *and* $\{u_1, u_2, \ldots, u_{\frac{k}{2}+1}\}$ *is any path of length* $\frac{k}{2} + 1$ *contained in no induced cycle in* G *or in any one induced cycle* S_i *of length* k *in* G *such that there does not exist an induced cycle* S_j *in* G *with* $V(S_i) \cap V(S_j) = \{w_1, \ldots, w_{\frac{k}{2}+1}\}$*,* $w_p = u_p$ *for some* $p \in \{1, \ldots, \frac{k}{2}+1\}$ *and for at least one* $q \in \{1, \ldots, \frac{k}{2} + 1\}$*,* $w_q \neq u_q$*.*

Throughout this subsection, the graph G refers to an SC_k graph, $k \geq 5$.

Definition 1. *Let* $\mu = (x_1, \ldots, x_s)$*,* $1 \leq s \leq n$*, be the ordering of* G*. If* $s = 1$*, then either* G *is a trivial graph or a cycle of length* k*. If* $s \geq 2$*, then the label(*x_i*),* $i < s$*, denotes the*
(a) pendant vertex if it satisfies the condition (iii) of Theorem 1*,*
(b) 0-pendant cycle if it satisfies the condition (iv) of Theorem 1 *and if* u *is not a part of any cycle in* G*,*
(c) 1-pendant cycle if it satisfies the condition (iv) of Theorem 1 *and if* u *is part of at least one cycle,*
(d) 2-pendant cycle if it satisfies the condition (v) of Theorem 1 *and*
(e) $(\frac{k}{2} + 1)$*-pendant cycle if it satisfies the condition (vi) of* Theorem 1 *w.r.t the induced graph on* $(x_i, x_{i+1}, \ldots, x_s)$*. Note that, in a* $(\frac{k}{2} + 1)$*-pendant cycle* S*,* S *can have either* u_1 *or* $u_{\frac{k}{2}+1}$ *as a cut vertex but not both.*

Definition 2. *Let* $\sigma = (x_1, \ldots, x_s)$ *be the ordering of vertices and cycles in a graph* G*. We say that* σ *is a Vertex Cycle Ordering (VCO) if each* x_i *is a pendant vertex or a* s*-pendant* C_k*,* $s \in \{0, 1, 2, \frac{k}{2} + 1\}$*.*

It is also proved that, SC_k graph has at least one pendant vertex or a s-pendant C_k, $s \in \{0, 1, 2, \frac{k}{2} + 1\}$. Also, since SC_k graphs preserves hereditary property, we can get a *Vertex Cycle Ordering (VCO)* for any $SC_k, k \geq 5$ graph G in at most n iterations, where n is the number of vertices in G [1].

Corollary 1. *A graph* G *is an* SC_k*,* $k \geq 5$ *graph if and only if it has a vertex cycle ordering.*

Definition 3. *Let* G *be a simple connected graph and* T *be the Breadth First Search (BFS) tree for* G*. The* $missing\ edges$ *of* T *is the set of all non-tree edges,* $E(G) \backslash E(T)$*.*

Definition 4. *An* SC_k *graph is said to be* n-C_k pyramid *if it has* $(k - 2)n + 2$ *vertices,* $(k - 1)n + 1$ *edges, exactly two adjacent vertices of degree* $n + 1$ *and every other vertex are of degree two. A 3-*C_5 *pyramid is shown in Figure 1.*

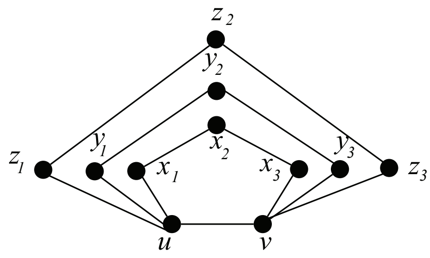


Figure 1. 3-C_5 Pyramid.

Definition 5. *A graph G is said to be a cage graph of size l denoted as $CAGE(l, s)$ if there exist $w, z \in V(G)$ such that $\{w, u_s^i\}, \{z, u_{s-2}^i\} \in E(G)$ for all $1 \leq i \leq n$ and there exist a path from u_1^i to u_{s-2}^i for all $1 \leq i \leq n$ of length $s-2$. The $CAGE(3, 4)$ is shown in* Figure 2. *A $CAGE(l, s)$ is maximum or a maximum cage if there is no $l' > l$ such that G has $CAGE(l', s)$.*
In particular, if the cage is an SC_k graph and k is even, then $s = \frac{k}{2} + 1$ and it has $(l+1) \times (\frac{k}{2} - 1) + 2$ vertices.

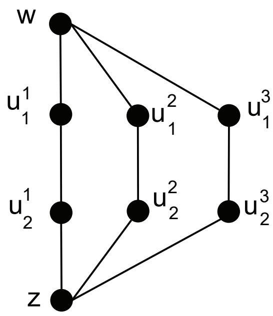


Figure 2. $CAGE(3, 4)$

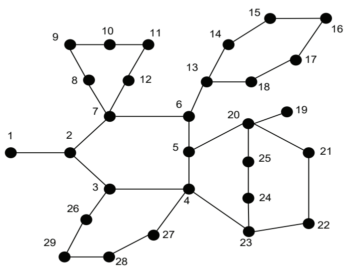
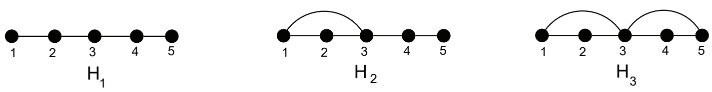


Figure 3. An example for an SC_6 graph. One of the vertex cycle ordering for this graph is ($\{1\}$, $\{19\}$, $\{13, 14, 15, 16, 17, 18\}$, $\{13\}$, $\{7, 8, 9, 10, 11, 12\}$, $\{3, 4, 27, 28, 29, 26\}$, $\{20, 21, 22, 23, 24, 25\}$, $\{4, 5, 20, 25, 24, 23\}$, $\{2, 3, 4, 5, 6, 7\}$), where the vertices 1 and 19 are said to be pendant, $(13, 14, 15, 16, 17, 18)$ is a 0-pendant cycle, $(7, 8, 9, 10, 11, 12)$ is a 1-pendant cycle, $(3, 4, 27, 28, 29, 26)$ as 2-pendant cycle and $(20, 21, 22, 23, 24, 25)$ is a 4-pendant cycle. The graph induced on the vertex set $\{4, 5, 20, 21, 22, 23, 24, 25\}$ is the $CAGE(3, 4)$.

2.3. *Definitions and properties on $2K_2$-free graphs*

The definitions and properties given in this section is from [14].

Lemma 1. *A connected graph is $2K_2$ free if and only if it forbids H_1, H_2 and H_3 as an induced subgraphs.*



Definition 1. *Let G be a graph and $S \subset V(G)$. A vertex $v \in V(G \backslash S)$ is said to be a* universal vertex *if $\forall\, x \in S, \{x, v\} \in E(G)$. An edge $\{u, v\}$ is said to be a* universal edge *if $\forall\, x \in S$, either $\{x, u\} \in E(G)$ or $\{x, v\} \in E(G)$.*

Theorem 2. *[14] Let G be a connected graph and S be any minimal vertex separator of G. Let $G_1, G_2, \ldots, G_l$, $(l \geq 2)$ be the connected components in $G \backslash S$. G is $2K_2$ free if and only if it satisfies the following conditions:*

- *(i) $G \backslash S$ contains at most one non-trivial component. Further, if $G \backslash S$ has a non-trivial component, say G_1, then the graph induced on $V(G_1)$ does not contain H_1, H_2, H_3 as an induced subgraphs.*
- *(ii) Every trivial component of $G \backslash S$ is universal to S.*
- *(iii) Every edge in the non-trivial component of $G \backslash S$ is universal to S.*
- *(iv) The graph induced on $V(S)$ is either connected or has at most one non-trivial component. Further, if the graph induced on $V(S)$ has a non-trivial component, say S_1, then the graph induced on $V(S_1)$ does not contain H_1, H_2, H_3 as an induced subgraphs.*
- *(v) If S and $G \backslash S$ has a non-trivial component, say S_1 and G_1, respectively, then every edge in S_1 is universal to $G_1 \backslash M$, where $M = \{v \in V(G_1) \mid N_G(v) \cap V(S) = \phi\}$.*

3. Algorithmic Results on SC_k graphs

In this section, we present a linear time algorithm to find a vertex cycle ordering (VCO) for an SC_k, $k \geq 5$ graph. We also present linear time algorithms for subset problems such as maximum independent set, minimum dominating set, odd cycle transversal, even cycle transversal, feedback vertex set, and Steiner tree problem.

3.1. *VCO in linear time*

In this subsection, we present an algorithm to find a VCO for the given SC_k, $k \geq 5$ graph followed by its proof of correctness and analysis of the algorithm. Also, we provide an upper bound for the number of edges and the number of cycles in an SC_k, $k \geq 5$ graph.

Lemma 2. *The cost for finding a pendant cycle in the* Algorithm 1 *is $O(k)$.*

Proof. Let $e = \{u, v\}$ be the missing edge encountered in *Step 6*.

- If e satisfies *Case 6.1*, then to find the induced cycle C through the missing edge e follow these three steps: 1. Iteratively, find the parent of u_1 until we find $u_{\frac{k-1}{2}}$. 2. Iteratively, find the parent of u_k until we find $u_{\frac{k+1}{2}+1}$. 3. Now, find $N(u_{\frac{k-1}{2}}) \cap N(u_{\frac{k+1}{2}+1})$, which will be just a vertex as C is a cycle, say $u_{\frac{k+1}{2}}$. Thus, $C = (u_1, \ldots, u_{\frac{k-1}{2}}, u_{\frac{k+1}{2}}, u_{\frac{k+1}{2}}+1, \ldots, u_k)$ forms an induced cycle of length k. Clearly, C is a pendant cycle as e is in the deepest level of T.
- If e satisfies *Case 6.2* or *Case 6.3*, then to find the induced cycle C through the missing edge e follow these four steps: 1. Let e satisfies *6.2*. Choose a missing edge incident on u other than $\{u, v\}$, say $\{u, u_2\}$ 2. Iteratively, find the parent of u_2 until we find $u_{\frac{k-1}{2}}$. 3. Iteratively, find the parent of u_k until we find $u_{\frac{k+1}{2}+1}$. 4. Now, find $N(u_{\frac{k-1}{2}}) \cap N(u_{\frac{k+1}{2}+1})$, which will be just a vertex as C

Algorithm 1 Vertex Cycle Ordering for an SC_k graph

1: **Input: An SC_k, $k \geq 5$ Graph G**
2: **Output:** σ, VCO for G
3: Construct a Breadth first search (BFS) tree T for the graph G rooted at a maximum degree vertex.
4: Unmark all the vertices in T.
5: Iteratively, remove all unmarked pendant vertices from T and add them to σ until there are no unmarked pendant vertices in T.
6: Let $\{u, v\}$ be the missing edge at the deepest level. Add the pendant cycle $C = (u = u_1, \ldots, u_k = v)$ to σ.

6.1 If there are no other missing edges incident on u and v, then remove degree two vertices present in C from T and mark the remaining vertices of C.

6.2 If u have more than one missing edge incident on it, then remove the internal vertices in the path P_{vw} from T and also remove v, where $w \in C$ and $deg_T(w) > 2$.

6.3 If v have more than one missing edge incident on it, then remove the internal vertices in the path P_{uw} from T and also remove u, where $w \in C$ and $deg_T(w) > 2$.

7: Iteratively, remove all marked pendant vertices from T until there are no marked pendant vertices in T. If $T \neq \emptyset$, then Goto Step 5.

is a cycle, say $u_{\frac{k+1}{2}}$. Thus, $C = (u_1, \ldots, u_{\frac{k-1}{2}}, u_{\frac{k+1}{2}}, u_{\frac{k+1}{2}} + 1, \ldots, u_k)$ forms an induced cycle of length k. Clearly, C is a pendant cycle as e is in the deepest level of T.

Clearly, the above steps take $O(k)$ time to find a pendant cycle. □

Lemma 3. *An ordering σ obtained from the Algorithm 1 is a VCO of G.*

Proof. We claim to prove that if a vertex is added to σ in the algorithm, it is a pendant vertex in the ordering and if a cycle is added to σ in the algorithm, it is a pendant cycle in the ordering. It is evident from *Step 5*, that every vertex added to σ is a pendant vertex. Let e be the missing edge encountered in *Step 6* and C be an induced cycle through e found by the method in *Lemma 2*. If $T = C$, then trivially C is a pendant cycle. Assume that $T \neq C$. The *Steps 5* and 7 ensures that T is free from pendant vertices. By the *Lemma 2*, and by the fact e is in the deepest level of T, it is true that C is a pendant cycle. □

Lemma 4. *Let G be an SC_k, $k \geq 5$ graph. The number of cycles in G is $O(\frac{n}{k})$ where n denotes the number of vertices in G.*

Proof. Let l be the number of cycles in G.
Case 1: k is odd.
Observe that, if G is a pyramid, then G has the maximum number of cycles. Hence, assume that G is a pyramid. Thus by the *definition 4* in *Section 2.2*, $n = l \times (k-2) + 2$. On simplification and by rearranging the terms we get, $l = \frac{n-2}{k-2}$. Therefore, the number of cycles in an SC_k, $k \geq 5$ graph is $O(\frac{n}{k})$ when k is odd.
Case 2: k is even.
Observe that, if G is a CAGE, then G has the maximum number of cycles. Hence, assume that G is a CAGE. Thus by the *definition 5* in *Section 2.2*, $n = (l+1) \times (\frac{k}{2} - 1) + 2$. On simplification and by

rearranging the terms we get, $l \times (\frac{k}{2} - 1) = n - \frac{k}{2} - 1 \leq n$. Thus, $l \leq \frac{n}{\frac{k}{2}-1}$. Therefore, the number of cycles in an SC_k, $k \geq 5$ graph is $O(\frac{n}{k})$ when k is even. □

Lemma 5. *Let G be an SC_k, $k \geq 5$ graph. The number of edges in G is $O(n)$ where n denotes the number of vertices in G.*

Proof. Let m be the number of edges in G. If G has no cycles, clearly $m = O(n)$. If we add an induced cycle C of length k to G such that C has a vertex intersection with G, then the cycle C contributes $(k-1)$ new vertices and k new edges to G. If we add an induced cycle C of length k to G such that C has an edge intersection with G, then the cycle C contributes $(k-2)$ new vertices and $(k-1)$ new edges to G. Similarly, if we add an induced cycle C of length k to G such that C has a path intersection with G of length $\frac{k}{2}+1$, then the cycle C contributes $(\frac{k}{2}-1)$ new vertices and $\frac{k}{2}$ new edges to G. This shows that the number of edges added to the graph is linear in terms of number of vertices. Thus, $m = O(n)$. □

Theorem 3. *The run time of* Algorithm 1 *is linear in the input size.*

Proof. The algorithm takes $O(n)$ time in *Step 3* and *Step 4* by *Lemma 5*. The *Steps 5-7* takes $O(k) \times O(\frac{n}{k}) = O(n)$, by *Lemma 2, Lemma4*. Thus, the algorithm takes $O(n)$ time, which is linear in the input size. □

Trace of the Algorithm 1

We trace the steps of *Algorithm 1* in *Figure 4* and in *Figure 5*.

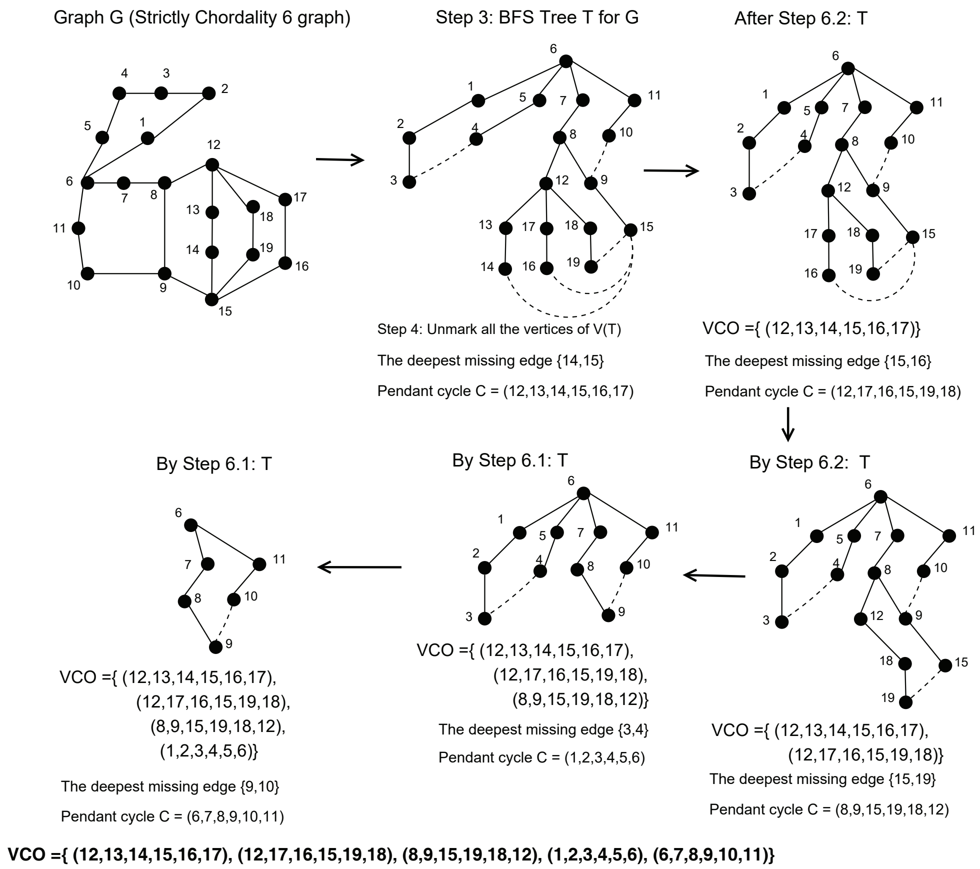


Figure 5. An illustration for Algorithm 1 when k is even

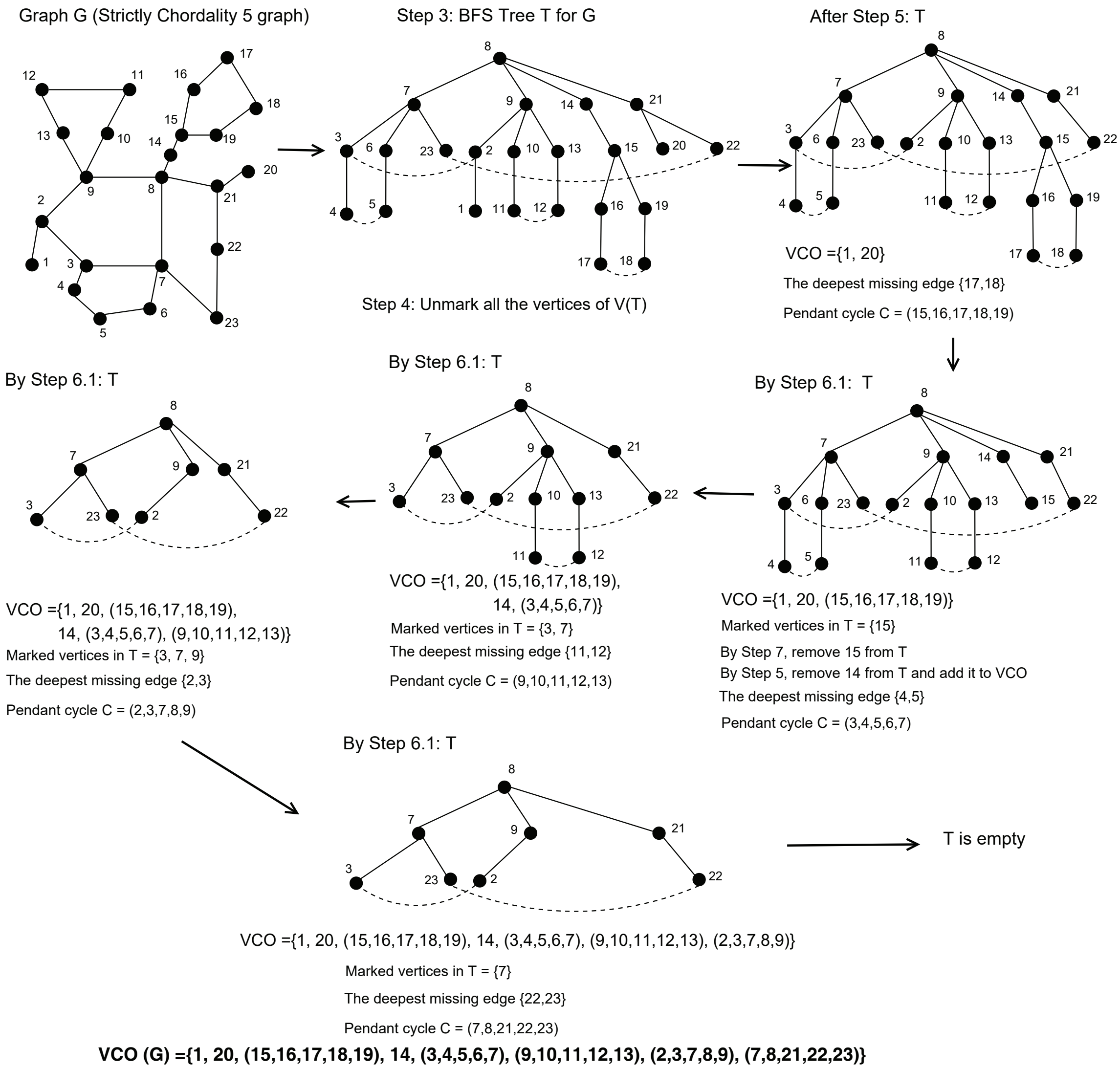


Figure 4. An illustration for Algorithm 1 when k is odd

3.2. *Subset problems on SC_k graphs, $k \geq 5$*

Let G be a strictly chordality-k graph, $k \geq 5$. Let $\mu = (x_1, \ldots, x_s)$ be the VCO of G, $1 \leq s \leq n$. Each algorithm makes use of a VCO and picks the *desired* vertices. At every stage of the algorithm, *pruning* of undesired vertices is also done. Our algorithms are based on dynamic programming paradigm.

For each x_i, $1 \leq i \leq s$, we define *label(x_i)* that denotes the associated vertices in x_i. For Figure 3, $\mu = (x_1, \ldots, x_9)$, where $label(x_1) = \{1\}$, $label(x_3) = \{13, 14, 15, 16, 17, 18\}, \ldots$, $label(x_9) = \{2, 3, 4, 5, 6, 7\}$.

Problem 1 *Maximum Independent Set (MIS).*

Given an SC_k graph G, $k \geq 5$, an independent set $S \subseteq V(G)$ such that $\forall\ u, v \in S,\ \{u, v\} \notin E(G)$. The objective is to find an independent set in G of maximum cardinality. We now present an algorithm to find a MIS.

1. Let $\mu = (x_1, \ldots, x_s)$, $1 \leq s \leq n$, be the VCO of G
2. Find an MIS S' for $label(x_1)$. Add S' to S.

3. Remove $S' \cup N_G(S')$ from G.

4. Update μ and repeat *Steps 2* and 3.

Let $I(G)$ denote the independent set of G with maximum size. Then, $I(G) = I(label(x_1)) \cup I(G \backslash M)$ where $M = S' \cup N_G(S')$.

Computing $I(label(x_1))$:

Lemma 6. *$I(label(x_1)) = \{u\}$ if $label(x_1) = \{u\}$ is a pendant vertex.*

Proof. On the contrary, assume that u is not a part of any maximum independent set of G. Since u is a pendant vertex, $N_G(u)$ is a singleton set, say $\{v\}$. If v is also a pendant vertex, then there is nothing to prove. Assume that v is not a pendant vertex. It is clear from the definition of $I(G)$ that either $u \in I(G)$ or $v \in I(G)$. By our assumption, $u \notin I(G)$. Thus, $v \in I(G)$. By choosing v, we are forced not to add the vertices in $N_G(v)$, whose cardinality is strictly greater than zero. This will contradict the maximality of $I(G)$ unless G is either $P_{2m}, m \geq 2$ or $|P_{ux}| \geq 2m-1, m \geq 2$ where x is the first vertex of degree at least three in G. $\square$

Lemma 7. *Let $label(x_1) = \{u_1, \ldots, u_k\}$ be the 0-pendant cycle (or 1-pendant cycle) where $deg_G(u_1) \geq 3$, $\{u_1, u_k\} \in E(G)$ and $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$. Then $I(label(x_1)) = \{u_2, u_4, \ldots, u_{k-1}\}$ if k is odd and $I(label(x_1)) = \{u_2, u_4, \ldots, u_k\}$ if k is even.*

Proof. It is clear that the maximum size of an independent set of a cycle C_k is $\lfloor \frac{k}{2} \rfloor$. The cardinality of the given set $I(label(x_1))$ is $\lfloor \frac{k}{2} \rfloor$. Thus, $I(label(x_1))$ is the maximum independent set of $label(x_1)$. It remains to show that the set $I(label(x_1))$ does not affect the maximality of $I(G)$. i.e., to prove that the maximality of $I(G)$ is affected if we choose $I(label(x_1)) = \{u_1, u_3, \ldots, u_{k-2}\}$ when k is odd and $I(label(x_1)) = \{u_1, u_3, \ldots, u_{k-1}\}$ when k is even. It is enough to prove that u_1 is not part of $I(G)$. Since $deg_G(u_1) \geq 3$, any MIS I' containing u_1 has the property that $I' < I$. Thus, if we choose u_1 for $I(label(x_1))$, then the cardinality of the resultant independent set for G is either $|I(G)|$ or less than $|I(G)|$. $\square$

Lemma 8. *Let $label(x_1) = \{u_1, \ldots, u_k\}$ be the 2-pendant cycle where $\{u_1, u_k\} \in E(G)$ and $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$, $deg_G(u_1) \geq 3$ and $deg_G(u_2) \geq 3$. Then $I(label(x_1)) = \{u_3, u_5, \ldots, u_k\}$ if k is odd and $I(G) = \max\{I_1(label(x_1)) \cup I(G \backslash M_1), I_2(label(x_1)) \cup I(G \backslash M_2), I_3(label(x_1)) \cup I(G \backslash M_3)\}$ if k is even, where $I_1(label(x_1)) = \{u_1, u_3, \ldots, u_{k-1}\}$, $I_2(label(x_1)) = \{u_2, u_4, \ldots, u_k\}$, $I_3(label(x_1)) = \{u_3, \ldots, u_{k-1}\}$ and $M_i = \bigcup_{u \in I_i(label(x_1))} (u \cup N_G(u))$, $i \in \{1, 2, 3\}$.*

Proof. We prove this lemma by splitting k into odd and even. **Case 1:** When k is odd. The size of the set $I(label(x_1))$ is $\lfloor \frac{k}{2} \rfloor$, which is the maximum size of an independent set in an odd cycle of length k. An argument similar to *Lemma 7* proves that the set $I(label(x_1))$ does not affect the maximality of $I(G)$. **Case 2:** When k is even. The size of both the sets $I_1(label(x_1))$ and $I_2(label(x_1))$ are $\frac{k}{2}$, which is the maximum size of an independent set in an even cycle of length k. In order to get the maximum independent set for G, the maximum is taken over $I_i(label(x_1)) \cup I(G \backslash M_i)$, $i = 1, 2, 3$, and the conclusion follows. We consider $I_3(label(x_1))$, not to contradict the maximality of $I(G)$ due to the presence of both u_1 and u_2. $\square$

Lemma 9. *Let $label(x_1) = \{u_1, \ldots, u_k\}$ be the ($\frac{k}{2}$+1)-pendant cycle where $\{u_1, u_k\} \in E(G)$, $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$, $deg_G(u_1) \geq 3$ and $deg_G(u_{\frac{k}{2}+1}) \geq 3$. Then $I(label(x_1)) = \{u_2, u_4, \ldots, u_k\}$ if $k = 4m+4, m \in \mathbb{N}$ and $I(G) = \max\{I_1(label(x_1)) \cup I(G \backslash M_1), I_2(label(x_1)) \cup I(G \backslash M_2)\}$ if $k = 4m+2, m \in \mathbb{N}$, where $I_1(label(x_1)) = \{u_1, u_3, \ldots, u_{k-1}\}$, $I_2(label(x_1)) = \{u_2, u_4, \ldots, u_k\}$ and $M_i = \bigcup_{u \in I_i(label(x_1))} (\{u\} \cup N_G(u))$, $i = 1, 2$.*

Proof. The ($\frac{k}{2}+1$)-pendant cycle forms a $CAGE(p, \frac{k}{2}+1)$, $p \geq 3$. It is clear from the definition of ($\frac{k}{2}+1$)-pendant cycle that either u_1 is a cut vertex or $u_{\frac{k}{2}+1}$ is a cut vertex but not both and the degree of each vertices in the set $\{u_2, \ldots, u_{\frac{k}{2}}, u_{\frac{k}{2}+2}, \ldots, u_k\}$ is two. We prove this lemma by partitioning the k into the following two cases: **Case 1:** $k = 4m+4, m \in \mathbb{N}$. The size of the set $I(label(x_1)) = \{u_2, u_4, \ldots, u_k\}$ is $\frac{k}{2}$, which is maximum. Moreover, the set does not include u_1 and $u_{\frac{k}{2}+1}$ and this concludes the proof of this case. **Case 2:** $k = 4m+2, m \in \mathbb{N}$. The size of both $I_1(label(x_1))$ and $I_2(label(x_1))$ are $\frac{k}{2}$, which is maximum, where $I_1(label(x_1))$ is the set containing u_1 and $I_2(label(x_1))$ is the set containing $u_{\frac{k}{2}+1}$. By the definition of ($\frac{k}{2}+1$)-pendant cycle, it is enough to take the maximum of $I_1(label(x_1)) \cup I(G \backslash M_1)$ and $I_2(label(x_1)) \cup I(G \backslash M_2)$, to get $I(G)$. $\square$

Theorem 4. *Let G be an SC_k graph. A maximum independent set can be found in $O(n)$ time. Further, a minimum vertex cover can be computed in $O(n)$ time.*

Proof. The claim follows from *Lemmas 6-9* can be computed in linear time and by the *Theorem 3*, VCO can be computed in $O(n)$ time. A minimum vertex cover for G can be obtained by taking the complement of a maximum independent set of G, which can be obtained in $O(n)$ time. Thus, the theorem. $\square$

Problem 2 *Minimum Dominating Set.*

Given an SC_k graph G, $k \geq 5$, the objective is to find a vertex subset S of G with minimum cardinality such that for every $v \in V(G)$, either $v \in S$ or $v \in N_G(x)$ for some $x \in S$.

The algorithm for a minimum dominating set: Start by finding the VCO for a given SC_k graph G, say $\mu = (x_1, \ldots, x_s)$, $1 \leq s \leq n$. Now, find the minimum dominating set for the first element in the ordering. This immediately suggests we to remove the chosen vertices along with their neighbors from G and we recursively compute the dominating set.

$$D(G) = D(label(x_1)) \cup D(G \backslash M)$$

where $D(G)$ denotes a dominating set of G with minimum size and $M = \bigcup_{u \in D(label(x_1))} (\{u\} \cup N_G(u))$

Computing $D(label(x_1))$:

Lemma 10. *$D(label(x_1)) = \{v\}$ if $label(x_1) = \{u\}$ is a pendant vertex and $N_G(u) = \{v\}$.*

Proof. The pendant vertex u can be dominated either by choosing its neighbor v or by choosing the vertex u itself. By choosing v, we can dominate more vertices in G, which helps us to minimize the size of the dominating set for G. $\square$

Lemma 11. *Let $label(x_1) = \{u_1, \dots, u_k\}$ be the 0-pendant cycle (or 1-pendant cycle) where $deg_G(u_1) \geq 3$, $\{u_1, u_k\} \in E(G)$ and $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$. Then $D(label(x_1)) = \{u_1, u_4, u_7, \dots, u_p\}$ where $k - 3 < p \leq k$.*

Proof. It is clear that the minimum size of a dominating set of a cycle C_k is $\lceil \frac{k}{3} \rceil$. The cardinality of the given set $D(label(x_1))$ is $\lceil \frac{k}{3} \rceil$. Thus, $D(label(x_1))$ is the minimum dominating set of x_1 and the set does not affect the minimality of $D(G)$ as $D(label(x_1))$ contains u_1. This completes the proof of the lemma. □

Lemma 12. *Let $label(x_1) = \{u_1, \dots, u_k\}$ be the 2-pendant cycle where $\{u_1, u_k\} \in E(G)$ and $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$, $deg_G(u_1) \geq 3$ and $deg_G(u_2) \geq 3$. Then $D(G) = \min_{i=1,2}\{D_i(label(x_1)) \cup D(G \backslash M_1)\}$, where $D_1(label(x_1)) = \{u_1, u_4, \dots, u_p\}$, $D_2(label(x_1)) = \{u_2, u_5, \dots, u_{p'}\}$, $M_i = \bigcup_{u \in D_i(label(x_1))} (\{u\} \cup N_G(u))$, $i \in \{1, 2\}$, $k - 3 < p, p' \leq k$.*

Proof. The size of both the sets $D_1(label(x_1))$ and $D_2(label(x_1))$ are $\lceil \frac{k}{3} \rceil$, which is the minimum dominating set in a cycle of length k. In order to get the minimum dominating set for G, the minimum is taken over $D_i(label(x_1)) \cup D(G \backslash M_i)$, $i = 1, 2$, and the conclusion follows. □

Lemma 13. *Let $label(x_1) = \{u_1, \dots, u_k\}$ be the $(\frac{k}{2} + 1)$-pendant cycle where $\{u_1, u_k\} \in E(G)$, $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$, $deg_G(u_1) \geq 3$ and $deg_G(u_{\frac{k}{2}+1}) \geq 3$. Then $D(G) = \min_{i=1,2}\{D_i(label(x_1)) \cup I(G \backslash M_i)\}$ where $D_1(x_1)$ is the minimum dominating set for x_1 including u_1, $D_2(x_1)$ is the minimum dominating set for x_1 including $u_{\frac{k}{2}+1}$, $M_i = \bigcup_{u \in D_i(label(x_1))} (\{u\} \cup N_G(u))$, $i \in \{1, 2\}$.*

Proof. The argument similar to *Lemma 12* establishes the claim. □

Thus, we get a linear time algorithm, $O(n)$, to find a minimum dominating set using *Lemmas 10-13*.

Problem 3 *Odd Cycle Transversal.*

Given an SC_k graph G, $k \geq 5$, the objective is to find a vertex subset S of G with minimum cardinality such that $G \backslash S$ is a bipartite graph (every induced cycle is even). Since the SC_k graphs do not contain an odd cycle when k is even, the set S is empty in this case. Hence, our problem is to find the set S for SC_{2k+1} graph, $k \geq 1$. Let $\mu = (x_1, \dots, x_s)$, $1 \leq s \leq n$, be the VCO of G. Thus, the recursive solution is:

$$OCT(G) = \begin{cases} OCT(G \backslash \{label(x_1)\}) & \text{if } label(x_1) \text{ is a pendant vertex} \\ \{u\} \cup OCT(G \backslash \{label(x_1)\}) & \text{if } label(x_1) \text{ is a 0(1)- pendant cycle} \\ & \text{where } deg_G(u) \geq 3, u \in label(x_1) \\ \min\{\{u\} \cup OCT(G \backslash \{label(x_1)\}), & \text{if } x_1 \text{ is a 2-pendant cycle where} \\ \quad \{v\} \cup OCT(G \backslash \{label(x_1)\})\} & \{u, v\} \in E(G) \text{ and, } deg_G(u) \geq 3 \\ & \text{and } deg_G(v) \geq 3, u, v \in label(x_1) \end{cases}$$

where, $OCT(G)$ is the required set S.

Problem 4 *Even Cycle Transversal*.

Given an SC_k graph G, $k \geq 5$, the objective is to find a vertex subset S of G with minimum cardinality such that $G \backslash S$ is a graph where every induced cycle is of odd length. Since the SC_k graphs do not contain an even cycle when k is odd, the set S is empty in this case. Let $\mu = (x_1, \ldots, x_s)$, $1 \leq s \leq n$, be the VCO of G. Thus, the recursive solution is:

$$ECT(G) = \begin{cases} ECT(G\backslash\{label(x_1)\}) & \text{if } label(x_1) \text{ is a pendant vertex} \\ \{u\} \cup ECT(G\backslash\{label(x_1)\}) & \text{if } label(x_1) \text{ is a 0(1)- pendant cycle where} \\ & deg_G(u) \geq 3, u \in label(x_1) \\ \min\{\{u\} \cup ECT(G\backslash\{label(x_1)\}), & \text{if } label(x_1) \text{ is a 2-pendant cycle where} \\ \quad \{v\} \cup ECT(G\backslash\{label(x_1)\})\} & \{u, v\} \in E(G) \text{ and, } deg_G(u) \geq 3 \\ & \text{ and } deg_G(v) \geq 3, u, v \in label(x_1) \\ \min\{\{u\} \cup ECT(G\backslash\{label(x_1)\}), & \text{if } label(x_1) \text{ is a } (\frac{k}{2}+1)\text{-pendant cycle where} \\ \quad \{w\} \cup ECT(G\backslash\{label(x_1)\})\} & deg_G(u) \geq 3 \text{ and } deg_G(w) \geq 3, u, w \in label(x_1) \end{cases}$$

where, $ECT(G)$ is the required set S.

Theorem 5. *$OCT(G)$ and $ECT(G)$ yield an optimum OCT and ECT, respectively.*

Proof. Arguments similar to *Lemmas 10-13* establishes this claim and thus, $OCT(G)$ and $ECT(G)$ can be computed in linear time as the VCO(G) can be found in linear time. □

Problem 5 *Feedback Vertex Set*.

Given an SC_k graph G, $k \geq 5$, the objective is to find a vertex subset S of G with minimum cardinality such that $G \backslash S$ is a forest. It is easy to see that FVS is precisely OCT when k is odd, and ECT when k is even. Thus, FVS can be computed in linear time.

Problem 6 *Steiner Tree*.

Given an SC_k graph G, $k \geq 5$, and a terminal set $R \subseteq V(G)$, Steiner tree asks for a tree T spanning the terminal set. The objective is to minimize the number of additional vertices ($S \subseteq V(G) \backslash R$, also known as Steiner vertices).

Definition 6. *Let S_i be the s-pendant cycle in G such that there exist a cycle S_j in G, where either $|E(S_i) \cap E(S_j)| = 0$ or $s - 1$ or $|V(S_i) \cap V(S_j)| = s$. Let $R = V(S_i) \backslash (V(S_i) \cap V(S_j))$. The* removal of a s-pendant cycle *S_i from G yields the induced subgraph $G \backslash R$. Note that for each S_i, there is a corresponding R and $G \backslash S_i$ corresponds to the graph $G \backslash R$.*

We now present an algorithm to find a minimum Steiner Set.

1. Remove all the pendant vertices and pendant cycles which do not contain any terminal vertex and update G. Return G if G is acyclic.

2. Let $\mu = (x_1, \ldots, x_s)$, $1 \leq s \leq n$, be the VCO of G

3. Find a Steiner set S' for $label(x_1)$. Add S' to S. A desired vertex x' for the $label(x_1)$ is added to R.

4. Remove $label(x_1)$ from G.
5. Update μ and repeat *Steps 1-4*.

Let $ST(G, R)$ denote the vertex set of Steiner tree T which spans $R \subseteq V(G)$ with a minimum number of Steiner vertices.

$$ST(G, R) = ST(G, (R \cap label(x_1)) \cup \{x'\}) \cup ST(G, (R \backslash label(x_1)) \cup \{x'\})$$

Computing $ST(G, (R \cap label(x_1)) \cup \{x'\})$:

Lemma 14. *If $label(x_1) = \{u\}$ is a pendant vertex, then $x' = v$ and $ST(G, (R \cap label(x_1)) \cup \{x'\}) = V(P_{uv})$, where v is the vertex of some C_k in G and the first vertex of $deg_G(v) \geq 3$ in a path from u in G.*

Proof. We add the vertex v to the terminal set because the required tree T should be connected. Now, the only possible Steiner tree T containing pendant vertex u and v is P_{uv}. ☐

Lemma 15. *Let $label(x_1) = \{u_1, \ldots, u_k\}$ be the 0-pendant cycle (or 1-pendant cycle) where $deg_G(u_1) \geq 3$, $\{u_1, u_k\} \in E(G)$ and $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$. Let $\{r_1, \ldots, r_s\} \subseteq \{u_1, \ldots, u_k\}$ be the set of terminal vertices in $label(x_1)$. Then $x' = u_1$ and $ST(G, (R \cap label(x_1)) \cup \{x'\}) = \min\limits_{0 \leq i \leq s} V(P_i) \cup ST(G, (R \backslash label(x_1)) \cup \{u_1\})$ where P_i is the induced path obtained by removing the internal vertices of $P_{r_i r_{i+1}}$, $1 \leq i \leq s-1$ from $label(x_1)$, P_0 and P_s is obtained by removing the internal vertices of $P_{u_1 r_1}$ and $P_{r_s u_1}$ from $label(x_1)$, respectively.*

Proof. We add the vertex u_1 to the terminal set because the required tree T should be connected. The minimum of all possibilities over the $label(x_1)$ is considered to get a minimum Steiner tree. ☐

Lemma 16. *Let $label(x_1) = \{u_1, \ldots, u_k\}$ be the 2-pendant cycle where $\{u_1, u_k\} \in E(G)$ and $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$, $deg_G(u_1) \geq 3$ and $deg_G(u_2) \geq 3$. Let $\{r_1, \ldots, r_s\} \subseteq \{u_1, \ldots, u_k\}$ be the set of terminal vertices in $label(x_1)$. Then x' is either u_1 or u_2 and $ST(G, (R \cap label(x_1)) \cup \{x'\}) = \min\limits_{j=1,2} \min\limits_{0 \leq i \leq s} V(P_i) \cup ST(G, (R \backslash label(x_1)) \cup \{u_j\})$ where P_i is the induced path obtained by removing the internal vertices of $P_{r_i r_{i+1}}$, $1 \leq i \leq s-1$ from $label(x_1)$, P_0 and P_s is obtained by removing the internal vertices of $P_{u_j r_1}$ and $P_{r_s u_j}$ from $label(x_1)$, respectively.*

Proof. We add either u_1 or u_2 to the terminal set to get the connected graph T. We list all the possibilities by adding u_1 to the terminal set and by adding u_2 to the terminal set, separately. Finally, we choose the minimum of all in order to get a minimum Steiner tree. ☐

Lemma 17. *Let $label(x_1) = \{u_1, \ldots, u_k\}$ be the $(\frac{k}{2}+1)$-pendant cycle where $\{u_1, u_k\} \in E(G)$, $\{u_i, u_{i+1}\} \in E(G), 1 \leq i \leq k-1$,$deg_G(u_1) \geq 3$ and $deg_G(u_{\frac{k}{2}+1}) \geq 3$. Let $\{r_1, \ldots, r_s\} \subseteq \{u_1, \ldots, u_k\}$ be the set of terminal vertices in $label(x_1)$. Then x' is either u_1 or $u_{\frac{k}{2}+1}$ and $ST(G, (R \cap label(x_1)) \cup \{x'\}) = \min\limits_{j=1,\frac{k}{2}+1} \min\limits_{0 \leq i \leq s} V(P_i) \cup ST(G, (R \backslash label(x_1)) \cup \{u_j\})$ where P_i is the induced path obtained by removing the internal vertices of $P_{r_i r_{i+1}}$, $1 \leq i \leq s-1$ from $label(x_1)$, P_0 and P_s is obtained by removing the internal vertices of $P_{u_1 r_1}$ and $P_{r_s u_{\frac{k}{2}+1}}$ from $label(x_1)$, respectively.*

Proof. The argument similar to *Lemma 16* establishes the claim. ☐

Thus, we get a linear time algorithm ($O(n)$) to find a minimum Steiner set using *Lemmas 14-17*. Steiner tree can be obtained by finding a minimum spanning tree of the induced subgraph on $ST(G, R)$.

4. Structural and Algorithmic Results on $2K_2$-free graphs

It is known from [9, 10] that Steiner tree and dominating set are NP-Complete on $2K_2$-free graphs. In this section, we study subclasses of $2K_2$-free graphs where these two problems are polynomial-time solvable. Further, on such subclasses, we show that FVS and OCT are also polynomial-time solvable. To the best of our knowledge, this line of study has not been explored in the literature on these problems.

4.1. $(2K_2, C_3, C_4)$-free graphs

$(2K_2, C_3, C_4)$-free graphs form a proper subclass of $2K_2$-free graphs, where every induced cycle is of length 5. We observed the following structural properties and conclude that it is a trivial graph class.

Theorem 6. *If G is a connected $(2K_2, C_3, C_4)$-free graph, then any minimal vertex separator S of G satisfies the following properties:*

(i) S is an independent set.

(ii) If $\mid S \mid > 1$, then $G \backslash S$ has exactly one trivial component.

(iii) If $G \backslash S$ has a non-trivial component, say G_1, then for every edge $\{u, v\} \in E(G_1)$, $(N_G(u) \cap S) \cap (N_G(v) \cap S) = \emptyset$ and $(N_G(u) \cap S) \cup (N_G(v) \cap S) = S$. i.e., For every vertex $x \in S$, $(N_G(x) \cap V(G_1))$ is an independent set.

(iv) Every vertex in a non-trivial component is adjacent to exactly one vertex in S.

Proof. (i) On the contrary, assume that S has at least one edge, say $\{x, y\}$. Let G_i be a trivial component in $G \backslash S$ and let $V(G_i) = \{w\}$. Since, G is a $2K_2$-free graph, $\{w, x\}, \{w, y\} \in E(G)$ (by *Theorem 2.(ii)*). Thus, (w, x, y) forms an induced C_3, which is a contradiction to the definition of G. Hence, S is an independent set.

(ii) On the contrary, assume that $G \backslash S$ has at least two trivial components, say G_i and G_j. Let $V(G_i) = \{w_i\}$ and $V(G_j) = \{w_j\}$. Let x, y be any two vertices in S. By (i), $\{x, y\} \notin E(G)$ and by *Theorem 2.(ii)*, $\{w_i, x\}$, $\{w_i, y\}$, $\{w_j, x\}$, $\{w_j, y\} \in E(G)$. Thus, (w_i, x, w_j, y) forms an induced C_4, which is a contradiction to the definition of G. Hence, $G \backslash S$ has exactly one trivial component if $\mid S \mid > 1$.

(iii) By *Theorem 2.(iii)*, every edge $\{u, v\} \in E(G_1)$ is universal to S, thus, $(N_G(u) \cap S) \cup (N_G(v) \cap S) = S$. Moreover, if $(N_G(u) \cap S) \cap (N_G(v) \cap S) \neq \emptyset$, then every vertex in $(N_G(u) \cap S) \cap (N_G(v) \cap S)$ forms an induced C_3 together with u and v. Hence, $(N_G(u) \cap S) \cap (N_G(v) \cap S) = \emptyset$.

(iv) On the contrary, assume that exist a vertex v in a non-trivial component such that $(N_G(v) \cap S) = \{x_1, x_2, \ldots, x_p\}$, $p \geq 2$. By (ii), there exist a trivial component in $G \backslash S$, say G_2. Let $V(G_2) = \{w\}$. Therefore, (v, x_1, x_2, w) forms an induced C_4, which is a contradiction to the definition of G.

□

Corollary 2. *If G is a connected $(2K_2, C_3, C_4)$-free graph, then G is either a tree or C_5.*

Proof. From *Theorem 6*, we can observe that the only possible structure of a non-trivial component after the removal of any minimal vertex separator from G is K_2 and $\mid S \mid \leq 2$. Further, if $\mid S \mid = 1$, then the graph is $(2K_2, cycle)$-free. If $\mid S \mid = 2$ and if $G \backslash S$ has a non-trivial component, then the graph is an induced C_5. $\square$

Thus, FVS, OCT, Steiner tree problem and a dominating set can be solved in $O(1)$ time when the input is restricted to $(2K_2, C_3, C_4)$-free graphs.

4.2. $(2K_2, C_3, C_5)$-free graphs

$(2K_2, C_3, C_5)$-free graphs are $2K_2$-free graphs which are either acyclic or every induced cycle is of length 4. Further, these graphs are $2K_2$-free chordal bipartite graphs. We shall study this graph class from MVS perspective.

Theorem 7. *If G is a connected $(2K_2, C_3, C_5)$-free graph, then any minimal vertex separator S of G satisfies the following properties:*

(i) S is an independent set.

(ii) If $G \backslash S$ has a non-trivial component, say G_1, then for every vertex $x \in S$, $(N_G(x) \cap V(G_1))$ is an independent set.

(iii) For every edge $\{u, v\}$ in a non-trivial component G_1 of $G \backslash S$, u is universal to S and $(N_G(v) \cap S) = \emptyset$.

(iv) Let T be the set of all vertices in the trivial components of $G \backslash S$. Then the graph induced on the vertex set $T \cup S$ is a complete bipartite graph.

(v) Let U and U' be the set of all vertices in a non-trivial component which are universal and non-universal to S, respectively. Then, there exists a vertex $u \in U$ such that u is universal to U'.

Proof. (i) The argument is similar to the proof in *Theorem 6.(i)*.

(ii) The argument is similar to the proof in *Theorem 6.(iii)*.

(iii) On the contrary, assume that there exists an edge $\{u, v\} \in E(G_1)$ such that $S \not\subseteq N_G(u)$, $(N_G(v) \cap S) \neq \emptyset$ and $(N_G(u) \cap S) \neq \emptyset$. Since, G is $2K_2$-free graph, $(N_G(u) \cap S) \cup (N_G(v) \cap S) = S$ and there exists a trivial component in $G \backslash S$, say G_2. Let $V(G_2) = \{w\}$. By our assumption, u is adjacent to some vertex in S, say x and v are adjacent to some vertex in S, say y, such that $x \neq y$. Thus, (u, v, y, w, x) forms an induced C_5, which is a contradiction to the definition of G.

(iv) This is true by the fact that S is independent and every trivial component is universal to S.

(v) By (iii), G_1 is a bipartite graph where U and U' are the independent sets. Let us prove the statement by mathematical induction on the cardinality of U.
Base Case: Since G_1 is connected, the statement is true for $|U| = 1$.
Hypothesis: Assume that the statement is true for $|U| = s,\ s \geq 1$.
Induction Step: Let $|U| = s + 1,\ s \geq 1$.

For some $u \in U$, the graph $G_1 \backslash \{u\}$ has a vertex $v \in U$ universal to U', by the hypothesis. If $N_{G_1}(u) \subset N_{G_1}(v)$, then there is nothing to prove. W.l.o.g. assume that $N_{G_1}(u) \backslash N_{G_1}(v) \neq \emptyset$. For arbitrary $x \in N_{G_1}(u) \backslash N_{G_1}(v)$. If $\{v, x\} \in E(G)$, then v is the required vertex which is universal to U'. If $\{v, x\} \notin E(G)$, then u is the required vertex which is universal to U'.

$\square$

Corollary 3. *Let G be a connected $(2K_2, C_3, C_5)$-free graph, $R \subseteq V(G)$ be the terminal set of G and S be any MVS of G. Let T be the set of all trivial components in $G \backslash S$, U and U' be the set of universal and non-universal vertices in a non-trivial component of $G \backslash S$, respectively. If R is connected, then the Steiner tree $ST(G, R)$ is the graph induced on the vertex set R. If R is not connected, then the Steiner tree $ST(G, R)$ is the graph induced on the vertex set*

- *$R \cup \{x\}$, for some $x \in S$, if $R \backslash T$ is connected or when R is the subset of T or U or $(T \cup U)$.*
- *$R \cup \{a\}$, for some $a \in T$, when R is the subset of S*
- *$R \cup \{v\}$, where $v \in U$ is universal to U', when R is the subset of U' or $(S \cup U')$ or $(U \cup U')$ or $(T \cup S \cup U')$ or $(S \cup U \cup U')$ or $(T \cup S \cup U \cup U')$.*
- *$R \cup \{v\} \cup \{x\}$, for some $x \in S$ and a vertex $v \in U$ universal to U', if $R \backslash T$ is connected or $R \subseteq T \cup U'$.*

Corollary 4. *Let G be a connected $(2K_2, C_3, C_5)$-free graph and S be any minimal vertex separator of G. Let T be the set of all trivial components in $G \backslash S$, U and U' be the set of universal and non-universal vertices in a non-trivial component of $G \backslash S$, respectively. If $G \backslash S$ has only trivial components, then the dominating set is $\{x, a\}$, for some $x \in S$ and $a \in T$ when $|S| \geq 2$, and the dominating set is S when $|S| = 1$. If $G \backslash S$ has a non-trivial component, then the dominating set is $\{x, u\}$, for some $x \in S$ and $u \in U$ is universal to U'.*

Although, it is known that the problem of finding a minimum feedback vertex set in chordal bipartite graphs, a super class of $2K_2$-free chordal bipartite graphs, is polynomial time solvable [2], using the above observation we provide a different approach for this problem in $(2K_2, C_3, C_5)$-free graph. Moreover, our approach takes linear time in terms of the input size. Also, it is easy to see that FVS is precisely ECT.

Theorem 8. *Let G be a connected $(2K_2, C_3, C_5)$-free graph and S be any minimal vertex separator of G, then the cardinality of any minimum feedback vertex set F is*

(i) *$min\{| S | -1, | T | -1\}$, if $G \backslash S$ has only trivial components, and T is the set of all trivial components in $G \backslash S$.*

(ii) *$min\{| S |, | U | +(| T | -1)\}$, if $G \backslash S$ has a non-trivial component G_1, which is cycle-free, and U is the set of all vertices in G_1 which are universal to S.*

(iii) *$min\{| U | +(| T | -1), (| U | -1) + (| S | -1)\}$, if $G \backslash S$ has a non-trivial component G_1 and G_1 has at least one cycle.*

Proof. (i) If G is a cycle-free graph, then either $\mid S \mid= 1$ or $\mid T \mid= 1$. Thus, $F = \emptyset$, which is minimum. Without loss of generality, assume that G has at least one cycle and $G \backslash S$ has only trivial components, say $G_1, G_2, \ldots, G_l$, $l \geq 2$. By our assumption, $\mid S \mid\geq 2$ and by *Theorem 7*, S is an independent set. Let $V(G_i) = \{u_i\}$. Clearly, $G \backslash F$ results in a forest, where F consists of $\mid S \mid$ - 1 vertices from S and $\mid T \mid$ - 1 vertices from T. Now, we claim to prove the set F is minimum.

- $F = min\{\mid S \mid -1, \mid T \mid -1\} =\mid S \mid -1$
  On the contrary, assume that F is not minimum, then the removal of S' vertices from G results in a forest, where $S' <\mid S \mid -1$. I.e., S has at least two vertices in $G \backslash F$, say $x, y \in S$. Clearly, (u_1, x, u_2, y) forms an induced C_4, which is a contradiction to the definition of F.
- $F = min\{\mid S \mid -1, \mid T \mid -1\} =\mid T \mid -1$
  On the contrary, assume that F is not minimum, then the removal of T' vertices from G results in a forest, where $T' <\mid T \mid -1$. I.e., T has at least two vertices in $G \backslash F$, say $u_1, u_2 \in T$. Let x and y be any two vertices in S. Clearly, (u_1, x, u_2, y) forms an induced C_4, which is a contradiction to the definition of F.

Hence, F is a minimum FVS if $G \backslash S$ has only trivial components.

(ii) All possible structures of G_1 are given in *Figure 6*. From the structures of G_1, it is clear that F is a minimum FVS. It follows from *Theorem 7* that no more structures of G_1 are possible.

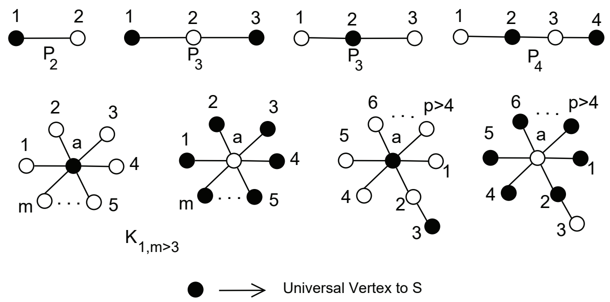


Figure 6. All Possible structures of G_1 when G_1 is cycle-free

(iii) We prove this case separately for $\mid S \mid= 1$ and $\mid S \mid> 1$.

- $\mid S \mid= 1$ and let $S = \{x\}$.
  It is clear that every cycle of G lies in G_1. Thus, $F = min\{\mid U \mid +(\mid T \mid -1), (\mid U \mid -1) + (\mid S \mid -1)\} =\mid U \mid -1$ and the removal of $\mid U \mid -1$ vertices from U results in a forest. Now, we claim to prove that F is minimum. On the contrary, assume that removing at most $\mid U \mid$ - 2 vertices from U results in a forest. I.e., $G \backslash F$ has at least two vertices in U, say $v, w \in U$. Since, G is $2K_2$-free $\mid P_{vw} \mid\leq 4$. Note that, $\mid P_{vw} \mid\neq 2$ because every edge in G_1 is between an universal vertex and a non-universal vertex in G_1, by *Theorem 7.(iii)*. Similarly, $\mid P_{vw} \mid\neq 4$. Thus, the only possibility is $\mid P_{vw} \mid= 3$. Therefore, (P_{vw}, x) forms an induced C_4, which is a contradiction to F.
- $\mid S \mid> 1$. S has at least two vertices, say $x, y \in S$. We claim to prove that S is minimum.
  - $F = min\{\mid U \mid +(\mid T \mid -1), (\mid U \mid -1) + (\mid S \mid -1)\} =\mid U \mid +(\mid T \mid -1)$
    On the contrary, assume that for some $a \in T$ there exists a set $M \subset (U \cup (T \backslash \{a\}))$ such that $|M| < F$ and $G \backslash M$ is a forest. Let $v \in U - M$. Then (a, x, v, y) forms an induced

C_4, which is a contradiction. Let $b \in T - M$ and $b \neq a$. Then (a, x, b, y) forms an induced C_4, which is a contradiction.

- $F = min\{| U | + (| T | - 1), (| U | - 1) + (| S | - 1)\} = (| U | - 1) + (| S | - 1)$
  On the contrary, assume that for some $v \in U$ there exists a set $M \subset (U \backslash \{v\}) \cup (S \backslash \{x\})$ such that $|M| < F$ and $G \backslash M$ is a forest. Let $w \in U - M$ and $w \neq v$. Then (P_{vw}, y) forms an induced C_4, which is a contradiction. Let $y \in S - M$. Then any $a \in T$, (a, x, v, y) forms an induced C_4, which is a contradiction.

From all the above cases, it is proved that F is a minimum FVS. Hence, the theorem. □

Corollary 3, *Corollary 4* and *Theorem 8* naturally yields an algorithm to find a minimum FVS, Steiner tree and dominating set, respectively, in $O(n)$ time, which is linear in the input size.

4.3. $(2K_2, C_4, C_5)$-free graphs

$(2K_2, C_4, C_5)$-free graphs are $2K_2$-free graphs where every induced cycle is of length 3. These graphs can also be called $2K_2$-free chordal graphs. Note that $2K_2$-free chordal graphs are known as split graphs. We know that the structural of any minimal (a, b)-vertex separator in chordal graphs is a clique. It is important to highlight that, the feedback vertex set problem is solvable in polynomial time, for chordal graphs [17], a superclass of split graphs.

Theorem 9. *Let G be a connected $(2K_2, C_4, C_5)$-free graph and S be any MVS of G, then a minimum FVS $FVS(G)$ is*

(i) $V(G) \backslash \{x, y\}$, if G is a complete graph, for some $x, y \in V(G)$.

(ii) $S \backslash \{v\}$, for some $v \in S$, if $G \backslash S$ has only trivial components.

(iii) $G \backslash S$ has a non-trivial component G_1 and G_1 is a tree. If there exist a vertex $v \in S$ such that $| N_G(v) \cap V(G_1) | = 1$, then $FVS(G) = S \backslash \{v\}$. If for every vertex $v \in S$, $| N_G(v) \cap V(G_1) | \geq 2$, then $FVS(G) = S$.

(iv) $\min\limits_{j \in S}\{| S \backslash \{j\} | + FVS(G_1 \cup \{j\})\}$, if $G \backslash S$ has a non-trivial component G_1 and G_1 has at least one cycle.

Proof. (i) The proof is obvious from the definition of complete graphs.

(ii) Since, S is a clique, we have to remove at least $| S | - 2$ vertices from S. Assume that the remaining edge in S is $\{u, v\}$, after the removal of $| S | - 2$ vertices. We know that $G \backslash S$ has at least two components and given that every component in $G \backslash S$ is a trivial component. Thus, we have to remove any one vertex from $\{u, v\}$ such that all cycles formed between trivial components and an edge $\{u, v\}$ are removed.

(iii) By (ii), it is clear that we have to remove at least $| S | - 1$ vertices from S. If there exists a vertex, v, in S whose neighborhood in a non-trivial component is a singleton set, then the removal of $M = S \backslash \{v\}$ from G creates a forest and thus, $FVS(G) = M$. If every vertex in S has more than one vertex in G_1 as its neighbor, then u forms at least one cycle along with G_1, thus, $FVS(G) = S$.

(iv) We enumerate all possible feedback vertex set in $S \cup G_1$, whose removal from G results in a forest, and we choose the minimum among them.

□

Theorem 9 naturally yields an algorithm to find a minimum FVS in $O(n^2\delta)$ time. It is important to highlight that, the feedback vertex set problem is solvable in polynomial time, $O(n^5)$, for chordal graphs [17], a superclass of split graphs.

4.4. $(2K_2, C_3)$-free graphs

$(2K_2, C_3)$-free graphs are $2K_2$-free graphs where every induced cycle is of length 4 or 5. A structural observation is given below:

Definition 7. *Let G be a connected graph and S be a minimal vertex separator for G. Let $G_1, \ldots, G_s$ be the connected components of $G \backslash S$. For some $u, v \in V(G_i)$, P^i_{uv} denotes the shortest path between u and v in a graph G such that all internal vertices belongs to $V(G_i)$.*

Theorem 10. *If G is a connected $(2K_2, C_3)$-free graph, then any minimal vertex separator S of G satisfies the following properties:*

(i) S is an independent set.

(ii) If $G \backslash S$ has a non-trivial component G_1, then for every vertex $x \in S$, $(N_G(x) \cap V(G_1))$ is an independent set. Moreover, $\mid P^1_{uv} \mid = 3$, for all $u, v \in (N_G(x) \cap V(G_1))$.

(iii) If $\mid S \mid \geq 2$ and $G \backslash S$ has a non-trivial component G_1, then $G_1 \backslash M$ is P_4-free, where $M = \{v \in V(G_1) \mid N_G(v) \cap S = \phi\}$. Moreover, M is independent and there exist an unique vertex $u \in G_1 \backslash M$ such that u is universal to M.

(iv) If $G \backslash S$ has a non-trivial component, say G_1, then G_1 is C_5-free. Further, the graph induced on $G_1 \cup S$ is C_5-free.

Proof. (i) The argument is similar to the proof in *Theorem 6.(i)*.

(ii) The argument is similar to the proof in *Theorem 6.(iii)*. Let u and v be any two vertices in $(N_G(x) \cap V(G_1))$. We claim to prove that $\mid P^1_{uv} \mid = 3$. On the contrary, assume that $\mid P^1_{uv} \mid = 4$ (Since, G is $2K_2$-free, $\mid P^1_{uv} \mid \ngeq 5$), say $P^1_{uv} = (u, w, s, v)$. We know that in a $2K_2$-free graph, every edge in a non-trivial component is universal to S. Thus, either $\{w, x\} \in E(G)$ or $\{s, x\} \in E(G)$. If $\{w, x\} \in E(G)$, then (u, w, x) forms a C_3 or if $\{s, x\} \in E(G)$, then (x, s, v) forms a C_3, which is a contradiction to the definition of G.

(iii) On the contrary, assume that $G_1 \backslash M$ has an induced P_4, say $P_4 = (u, v, w, s)$. Choose any two vertices x and y from S. Either $\{x, u\}, \{x, w\}, \{y, v\}, \{y, s\} \in E(G)$, where $P = (x, u, v, y, s)$ forms an induced P_5 (P is induced by (ii)) or $\{y, u\}, \{y, w\}, \{x, v\}, \{x, s\} \in E(G)$, where $P' = (y, u, v, x, s)$ forms an induced P_5 (P' is induced by (ii)), which is a contradiction to the definition of G. M is independent because of the fact every edge in G_1 is universal to S. The existence of universal vertex to M in $G_1 \backslash M$ is true by the fact G is $2K_2$-free and it is unique by (ii).

(iv) On the contrary, assume that G_1 has an induced $C_5 = (u_1, u_2, u_3, u_4, u_5)$. Choose a vertex $x \in S$. Since, every edge in G_1 is universal to S, any one of the following is true:

- $\{u_1, x\}, \{u_3, x\}, \{u_5, x\} \in E(G)$, then (u_1, u_5, x) forms a C_3.
- $\{u_2, x\}, \{u_4, x\} \in E(G)$, then the edge $\{u_1, u_5\}$ is not universal to S.

Both contradicts the definition of G. Since S is independent, the graph induced on $G_1 \cup S$ is also C_5-free. □

Theorem 10 naturally yields an algorithm to find the FVS, which is described as follows. Finding a FVS in a $(2K_2, C_3)$-free graph is same as finding a FVS in $(S \cup G_1)$, say A, and in $G \backslash A$, which is a recursive call and the recursion bottoms out when it returns a bipartite graph, $(2K_2, C_3, C_5)$-free graph. This can be done in polynomial time.

Theorem 11. *Let G be a connected $(2K_2, C_3)$-free graph, $R \subseteq V(G)$ be the terminal set of G and S be any minimal vertex separator of G. Let T be the set of all trivial components in $G \backslash S$. If R is connected, then the Steiner tree $ST(G, R)$ is the graph induced on the vertex set R. If R is not connected, then the Steiner tree $ST(G, R)$ is the graph induced on the vertex set*

- *$R \cup \{x\}$, for some $x \in S$, if $R \subseteq T$.*
- *$R \cup \{a\}$, for some $a \in T$, if $R \subseteq S$.*
- *$\min\limits_{\forall\, x_i \in S} \{ST([S \cup V(G_1)], R \cup \{x_i\})\}$, if $R \subseteq (T \cup G_1)$.*
- *$ST([S \cup V(G_1)], R)$, if R is the subset of G_1 or $(S \cup G_1)$ or $(T \cup S \cup G_1)$.*

Proof. Trivially follows from *Theorem 10*. □

Theorem 12. *Let G be a connected $(2K_2, C_3)$-free graph and S be any minimal vertex separator of G. Let T be the set of all trivial components in $G \backslash S$. If $G \backslash S$ has only trivial components, then the dominating set is $\{x, a\}$, for some $x \in S$ and $a \in T$ when $|S| \geq 2$, and the dominating set is S when $|S| = 1$. If $G \backslash S$ has a non-trivial component, then the dominating set is $\min\limits_{\forall\, x_i \in S} \{\{x_i\} \cup \{a\} \cup D_i\}$, where $a \in T$ and D_i is the dominating set of the graph induced on $(S \cup V(G_1)) \backslash (x_i \cup N_G(x_i))$, which is $2K_2$-free chordal bipartite graph.*

Proof. Trivially follows from *Theorem 10*. □

It is easy to see that the *Theorem 11* and *Theorem 12* yields a linear time algorithm to find a Steiner tree and dominating set, respectively.

4.5. $(2K_2, C_4)$-free graphs

$(2K_2, C_4)$-free graphs are $2K_2$-free graphs where every induced cycle is of length 3 or 5. The structural observations for this graph class are as follows:

Theorem 13. *If G is a connected $(2K_2, C_4)$-free graph, then any minimal vertex separator S of G satisfies the following properties:*

(i) *S is connected except if G is an induced C_5 or $K_{1,m}, m \geq 2$.*

(ii) *S is connected and has a non-trivial component, G_1, in $G\backslash S$. If a vertex $x \in V(G_1)$ is adjacent to a vertex $u \in S$, then $(N_G(u) \cap S) \subseteq N_G(x)$.*

(iii) *If S is not a clique, then $G\backslash S$ has exactly one trivial component. Moreover, every vertex in a non-trivial component of $G\backslash S$ is not universal to any non-adjacent pair of vertices in S.*

(iv) *If S is not a clique, then the only possibility of a non-trivial component of $G\backslash S$ is K_2.*

(v) *The size of the maximum independent set of the graph induced on S is at most two.*

(vi) *S contains neither P_4 nor $K_{1,m}$, $m \geq 3$.*

Proof. (i) On the contrary, assume that $G[S]$ has at least two components. Choose two vertices x and y from different components of $G[S]$. If $G\backslash S$ has only trivial components, then x, y and any two trivial components from $G\backslash S$ forms C_4, which is a contradiction. If $G\backslash S$ has a non-trivial component, G_1, then choose an edge $\{u, v\} \in E(G_1)$. If $\mid S \mid= 2$, then either u is universal to S or v is universal to S. W.l.o.g, assume that u is universal to S. Thus, u, x, y and a trivial component in $G\backslash S$ forms a C_4, which is a contradiction to the definition of G. If $\mid S \mid\geq 3$, then either $\mid N_G(u) \cap S \mid\geq 2$ or $\mid N_G(v) \cap S \mid\geq 2$. W.l.o.g, assume that $\mid N_G(u) \cap S \mid\geq 2$. Let $x, y \in (N_G(u) \cap S)$. Thus, u, x, y and a trivial component in $G\backslash S$ forms a C_4, which is a contradiction to the definition of G.

(ii) On the contrary, assume that $\{x, v\} \notin E(G)$ for some $v \in (N_G(u) \cap S)$. Since, G is $2K_2$-free and G_1 is a non-trivial component, there exists a vertex $y \in G_1$ such that $\{x, y\}, \{y, v\} \in E(G)$. Thus, (x, u, v, y) forms an induced C_4, which is a contradiction.

(iii) On the contrary, assume that $G\backslash S$ has more than one trivial component. Let $\{u\}$ and $\{v\}$ be any two trivial components in $G\backslash S$. Since S is not a clique, S contains a $P_3 = (x, y, z)$. Since, G is a $2K_2$-free graph, $\{u, x\}, \{u, z\}, \{v, x\}, \{v, z\} \in E(G)$. Thus, (u, x, v, z) forms an induced C_4, which is a contradiction to the definition of G. Moreover, if there exists a vertex, u, in a non-trivial component of $G\backslash S$ is universal to some non-adjacent pair (x, z) in S and if $\{v\}$ is a trivial component of $G\backslash S$, then (u, x, v, z) forms an induced C_4, which is a contradiction.

(iv) Since S is not a clique, S contains a P_3, say $P_3 = (x, y, z)$. On the contrary, assume that the non-trivial component of $G\backslash S$, G_1, contains either $K_3 = (u, v, w)$ or $P_3 = (u, v, w)$. Consider an edge $\{u, v\}$, since every edge in G_1 is universal to S, either $\{u, x\}, \{u, y\}, \{v, y\}, \{v, z\} \in E(G)$ or $\{v, x\}, \{v, y\}, \{u, y\}, \{u, z\} \in E(G)$. W.l.o.g, assume that, $\{u, x\}, \{u, y\}, \{v, y\}, \{v, z\} \in E(G)$. Now, consider the edge $\{v, w\}$, since, $\{v, y\}, \{v, z\} \in E(G)$ either $\{x, v\} \in E(G)$ or $\{x, w\} \in E(G)$. By (iii), $\{x, v\} \notin E(G)$. Thus, the only possibility is $\{x, w\} \in E(G)$. If (u, v, w) is a path, then (x, u, v, w) forms an induced C_4, which is a contradiction. If (u, v, w) is K_3, then consider the edge $\{u, w\}$, either $\{u, z\} \in E(G)$ or $\{w, z\} \in E(G)$. By (iii), both $\{u, z\}, \{w, z\} \notin E(G)$. Thus, the edge $\{u, w\}$ is not universal to S, which is a contradiction.

(v) On the contrary, assume that there exist at least three mutually independent vertices, say $\{x, y, z\}$ in S. It is clear that S is not a clique, therefore by (iii) and (iv), there exists a trivial component and a non-trivial component, i.e., a $K_2 = \{u, v\}$, in $G \backslash S$. By (iii), $u(v)$ can be adjacent to at most one vertex in $\{x, y, z\}\}$. W.l.o.g, assume that $\{u, x\}, \{v, y\} \in E(G)$. This implies, neither u is adjacent to the vertex z nor v is adjacent to the vertex z, which is a contradiction to *Theorem 1.(iii)*.

(vi) On the contrary, S contains either P_4 or $K_{1,m}$, $m \geq 3$. If S contains a $P_4 = (x, y, z, s)$: By (iii), $G \backslash S$ has exactly one trivial component and a non-trivial component K_2, say $G_1 = \{u, v\}$ (by (iv)). By (iii), either $\{u, x\}, \{u, y\} \in E(G)$ or $\{u, z\}, \{u, s\} \in E(G)$. W.l.o.g, assume that $\{u, x\}, \{u, y\} \in E(G)$. Since, G is $2K_2$-free, every edge in G_1 is universal to S. Therefore, $\{v, z\}, \{v, s\} \in E(G)$. By (iii), $\{u, z\}, \{v, y\} \notin E(G)$. Hence, (u, y, z, v) forms an induced C_4, which is a contradiction. Proof for S does not contain $K_{1,m}, m \geq 3$ directly follows from (v).

□

By the *Theorem 13*, it is clear that S is C_5-free. Hence, the feedback vertex set in a $(2K_2, C_4)$-free graph can be determined as follows:

Theorem 14. *Let G be a connected $(2K_2, C_4)$-free graph and S be any minimal vertex separator of G, then the cardinality of a minimum FVS, F, is equal to*

(i) $\mid G \backslash \{i, j\} \mid$, *if G is a complete graph, for some $i, j \in V(G)$.*

(ii) $\mid S \backslash \{j\} \mid$, *if S is an independent set, for some $j \in S$.*

(iii) $\mid S \backslash \{j\} \mid$, *if S is neither clique nor an independent set, for some $j \in S$.*

(iv) $\min\limits_{j \in S}\{\mid S \backslash \{j\} \mid + \mid FVS(G_1 \cup \{j\}) \mid\}$, *if S is a clique, where G_1 is a non-trivial component in $G \backslash S$.*

Proof. (i) The proof is obvious from the definition of complete graphs.

(ii) From *Theorem 13.(i)*, it is clear that S is independent only when $G = C_5$. Also, $\mid S \mid = 2$ and $\mid S \mid -1$ says that $FVS(G) = \{j\}$, for some $j \in V(G)$ i.e., $\mid FVS(G) \mid = 1$. Thus, the removal of a vertex from G makes G a tree and it is minimum.

(iii) We know that, S is a split graph. Since S is not complete, $G \backslash S$ has a non-trivial component, $G_1 = K_2 = \{u, v\}$ and a trivial component, $G_2 = \{w\}$. Thus, finding $FVS(G)$ is equivalent to finding $W = FVS(S)$ and $FVS((S \backslash W) \cup G_1 \cup G_2)$. The possible structures of $S \backslash W$ are (a) $2K_1$ (b) $K_1 \cup K_2$ (c) $K_1 \cup P_3$ (d) K_2 (e) P_3 (by *Theorem 13*). In all the cases, we are forced to pick exactly $(S \backslash W) \backslash \{i\}$, for some $i \in S \backslash W$, vertices. Thus, $FVS(G) = S \backslash \{j\}$, for some $j \in S$.

(iv) Since, S is a clique and $G \backslash S$ has at least one trivial component, $FVS(G)$ contains at least $S \backslash \{j\}$, for some $j \in S$, vertices. If $G \backslash S$ has a non-trivial component, G_1, then we are forced to find $FVS(G_1 \cup \{j\})$ in order to compute $FVS(G)$. Thus, $FVS(G) = \min\limits_{j \in S}\{\mid S \backslash \{j\} \mid + \mid FVS(G_1 \cup \{j\}) \mid\}$. It is minimum because we are varying j for all vertices in S and picking up the minimum.

□

Remark: Minimum dominating set and Steiner tree problem are NP-Complete restricted to $2K_2$-free graphs [10]. In this paper, we have identified two non-trivial subclasses of $2K_2$-free graphs where these problems are polynomial time solvable.

5. Applications

In this section, we consider the complexity of connected dominating set and the connected FVS using the results presented in *Sections 3-4*. It is interesting to observe that every minimum connected dominating set contains a minimum dominating set as a vertex subset. It is natural to ask, can we use a minimum dominating set as a terminal set and call Steiner tree algorithm as a black box to get a minimum connected dominating set. Surprisingly, this observation holds good for SC_k graphs and subclasses of $2K_2$-free graphs. A similar observation is true for the connected vertex cover and for the connected FVS. Further, the maximum leaf spanning tree problem is also polynomial time solvable restricted to SC_k and subclasses of $2K_2$-free graphs.

6. Conclusion

We have extended the study of strictly chordality-k graphs in the algorithmic perspective and solved the subset problems such as MIS, minimum vertex cover, minimum dominating set, FVS, OCT, ECT and Steiner tree problem. Also, we studied the complexity status of FVS, OCT, ECT and Steiner tree problem on the subclasses of $2K_2$-free graphs. We conclude the paper by stating few open problems for further research. Complexity of the problems like Steiner path and Total Dominating set on SC_k graphs, $k \geq 5$, are considered to be significant directions for further research. Also, the study of OCT, ECT and FVS in $2K_2$-free graphs would be interesting.

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