Advances in Pure and Applied Mathematics
2022, vol. 13, n° 4, 58-65 pages, DOI: 10.21494/ISTE.OP.2022.0888 ISTE OpenScience

Critical Point Approaches for Discrete Anisotropic Kirchhoff Type Problems

Approches de points critiques pour les problèmes discrets anisotropes de type Kirchhoff

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ABSTRACT. In this paper, we study a discrete anisotropic Kirchhoff type problem using variational methods and critical point theory, and we discuss the existence of two solutions for the problem. A example is presented to demonstrate the application of our main results.
2010 Mathematics Subject Classification. 39A05, 39A10, 39A70, 46E39.
KEYWORDS. Palais-Smale condition, Discrete boundary value problem, Kirchhoff type equation, Variational methods.

1. Introduction

In this work, letting $T \geq 2$ be an integer, set $[1, T]_{\mathbb{Z}}$ the discrete interval $\{1, 2, ..., T\}$ we deal with the following anisotropic discrete Kirchhoff type problem

$$\begin{cases} -M(\rho(u))\Delta(\phi_{p(k-1)}(\Delta u(k-1))) = \lambda f(k, u(k)), & k \in [1, T]_{\mathbb{Z}} \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where $\phi_{\rho(k)}(t) = |t|^{p(k)-2}t$, $\forall(k, t) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}$, λ is positive real parameter, $\Delta u(k-1) = u(k) - u(k-1)$ is the forward difference operator, $f : [1, T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous functions and the functional $\rho : \mathbb{R} \to [0, \infty)$ is defined by

$$\rho(u) := \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)}.$$

Moreover, we define the function $p : [0, T]_{\mathbb{Z}} \to [2, +\infty)$ such that

$$p^- = \min_{k \in [0,T]_{\mathbb{Z}}} p(k) \leq p^+ = \max_{k \in [0,T]_{\mathbb{Z}}} p(k).$$

The function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous satisfying the following condition:

(M_0) There is a positive constant m_0 such that $m_0 \leq M(t)$ for all $t \geq 0$.

Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

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which was presented by Kirchhoff in 1883 (see [14]) describing the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. This model is an extension of the classical d'Alembert wave equation, by considering the effect of the changing in the length of the string during the vibrations. The parameters in the Kirchhoff's model have the following meanings: L is the length of the string, h is the area of cross-section, X is the Young modulus of the material, ρ is the mass density and ρ_0 is the initial tension. A distinguishing feature of the Kirchhoff equation is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx$ which depends on the average $\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx$ of the kinetic energy $\frac{1}{2}|\frac{\partial u}{\partial x}|^2dx$ on $[0,L]$ and hence the equation is no longer a point wise identity. On the other hand, the stationary analogue of the equation (1.2) is given as

$$\begin{cases} (a+b\int_\Omega|\nabla u|^2dx)\Delta u=f(x,u), & \text{in } \Omega \\ u=0, & \text{on } \ \Omega \end{cases} \tag{1.3}$$

which received much attention only after Lions [16] proposed an abstract framework to the problem, which is related to the stationary analog of the equation of Kirchhoff-type

$$u_{tt}-(a+b\int_\Omega|\nabla u|^2dx)\Delta u=f(x,u), \text{in } \Omega$$

where u denotes the displacement, $f(x,u)$ the external force, b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus).

In recent years, the nonlocal differential equations with boundary conditions have been studied much in the literature, especially focusing on the Kirchhoff equations, due to their wide applications in applied sciences, in [1,2,10].

On the other hand, difference equations play a relevant role in modelling problems that arise in physics, engineering, biology, economics, finance, and many other areas. The study of discrete boundary value problems has captured special attention in the last years. The modelling of certain nonlinear problems from biological neural networks, economics models, optimal control and other areas of study have led to the rapid development of the theory of difference equations; see [3,7,8,13,18,20,21,26]. For example, Gao [8] has studied

$$\begin{cases} -\Delta[p(k-1)\Delta u(k-1)]+q(k)u(k)=ra(k)f(u(k)), & k\in[1,T]_{\mathbb{Z}} \\ \Delta u(0)=\Delta u(T), \end{cases}$$

and has discussed the existence of positive solutions by means of the Rabinowitzs bifurcation theorem. In [13] the existence of multiple solutions were investigated using critical point theorems for the following discrete fourth-order boundary value problem

$$\begin{cases} \Delta^4u(t-2)+\delta\Delta^2u(t-1)-\xi u(t)=\lambda f(t,u(t))+\mu g(t,u(t))+h(u(t)), & t\in[1,N]_{\mathbb{Z}} \\ u(0)=\Delta^2u(-1)=0, \ u(N+1)=\Delta u(N+1)=0. \end{cases}$$

Recently, many authors have studied the existence of solutions for various discrete boundary value problems of Kirchhoff type because due to their wide applications in engineering and physics, for instance see [5,11,12,15,22]. For example, Yang and Liu in [22], by using variational methods and the computations of critical groups, have obtained the existence of nontrivial solutions for the following problem

$$\begin{cases} -(a+b\sum_{k=1}^{N+1}|\Delta(u(k-1)|^2)\Delta^2u(k-1)=\lambda f(k,u(k)), & k\in[1,N]_{\mathbb{Z}} \\ u(0)=u(N+1)=0, \end{cases}$$

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where $N \geq 3$ is a fixed positive integer, $a, b > 0$ are real constants, $[1, N]_{\mathbb{Z}}$ denotes the discrete interval $\{1, 2, ..., N\}$. As usual, Δ denotes the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2 u(k) = \Delta(\Delta u(k))$, and for all $k \in [1, N]_{\mathbb{Z}}$, $f(k, .) \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $f(k, 0) = 0$. Kone et al. in [15], have proved the existence of solutions for the following problem

$$\begin{cases} -M(A(k-1, \Delta(u(k-1))\Delta(a(k-1, \Delta u(k-1))) = f(k), & k \in [1, T]_{\mathbb{Z}} \\ u(0) = \Delta u(T) = 0, \end{cases}$$

where $T \geq 2$ is a positive integer and $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator. In [11, 12] the existence of solutions for the problem (1.1) have been discussed using variational methods and critical point theory.

Here, we use the variational methods to prove existence results for the problem (1.1) under suitable conditions imposed on M and f (see, the conditions (M_0), (A_1), (f_0) and (f_1) of Theorem 3.1. In Theorem 3.1 we establish the existence of at least two solutions for the problem (1.1).

The present paper is organized as follows. In Section 2, we recall some basic definitions and our main tools. In Section 3, we state and prove the main results of the paper. Then, we give A example to illustrate our results.

2. Preliminaries and Basic Notation

In this section, we first introduce some notations and some necessary definitions. Set

$$X := \{u : [0, T+1] \to \mathbb{R}; u(0) = u(T+1) = 0\}$$

endowed with the norm

$$\|u\| = (\sum_{k=1}^{T+1} |\Delta u(k-1)|^2)^{\frac{1}{2}}.$$

Moreover, it is useful to introduce other norms on X, namely

$$|u|_m = (\sum_{k=1}^{T} |u(k)|^m)^{\frac{1}{m}}, \ \forall u \in X, \ \ m \geq 2.$$

We imply that:

$$T^{\frac{2-m}{2m}} |u|_2 \leq |u|_m \leq T^{\frac{1}{m}} |u|_2,$$

(see [2]). In the sequel, we will mention some auxiliary results used through the paper.

Lemma 2.1. *([18, 20]). We have the following assertions:*

(i_1) *for every $u \in X$ with $\|u\| \leq 1$ one has*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{p^+-2}{2}} \|u\|^{p^+}$$

(i_2) *For every $u \in X$ and for any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\max_{k \in [1,T]} |u(k)| < (T+1)^{\frac{1}{q}} (\sum_{k=1}^{T+1} |\Delta u(k-1)|^p)^{\frac{1}{p}}. \tag{2.1}$$

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Lemma 2.2. *(see [9]). We have the following inequality in space X:*

(j_1) *for every* $u \in X$ *and for every* $m > 1$ *we have*

$$\sum_{k=1}^{T} |u(k)|^m \leq T(T+1)^{m-1} \sum_{k=1}^{T+1} |\Delta u(k-1)|^m$$

(j_2) *for every* $u \in X$ *and for every* $m \geq 1$ *we have*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \leq (T+1)\|u\|^m$$

(j_3) *For any* $u \in X$ *and for every* $m \geq 2$ *we have*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \geq (T+1)^{\frac{2-m}{2}} \|u\|^m$$

(j_4) *For any* $u \in X$ *whit* $\|u\| \geq 1$ *we have*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{2-p^-}{2}} \|u\|^{p^+} \tag{2.2}$$

(j_5) *For every* $u \in X$ *we have*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq (T+1)\|u\|^{p^+} + (T+1).$$

Definition 2.3. *We say that* $u \in X$ *is a solution of problem* (1.1) *if*

$$M(\rho(u)) \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) - \lambda \sum_{k=1}^{T} f(k, u(k)) v(k) = 0$$

for all $v \in X$.

Set

$$\varphi(u) = \Phi(u) - \lambda \Psi(u)$$

for every $u \in X$, where

$$\Phi(u) = \widehat{M}(\rho(u)) \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)}, \;\; \Psi(u) = \sum_{k=1}^{T} (F(k, u(k)),$$

$$\widehat{M}(t) = \int_0^t M(\xi) d\xi, \;\; t \in \mathbb{R}$$

and

$$F(k,t) = \int_0^t f(k,s) ds, \;\; \forall (k,t) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}.$$

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Definition 2.4. *Let E be a real reflexive Banach space. If any sequence $\{u_k\} \subset E$ for which $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence, then we say φ satisfies Palais-Smale condition.*

The proofs on our theorems are based on Theorems 2.5 and 2.6 below.

Theorem 2.5. *[19, Theorem 4.10] Let $\varphi \in C^1(X, \mathbb{R})$, and φ satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and*

$$\inf_{\nu \in \partial\Omega} \varphi(\nu) > \max\{\varphi(u_0), \varphi(u_1)\},$$

then there exists a critical point u of φ, i.e. $\varphi'(u) = 0$ with $\varphi(u) > \max\{\varphi(u_0), \varphi(u_1)\}$.

Theorem 2.6. *[23, Theorem 38] For the functional $F : M \subseteq X \longrightarrow [-\infty, +\infty]$ with $M \neq \emptyset, \min_{u \in M} F(u) = \alpha$ has a solution in case the following conditions hold:*

- (h_1) *X is a real reflexive Banach space,*
- (h_2) *M is bounded and weak sequentially closed,*
- (h_3) *F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $F(u) \leq \lim_{n\to\infty} \inf F(u_n)$ holds.*

In Theorem 2.6 for a finite dimensional space setting one uses only norm topologies.

We refer to the papers [4, 24], Theorem 2.5 in which has been applied to obtain the multiple solutions boundary value problem. Moreover, in the paper [25], Theorems 2.6 and 2.5 have been successfully applied to obtain the existence of two solutions for a boundary value problem.

3. Main results

We utilize the following assumptions throughout this paper:

- (A_1) there exist constants $\nu > p^+$ and $L > 0$ such that for every $k \in [1, T]_{\mathbb{Z}}$, $0 < \nu F(k, t) \leq t f(k, t)$, for $|t| > L$,

- (f_1) $\lim_{t \longrightarrow 0} \frac{f(k,t)}{|t|^{p^+-1}} = 0$, for $k \in [1, T]_{\mathbb{Z}}$ uniformly.

The main result of this paper is the following theorem.

Theorem 3.1. *Assume that the assumptions $(M_0), (A_1)$ and (f_1) hold. Then:*

if $f(k, t) \geq 0$ for all $(k, t) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}$, the problem (1.1) *has at least two solutions.*

We need the following lemma to prove our main result.

Lemma 3.2. *Assume that (A_1) and (M_0) hold. Then $\varphi(u)$ satisfies the (PS)-condition.*

Proof. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|\varphi(u_n)| \le c_0$ and $|\varphi'(u_n)| \le c_0$ for all $n \in \mathbb{N}$. Therefore, from the definition of φ' and the assumption (A_1) and inequality (2.2) of Lemma 2.2 for some $c_1 > 0$ we have,

$$
\begin{aligned}
c_0 + c_1\|u_n\| &\ge \nu\varphi(u_n) - \varphi'(u_n)(u_n) \\
&\ge m_0(\frac{\nu}{p^+} - 1)T^{\frac{2-p^-}{2}}\|u_n\|^{p^+} - \lambda\nu\sum_1^{T+1} F(k, u_n(k)) \\
&+ \lambda\sum_1^{T+1} f(k, u_n(k))u_n(k) \ge m_0(\frac{\nu}{p^+} - 1)T^{\frac{2-p^-}{2}}\|u_n\|^{p^+}.
\end{aligned}
$$

Since $\nu > p^+$ this implies that (u_n) is bounded. Consequently, since X is a finite dimensional Banach space we have, up to a subsequence,

$$u_n \to u \ in \ X.$$

Consequently, φ satisfies the (PS)-condition. □

3.1. *The proof of Theorem 3.1*

Proof. In our case it is clear that $\varphi(0) = 0$. Lemma 3.2 shows that φ satisfies the (PS)-condition.
Step 1. We will show that there exists $M > 0$ such that the functional φ has a local minimum $u_0 \in B_M = \{u \in X; \|u\| < M\}$. Then by Theorem 2.6 we conclude that φ has a local minimum $u_0 \in \bar{B}_M$. We assume that $\varphi(u_0) = min_{u\in\bar{B}_M}\varphi(u)$. Now we will show that

$$\varphi(u_0) < inf_{u\in\partial B_M}\varphi(u).$$

From (2.1) we have

$$\|u\|_\infty \le \sqrt{1+T}\|u\|, \quad u \in X.$$

When $\|u\| \to 0$, by assumptions (f_0) we implies there exists $\varepsilon > 0$ be small enough, such that $\varepsilon < \frac{m_0}{\lambda p^+(1+T)^{\frac{p^++2}{2}}}T^{\frac{p^+-2}{2}}$ and $F(k,t) \le \varepsilon|t|^{p^+}$, therefore, one has

$$
\begin{aligned}
\varphi(u) &\ge \frac{m_0}{p^+}T^{\frac{p^+-2}{2}}\|u\|^{p^+} - \lambda\varepsilon\sum_1^{T+1}|u|^{p^+} - \lambda c\sum_1^{T+1}|u|^q \\
&\ge \frac{m_0}{p^+}T^{\frac{p^+-2}{2}}\|u\|^{p^+} - \lambda\varepsilon\sum_1^{T+1}\|u\|_\infty^{p^+} \\
&\ge \frac{m_0}{p^+}T^{\frac{p^+-2}{2}}\|u\|^{p^+} - \lambda\varepsilon(T+1)(1+T)^{\frac{p^+}{2}}\|u\|^{p^+} \\
&\ge \left(\frac{m_0}{p^+}T^{\frac{p^+-2}{2}} - \lambda\varepsilon(1+T)^{\frac{P^++2}{2}}\right)\|u\|^{p^+},
\end{aligned}
$$

therefore, there exist $r > 0$, $\delta > 0$ such that $\varphi(u) \ge \delta > 0$ for every $\|u\| = r$, We choosing $M = r$, so $\varphi(u) > 0 = \varphi(0) \ge \varphi(u_0)$ for $u \in \partial B_M$. Hence $u_0 \in B_M$ and $\varphi'(u_0) = 0$.
Step 2. Since u_0 is a minimum point of φ on X, we can consider $M > 0$ sufficiently large such that $\varphi(u_0) \le 0 < \inf_{u\in\partial B_M}\varphi(u)$, where $B_M = \{u \in X; \|u\| < M\}$. Now we will illustrate that there exists

u_1 with $\|u_1\| > M$ such that $\varphi(u_1) < \inf_{\partial B_M} \varphi(u)$. For this, Consider the function $e_1 : [1, T]_{\mathbb{Z}} \to \mathbb{R}$ such that there exists k_0 an integer satisfying $1 \leq k_0 \leq T$ for which $e_1(k_0) = 1$ and $e_1(k) = 0$ for any $k \in [1, T]_{\mathbb{Z}} \setminus \{k_0\}$. Thus, we deduce that $e_1 \in X$. let $u_1 = re_1, r > 0$ and $\|e_1\| = 1$. By [4, Remark 3.1] and (A_0) there exist constants $a_1, a_2 > 0$ such that $F(k, u(k)) \geq a_1|u(k)|^\nu - a_2$ for all $k \in [1, T]$. Thus

$$\begin{aligned}
\varphi(u_1) &= (\Phi - \lambda\Psi)(re_1) \\
&\leq \frac{m_1}{p^-} T(\|re_1\|^{p^+} + 1) - \lambda \sum_{1}^{T+1} F(k, re_1(k)) \\
&\leq \frac{m_1}{p^-} T(r^{p^+}\|e_1(k_0)\|^{p^+} + 1) - \lambda r^\nu a_1 |e_1(k_0)|^\nu + \lambda(T+1)a_2.
\end{aligned}$$

Since $\nu > p^+$, there exists sufficiently large $r > M > 0$ so that $\varphi(re_1) < 0$. Hence, $\max\{\varphi(u_0), \varphi(u_1)\} < \inf_{\partial B_M} \varphi(u)$. Then, Theorem 2.5 gives the critical point u^*. Therefore, u_0 and u^* are two critical points of φ, which are two solutions of the problem (1.1). □

We now present the following example to illustrate Theorem 3.1.

Example 3.3. *Consider* $T = 3$, $M(t) = 2 + \cos t$, *for* $t \in \mathbb{R}^+$, $p(k) = \frac{2}{3}k + 4$ *for* $k \in [0, 3]_{\mathbb{Z}}$ *and*

$$f(k,t) = \begin{cases} 7t^6, & |t| > 1, \\ 7t^8, & |t| \leq 1 \end{cases}$$

for all $k \in [0, 3]_{\mathbb{Z}}$. *By the expression of* f, *we have*

$$F(t) = \begin{cases} t^7, & |t| > 1, \\ \frac{7}{9}t^9 + \frac{2}{9}, & |t| \leq 1. \end{cases}$$

We observe that $p^+ = 6, p^- = 4$ *and* M *satisfies the condition* (M_0) *with* $m_0 = 1$. *Also,* M *and* f *are continuous functions and* $f(k,t) \geq 0$ *for all* $t \in \mathbb{R}$. *We have* $\lim_{\xi \to 0} \frac{f(k,\xi)}{\xi^{p^+-1}} = \lim_{\xi \to 0} \frac{7\xi^8}{\xi^5} = 0$ *and letting* $c = 7$, $q = 6 > 5 = p^+$ *one has* $|f(x,t)| < c(1 + |t|^6)$, *for* $|t| \leq 1$. *Since* $\lim_{\xi \to \infty} \frac{\xi f(k,\xi)}{F(k,\xi)} = 7$, *by choosing* $T = 1$ *and* $\nu = 7 > 6 = p^+$, *we have* $7F(k,t) \leq tf(k,t)$, *for* $|t| > 1$. *So we see that all conditions* $(A_1), (f_0), (f_1)$ *are fulfilled, therefore, by applying Theorem 3.1, for every* $\lambda > 0$ *the problem*

$$\begin{cases} -(2 + \cos(\rho(u(k))))\Delta(\phi_{p(k-1)}(\Delta u(k-1))) = \lambda f(k, u(k)), & k \in [1, 3] \\ u(0) = u(4) = 0, \end{cases} \tag{3.1}$$

has at least two solutions.

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