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Existence and multiplicity of solutions to a nonlocal elliptic PDE with variable exponent and nonlinear boundary conditions

Existence et multiplicité de solutions d'un problème elliptique non-local à exposant variable avec conditions aux limites non-lineaires

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ABSTRACT. Using the Nehari manifold approach and some variational techniques, we discuss the multiplicity of positive solutions for the $p(x)$-Laplacian elliptic systems with Nonlinear boundary conditions.
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1. Introduction

In this paper, we study the following elliptic system

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda(a(x)|u|^{r_1(x)-2}u + |u|^{h_1(x)-2}u), & x \in \Omega, \\ -\Delta_{q(x)}v + |v|^{q(x)-2}v = \mu(b(x)|v|^{r_2(x)-2}v + |v|^{h_2(x)-2}v), & x \in \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = \frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}, & x \in \partial\Omega, \\ |\nabla v|^{q(x)-2}\frac{\partial v}{\partial n} = \frac{\beta(x)}{\alpha(x)+\beta(x)}c(x)|v|^{\beta(x)-2}v|u|^{\alpha(x)}, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded domain with smooth boundary, n is the outer unit normal to $\partial\Omega$ and $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, $\lambda, \mu > 0$, the functions p, q, r_1, r_2, h_1, h_2, $a, b \in C(\bar{\Omega})$ and $c, \alpha, \beta \in C(\partial\Omega)$. In this paper, for any $v \in C(\bar{\Omega})$ we denote

$$v^+ = \text{ess}\sup_{x\in\Omega} v(x), \qquad v^- = \text{ess}\inf_{x\in\Omega} v(x).$$

Let us define for every $x \in \Omega$, $p^*(x) = \frac{Np(x)}{N-p(x)}$, $q^*(x) = \frac{Nq(x)}{N-q(x)}$, and for every $x \in \partial\Omega$, $p^*_\partial(x) = \frac{(N-1)p(x)}{N-p(x)}$ and $q^*_\partial(x) = \frac{(N-1)q(x)}{N-q(x)}$, when $p(x) < N$, $q(x) < N$.

Through the paper, we always assume that

(H$_0$) $\forall x \in \bar{\Omega}, 1 < p(x) < h_1(x) < r_1(x) < p^*(x)$ and

$$1 < p^- \leq p^+ < h_1^- \leq h_1^+ < r_1^- \leq r_1^+.$$

(H$_1$) $\forall x \in \bar{\Omega}, 1 < q(x) < h_2(x) < r_2(x) < q^*(x)$ and

$$1 < q^- \leq q^+ < h_2^- \leq h_2^+ < r_2^- \leq r_2^+.$$

(H$_2$) $\min\{h_1^-, h_2^-\} > \max\{p^+, q^+\}$.

(H$_3$) $\forall x \in \partial\Omega, 1 < \alpha(x), \beta(x)$ such that $\alpha(x) + \beta(x) < p(x) < p_\partial^*(x)$ and

$$\alpha^- + \beta^- \le \alpha^+ + \beta^+ < p^- \le p^+,$$

(H$_4$) $\forall x \in \partial\Omega,\ \alpha(x) + \beta(x) < q(x) < q_\partial^*(x)$ and

$$\alpha^- + \beta^- \le \alpha^+ + \beta^+ < q^- \le q^+,$$

(H$_5$) $a(x), b(x), c(x) \ge 0, a(x) \in L^{k_1(x)}(\Omega), b(x) \in L^{k_2(x)}(\Omega), c(x) \in L^{k_3(x)}(\partial\Omega), k_i \in C(\bar{\Omega})$ $(i = 1,2), k_1^-, k_2^- > 1, k_3 \in C(\partial\Omega), k_3^- = \operatorname{ess}\inf_{x\in\partial\Omega} k_3(x) > 1$ and there are $s_i(x) (i = 1,2,3,4)$ such that $s_1(x), s_2(x) \in L^\infty(\Omega)$ and $\forall x \in \Omega, p(x) \le s_1(x) \le p^*(x), q(x) \le s_2(x) \le q^*(x)$ and $s_3(x), s_4(x) \in L^\infty(\partial\Omega), \forall x \in \partial\Omega, p(x) \le s_3(x) \le p_\partial^*(x), q(x) \le s_4(x) \le q_\partial^*(x)$ where

$$\frac{1}{k_1(x)} + \frac{r_1(x)}{s_1(x)} = 1,$$
$$\frac{1}{k_2(x)} + \frac{r_2(x)}{s_2(x)} = 1,$$
$$\frac{1}{k_3(x)} + \frac{\alpha(x)}{s_3(x)} + \frac{\beta(x)}{s_4(x)} = 1.$$

There have been many authors used the Nehari manifold and fibering maps to solve semilinear and quasilinear problems (see [2, 3, 4, 5, 6, 13, 20]). Wu in [21] proved that, there exists $C_0 > 0$ such that if the parameter λ, μ satisfy $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C_0$, then problem (1.1) for $p = 2, r(x) = r, \alpha(x) = \alpha, \beta(x) = \beta$ and $1 < r < 2 < \alpha + \beta < 2^*$, has at least two solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $u_0^\pm \ge 0, v_0^\pm \ge 0$ in Ω and $u_0^\pm \ne 0, v_0^\pm \ne 0$. By the fibering method, Drabek and Pohozaev [6], Bozhkov and Mitidieri [4] studied respectively the existence of multiple solution to a p-Laplacian single equation and (p,q)-Laplacian system. In [5] Brown and Zhang used the relationship between the Nehari manifold and fibering maps to show how existence and nonexistence results of positive solutions of the equation are linked to properties of the Nehari manifold. In [2] Afrouzi and Rasouli for the case $p(x) = p, r(x) = r, \alpha(x) = \alpha, \beta(x) = \beta$ discussed the existence and multiplicity results of nontrivial nonnegative solutions for the system. In [16] Mashiyev, Ogras, Yucedag and Avci studied the multiplicity of positive solutions for the following elliptic equation

$$\begin{cases} -\Delta_{p(x)} u = \lambda a(x)|u|^{q(x)-2}u + b(x)|u|^{h(x)-2} & \text{in } \Omega \\ u(x) = 0 \qquad \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary in R^N, $p, q, h \in C^1(\bar{\Omega})$ such that $1 < q(x) < p(x) < h(x) < p^*(x) (p^*(x) = \frac{Np(x)}{N-p(x)}$ if $N > p(x), p^*(x) = \infty$ if $N \le p(x)), 1 < p^- :=$ ess $\inf_{x\in\Omega} p(x) \le$ ess $\sup_{x\in\Omega} p(x) < \infty, 1 < q^- \le q^+ < p^- \le p^+ < h^- \le h^+, \lambda > 0 \in R$ and $a, b \in C(\bar{\Omega})$ are non-negative weight functions with compact support in Ω.

In this paper, we have generalized the articles of Afrouzi-Rasouli [2] and Mashiyev, Ogras, Yucedag and Avci [16], to the $p(x)$-Laplacian by using the Nehari manifold under the similar conditions. We shall discuss the multiplicity of positive solutions for the problem (1.1) and prove the existence of at least two positive solutions.

If we consider all above-mentioned papers the use of the Nehari manifold for the system (1.1) makes our study very interesting.

This paper is divided into three parts. In the second part we introduce some basic properties of the variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$, where $\Omega \subset R^N$ is a domain, section 3 gives main results and proofs.

2. Preliminaries

In order to deal with $p(x)$-Laplacian problem, we need some theories on spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [10]). If $\Omega \subset R^N$ is an open bounded domain, write

$$\begin{aligned} L^{\infty}_{+}(\Omega) &= \{p \in L^{\infty}(\Omega) : \text{ess}\inf_{x\in\Omega} p(x) \geq 1\}, \\ S(\Omega) &= \{u | u \text{ is a measurable real-valued function on } \Omega\} \end{aligned}$$

For any $p \in L^{\infty}_{+}(\Omega)$, we denote the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u \in S(\Omega) | \int_{\Omega} |u|^{p(x)} dx < \infty\}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\{\lambda > 0 | \int_{\Omega} |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\},$$

and $(L^{p(x)}(\Omega), |.|_{p(x)})$ becomes a Banach space, we call it variable exponent Lebesgue space.

Proposition 2.1. *(See [10]). The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a separable, reflexive (if $1 < p^- \leq p^+ < \infty$) and uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$|\int_{\Omega} uv dx| \leq (\frac{1}{p^-} + \frac{1}{p'^-}) |u|_{p(x)} |v|_{p'(x)}.$$

Proposition 2.2. *(See [9]). If $\frac{1}{p(x)} + \frac{1}{q(x)} + \frac{1}{r(x)} = 1$, then for any $u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega)$ and $w \in L^{r(x)}(\Omega)$,*

$$|\int_{\Omega} uvw dx| \leq (\frac{1}{p^-} + \frac{1}{q^-} + \frac{1}{r^-}) |u|_{p(x)} |v|_{q(x)} |w|_{r(x)} \leq 3 |u|_{p(x)} |v|_{q(x)} |w|_{r(x)}.$$

Proposition 2.3. *(See [10]). Set*

$$\rho(u) = \int_{\Omega} |u|^{p(x)} dx, \qquad \forall u \in L^{p(x)}(\Omega),$$

then,

(i) $|u|_{p(x)} < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \leq \rho(u) \leq |u|^{p^+}_{p(x)}; |u|_{p(x)} < 1 \Rightarrow |u|^{p^-}_{p(x)} \geq \rho(u) \geq |u|^{p^+}_{p(x)}$;

(iii) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0; |u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

Proposition 2.4. *(See [10]). If $u, u_n \in L^{p(x)}(\Omega), n = 1, 2, ...,$ then the following statements are equivalent to each other:*

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$;

(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$;

(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$

and the norm

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \qquad \forall u \in W^{1,p(x)}(\Omega),$$

makes $W^{1,p(x)}(\Omega)$ a separable and reflexive Banach space. The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W_0^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. In this paper we will use the equivalent norm on $W^{1,p(x)}(\Omega)$;

$$\|u\|_{p(x)} = \inf\{\lambda > 0 \mid \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{\lambda^{p(x)}} dx \leq 1\}.$$

Proposition 2.5. *(See [9]).If we define $I(u) = \int_\Omega |\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} dx$, then for $u, u_k \in W^{1,p(x)}(\Omega)$:*

(1) $\|u\|_{p(x)} < 1(= 1; > 1) \Leftrightarrow I(u) < 1(= 1; > 1)$;

(2) $\|u\|_{p(x)} > 1$, *then* $\|u\|^{p^-}_{p(x)} \leq I(u) \leq \|u\|^{p^+}_{p(x)}$;

(3) $\|u\|_{p(x)} < 1$, *then* $\|u\|^{p^+}_{p(x)} \leq I(u) \leq \|u\|^{p^-}_{p(x)}$;

(4) $\|u_k\|_{p(x)} \to 0(\to \infty) \Leftrightarrow I(u_k) \to 0(\to \infty)$.

Proposition 2.6. *(See [8]). If $s(x) \in C(\bar{\Omega}), 1 < p(x) \leq s(x) < p^*(x)$ and*

$$ess\, inf_{x\in\overline{\Omega}}(p^*(x) - s(x)) > 0,$$

for all $x \in \bar{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous.

Define the variable exponent Lebesgue space by

$$L^{p(x)}(\partial\Omega)=\{u|u:\partial\Omega\to R \text{ is a measurable and } \int\limits_{\partial\Omega}|u|^{p(x)}d\sigma_x<+\infty\}$$

with the norm

$$|u|_{L^{p(x)}(\partial\Omega)}=\inf\{\lambda>0|\int\limits_{\partial\Omega}|\frac{u(x)}{\lambda}|^{p(x)}d\sigma_x\le 1\},$$

where $d\sigma_x$ is the measure on the boundary. Then, $L^{p(x)}(\partial\Omega)$ is a Banach space and Proposition 2.3 is satisfied for $\rho(u)=\int_{\partial\Omega}|u|^{p(x)}dx,\ \forall u\in L^{p(x)}(\partial\Omega)$.

Proposition 2.7. *(See [13]).*

(i) *If $q\in C(\bar{\Omega})$ and $q(x)<p(x)$ for any $x\in\bar{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous;*

(ii) *If $q\in C(\bar{\Omega})$ and $q(x)<p_\partial(x)$ for any $x\in\bar{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\partial\Omega)$ is compact and continuous.*

Proposition 2.8. *(See [9]). If $|u|^{q(x)}\in L^{s(x)/q(x)}(\Omega)$, where $s(x),q(x)\in L_+^\infty(\Omega), q(x)\le s(x)$, then $u\in L^{s(x)}(\Omega)$ and there is a number $\overline{q}\in[q^-,q^+]$ such that $\left||u|^{q(x)}\right|_{s(x)/q(x)}=(|u|_{s(x)})^{\overline{q}}$.*

In what follows, W will denote the Cartesian product of two Sobolev spaces $W^{1,p(x)}(\Omega)$ and $W^{1,q(x)}(\Omega)$, i.e., $W=W^{1,p(x)}(\Omega)\times W^{1,q(x)}(\Omega)$. Let us choose on W the norm $\|.\|$ defined by

$$\|(u,v)\|=\max\{\|u\|_p,\|v\|_q\},$$

where $\|.\|_p$ is the norm of $W^{1,p(x)}(\Omega)$ and $\|.\|_q$ is the norm of $W^{1,q(x)}(\Omega)$.

3. Main results

Definition 3.1. We say that $(u,v)\in W$ is a weak solution of problem (1.1) if for all $(\xi,\eta)\in W$ we have

$$\begin{aligned}&\int\limits_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla\xi dx+\int\limits_\Omega|u|^{p(x)-2}u\xi dx\\ =\ &\lambda\Big(\int\limits_\Omega a(x)|u|^{r_1(x)-2}u\xi dx+\int\limits_\Omega|u|^{h_1(x)-2}u\xi dx\Big)+\int\limits_{\partial\Omega}\frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}\xi dx,\end{aligned}$$

and

$$\begin{aligned}&\int\limits_\Omega|\nabla v|^{q(x)-2}\nabla v.\nabla\eta dx+\int\limits_\Omega|v|^{q(x)-2}v\eta dx\\ =\ &\mu\Big(\int\limits_\Omega b(x)|v|^{r_2(x)-2}v\eta dx+\int\limits_\Omega|v|^{h_2(x)-2}v\eta dx\Big)+\int\limits_{\partial\Omega}\frac{\beta(x)}{\alpha(x)+\beta(x)}c(x)|v|^{\beta(x)-2}v|u|^{\alpha(x)}\eta dx.\end{aligned}$$

It is clear that problem (1.1) has a variational structure. Let $J_{\lambda,\mu} : W \to R$ be the corresponding energy functional of problem (1.1) defined by

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega \frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})dx \\ &- \lambda\Big(\int_\Omega \frac{1}{r_1(x)}a(x)|u|^{r_1(x)} + \int_\Omega \frac{1}{h_1(x)}|u|^{h_1(x)}dx\Big) \\ &- \mu\Big(\int_\Omega \frac{1}{r_2(x)}b(x)|v|^{r_2(x)}dx + \int_\Omega \frac{1}{h_2(x)}|v|^{h_2(x)}dx\Big) \\ &- \int_{\partial\Omega} \frac{1}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx. \end{aligned}$$

Let

$$\begin{aligned} P(u,v) &= \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega (|\nabla v|^{q(x)} + |v|^{q(x)})dx, \\ Q(u,v) &= \lambda\Big(\int_\Omega a(x)|u|^{r_1(x)}dx + \int_\Omega |u|^{h_1(x)}dx\Big) + \mu\Big(\int_\Omega b(x)|v|^{r_2(x)}dx + \int_\Omega |v|^{h_2(x)}dx\Big), \\ R(u,v) &= \int_{\partial\Omega} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx. \end{aligned}$$

It is well known that the weak solution of problem (1.1) is of the critical points of the energy functional $J_{\lambda,\mu}$. Let I be the energy functional associated with an elliptic problem on a Banach space X. If I is bounded below and I has a minimizer on X, then this minimizer is a critical point of I. So it is a solution of the corresponding elliptic problem. However, the energy functional $J_{\lambda,\mu}$ is not bounded below on the whole space W, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives rise to solution to (1.1). A good candidate for an appropriate subset of W is the Nehari manifold.

Then we introduce the following notation: for any functional $f : W \to R$ we denote by $f'(u,v)(h_1,h_2)$ the Gateaux derivative of f at $(u,v) \in W$ in the direction of $(h_1,h_2) \in W$, and

$$f^{(1)}(u,v)h_1 = f'(u+\epsilon h_1, v)|_{\epsilon=0}, \qquad f^{(2)}(u,v)h_2 = f'(u, v+\delta h_2)|_{\delta=0}.$$

Consider the Nehari minimization problem for $\lambda, \mu > 0$,

$$\alpha_0(\lambda,\mu) = \inf\{J_{\lambda,\mu}(u,v) : (u,v) \in M_{\lambda,\mu}\}$$

where $M_{\lambda,\mu} = \{(u,v) \in W\backslash\{(0,0)\} : \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = \langle J^1_{\lambda,\mu}(u,v)u, J^2_{\lambda,\mu}(u,v)v\rangle = 0\}$. It is clear that all critical points of $J_{\lambda,\mu}$ must lie on $M_{\lambda,\mu}$ which is known as the Nehari manifold and local minimizers on $M_{\lambda,\mu}$ are usually critical points of $J_{\lambda,\mu}$. Thus $(u,v) \in M_{\lambda,\mu}$ if and only if

$$
\begin{aligned}
I_{\lambda,\mu}(u,v):=\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle &= \int_{\Omega}(|\nabla u|^{p(x)}+|u|^{p(x)})dx+\int_{\Omega}(|\nabla v|^{q(x)}+|v|^{q(x)})dx\\
&- \lambda\Big(\int_{\Omega}a(x)|u|^{r_1(x)}dx+\int_{\Omega}|u|^{h_1(x)}dx\Big)\\
&- \mu\Big(\int_{\Omega}b(x)|v|^{r_2(x)}dx+\int_{\Omega}|v|^{h_2(x)}dx\Big)\\
&- \int_{\partial\Omega}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx=0. \qquad (3.1)
\end{aligned}
$$

Then for $(u,v)\in M_{\lambda,\mu}$, we have

$$
\begin{aligned}
\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle &= \int_{\Omega}p(x)(|\nabla u|^{p(x)}+|u|^{p(x)})dx+\int_{\Omega}q(x)(|\nabla v|^{q(x)}+|v|^{q(x)})dx\\
&- \lambda\Big(\int_{\Omega}r_1(x)a(x)|u|^{r_1(x)}dx+\int_{\Omega}h_1(x)|u|^{h_1(x)}dx\Big)\\
&- \mu\Big(\int_{\Omega}r_2(x)b(x)|v|^{r_2(x)}dx+\int_{\Omega}h_2(x)|v|^{h_2(x)}dx\\
&- \int_{\partial\Omega}(\alpha(x)+\beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.
\end{aligned}
$$

Now, we split $M_{\lambda,\mu}$ into three parts:

$$M^{+}_{\lambda,\mu}=\{(u,v)\in M_{\lambda,\mu}:\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle>0\},$$

$$M^{0}_{\lambda,\mu}=\{(u,v)\in M_{\lambda,\mu}:\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle=0\},$$

$$M^{-}_{\lambda,\mu}=\{(u,v)\in M_{\lambda,\mu}:\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle<0\}.$$

Theorem 3.2. *Suppose that (u_0,v_0) is a local maximum or minimum for $J_{\lambda,\mu}$ on $M_{\lambda,\mu}$. If $(u_0,v_0)\notin M^0_{\lambda,\mu}$, then (u_0,v_0) is a critical point of $J_{\lambda,\mu}$.*

Proof. The proof of theorem 3.2 can be obtained directly from the following lemmas. □

Lemma 3.3. *There exists $\delta>0$ such that for $0<\lambda+\mu<\delta$, we have $M^0_{\lambda,\mu}=\emptyset$.*

Proof. Suppose otherwise, then for

$$\delta=\frac{(\min\{p^-,q^-\}-\alpha^+-\beta^+)}{(\max\{r_1^+,r_2^+\}-\alpha^+-\beta^+)C_4}\left[\frac{(\min\{h_1^-,h_2^-\}-\max\{p^+,q^+\})}{C_5(\min\{h_1^-,h_2^-\}-\alpha^--\beta^-)}\right]^{\frac{\max\{r_1^+,r_2^+\}-\min\{p^-,q^-\}}{\min\{p^-,q^-\}-\alpha^+-\beta^+}},$$

where C_4, C_5 are positive constants and specified later, there exists (λ, μ) with $0 < \lambda + \mu < \delta$ such that $M^0_{\lambda,\mu} \neq \emptyset$. Then for $(u, v) \in M^0_{\lambda,\mu}$ we have

$$\begin{aligned}
0 &= \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = \int_\Omega p(x)(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega q(x)(|\nabla v|^{q(x)} + |v|^{q(x)})dx \\
&- \lambda\Big(\int_\Omega a(x)r_1(x)|u|^{r_1(x)}dx + \int_\Omega h_1(x)|u|^{h_1(x)}dx\Big) \\
&- \mu\Big(\int_\Omega b(x)r_2(x)|v|^{r_2(x)}dx + \int_\Omega h_2(x)|v|^{h_2(x)}dx\Big) \\
&- \int_{\partial\Omega}(\alpha(x)+\beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\geq \min\{p^-, q^-\}\Big[\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega (|\nabla v|^{q(x)} + |v|^{q(x)})dx\Big] \\
&- \max\{r_1^+, r_2^+\}\Big[\lambda\Big(\int_\Omega a(x)|u|^{r_1(x)}dx + \int_\Omega |u|^{h_1(x)}dx\Big) \\
&+ \mu\Big(\int_\Omega b(x)|v|^{r_2(x)}dx + \int_\Omega |v|^{h_2(x)}dx\Big)\Big] \\
&- (\alpha^+ + \beta^+)\int_{\partial\Omega} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&= \Big(\min\{p^-, q^-\} - \alpha^+ - \beta^+\Big)P(u,v) + \Big(\alpha^+ + \beta^+ - \max\{r_1^+, r_2^+\}\Big)Q(u,v),
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
0 &= \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = \int_\Omega p(x)(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega q(x)(|\nabla v|^{q(x)} + |v|^{q(x)})dx \\
&- \lambda\Big(\int_\Omega a(x)r_1(x)|u|^{r_1(x)}dx + \int_\Omega h_1(x)|u|^{h_1(x)}dx\Big) \\
&- \mu\Big(\int_\Omega b(x)r_2(x)|v|^{r_2(x)}dx + \int_\Omega h_2(x)|v|^{h_2(x)}dx\Big) \\
&- \int_{\partial\Omega}(\alpha(x)+\beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\leq \max\{p^+, q^+\}\Big[\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega (|\nabla v|^{q(x)} + |v|^{q(x)})dx\Big] \\
&- \min\{h_1^-, h_2^-\}\Big[\lambda\Big(\int_\Omega a(x)|u|^{r_1(x)} + \int_\Omega |u|^{h_1(x)}dx\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \mu\Bigg(\int_\Omega b(x)|v|^{r_2(x)}dx + \int_\Omega |v|^{h_2(x)}dx \Bigg)\Bigg] \\
&- \quad (\alpha^- + \beta^-) \int_{\partial\Omega} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&= \quad \Bigg(\max\{p^+, q^+\} - \min\{h_1^-, h_2^-\} \Bigg) P(u,v) + \Bigg(\min\{r_1^-, r_2^-\} - \alpha^- - \beta^- \Bigg) R(u,v). \qquad (3.3)
\end{aligned}
$$

By Propositions 2.1, 2.2, 2.6, 2.7 and 2.8 we have

$$
\begin{aligned}
Q(u,v) &= \lambda\Bigg(\int_\Omega a(x)|u|^{r_1(x)} + \int_\Omega |u|^{h_1(x)}dx \Bigg) \\
&+ \mu\Bigg(\int_\Omega b(x)|v|^{r_2(x)}dx + \int_\Omega |v|^{h_2(x)}dx \Bigg) \\
&\leq \lambda\Bigg(2|a(x)|_{k_1(x)} \left| |u|^{r_1(x)} \right|_{\frac{s_1(x)}{r_1(x)}} + |u|^{p^+}_{p(x)} \Bigg) \\
&+ \mu\Bigg(2|b(x)|_{k_2(x)} \left| |v|^{r_2(x)} \right|_{\frac{s_2(x)}{r_2(x)}} + |v|^{q^+}_{q(x)} \Bigg) \\
&\leq \lambda\Bigg(2|a(x)|_{r_1(x)} (|u|_{s_1(x)})^{\bar{r_1}} + \|u\|^{p^+}_p \Bigg) \\
&+ \mu\Bigg(2|b(x)|_{k_2(x)} (|v|_{s_2(x)})^{\bar{r_2}} + \|v\|^{q^+}_q \Bigg) \\
&\leq \lambda C_1 \|(u,v)\|^{r_1^+} + \mu C_2 \|(u,v)\|^{r_2^+} \\
&\leq (\lambda + \mu) C_3 \|(u,v)\|^{\max\{r_1^+, r_2^+\}}, \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
R(u,v) &= \int_{\partial\Omega} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\leq 3|c(x)|_{k_3(x)} \left| |u|^{\alpha(x)} \right|_{\frac{s_3(x)}{\alpha(x)}} \left| |v|^{\beta(x)} \right|_{\frac{s_4(x)}{\beta(x)}} \\
&\leq 3|c(x)|_{k_3(x)} (|u|_{s_3(x)})^{\overline{\alpha}} (|v|_{s_4(x)})^{\overline{\beta}} \\
&\leq C_4 \|u\|^{\overline{\alpha}}_p \|v\|^{\overline{\beta}}_q \\
&\leq C_5 \|(u,v)\|^{\alpha^+ + \beta^+}, \qquad (3.5)
\end{aligned}
$$

By using (3.4), (3.5) in (3.2) and (3.3) we get

$$
\|(u,v)\| \geq \left[\frac{(\min\{p^-, q^-\} - \alpha^+ - \beta^+)}{(\lambda + \mu) C_4 (\max\{r_1^+, r_2^+\} - \alpha^+ - \beta^+)} \right]^{\frac{1}{\max\{r_1^+, r_2^+\} - \min\{p^-, q^-\}}}
$$

and

$$
\|(u,v)\| \leq \left[\frac{C_5 (\min\{r_1^-, r_2^-\} - \alpha^- - \beta^-)}{(\min\{r_1^-, r_2^-\} - \max\{p^+, q^+\})} \right]^{\frac{1}{\min\{p^-, q^-\} - \alpha^+ - \beta^+}}.
$$

This implies $\lambda + \mu \geq \delta$ which is a contradiction. Thus we can conclude that there exists $\delta > 0$ such that for $0 < \lambda + \mu < \delta$, we have $M^0_{\lambda,\mu} = \emptyset$. □

Lemma 3.4. *The energy functional $J_{\lambda,\mu}$ is coercive and bounded below on $M_{\lambda,\mu}$.*

Proof. If $(u, v) \in M_{\lambda,\mu}$ and $\|(u, v)\| > 1$. Without loss of generality, we may assume $\|u\|_{p(x)}, \|v\|_{q(x)} > 1$, we have

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega \frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})dx \\
&- \lambda\Big(\int_\Omega \frac{1}{r_1(x)} a(x)|u|^{r_1(x)}dx + \int_\Omega \frac{1}{h_1(x)}|u|^{h_1(x)}dx\Big) \\
&- \mu\Big(\int_\Omega \frac{1}{r_2(x)} b(x)|v|^{r_2(x)}dx + \int_\Omega \frac{1}{h_2(x)}|v|^{h_2(x)}dx\Big) \\
&- \int_{\partial\Omega} \frac{1}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\geq \frac{1}{p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx + \frac{1}{q^+}\int_\Omega (|\nabla v|^{q(x)} + |v|^{q(x)})dx \\
&- \frac{\lambda}{h_1^-}\Big(\int_\Omega a(x)|u|^{r_1(x)}dx + \int_\Omega |u|^{h_1(x)}dx\Big) \\
&- \frac{\mu}{h_2^-}\Big(\int_\Omega b(x)|v|^{r_2(x)}dx + \int_\Omega |v|^{h_2(x)}dx\Big) \\
&- \frac{1}{\alpha^- + \beta^-}\int_{\partial\Omega} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\geq \Big(\frac{1}{\max\{p^+, q^+\}} - \frac{1}{\min\{h_1^-, h_2^-\}}\Big)\Big[\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx \\
&+ \int_\Omega (|\nabla v|^{q(x)} + |v|^{q(x)})dx\Big] \\
&- \Big(\frac{1}{\alpha^- + \beta^-} - \frac{1}{\min\{h_1^-, h_2^-\}}\Big)\int_{\partial\Omega} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\geq \Big(\frac{1}{\max\{p^+, q^+\}} - \frac{1}{\min\{h_1^-, h_2^-\}}\Big)\|(u,v)\|^{\min\{p^-, q^-\}} \\
&- \Big(\frac{1}{\alpha^- + \beta^-} - \frac{1}{\min\{h_1^-, h_2^-\}}\Big)C_6\|(u,v)\|^{\alpha^+ + \beta^+}.
\end{aligned}$$

Since $p^-, q^- > (\alpha^+ + \beta^+)$ so, $J_{\lambda,\mu}(u, v) \to \infty$ as $\|(u, v)\| \to \infty$. This implies $J_{\lambda,\mu}(u, v)$ is coercive and bounded below on $M_{\lambda,\mu}$. □

By Lemma 3.3, for $0 < \lambda + \mu < \delta$, we can write $M_{\lambda,\mu} = M^+_{\lambda,\mu} \cup M^-_{\lambda,\mu}$ and define

$$\alpha_0^+(\lambda,\mu) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v) \qquad \alpha_0^-(\lambda,\mu) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).$$

Lemma 3.5. *If $0 < \lambda + \mu < \delta$, then for all $(u,v) \in M^+_{\lambda,\mu}$, $J_{\lambda,\mu}(u,v) < 0$.*

Proof. Let $(u,v) \in M^+_{\lambda,\mu}$. We have

$$\begin{aligned}
&\max\{p^+,q^+\}\Big(\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega (|\nabla v|^{q(x)} + |v|^{q(x)})dx\Big) \\
&- \min\{h_1^-,h_2^-\}\Big[\lambda\Big(\int_\Omega \frac{1}{r_1(x)} a(x)|u|^{r_1(x)}dx + \int_\Omega \frac{1}{h_1(x)}|u|^{h_1(x)}dx\Big) \\
&+\mu\Big(\int_\Omega \frac{1}{r_2(x)} b(x)|v|^{r_2(x)}dx + \int_\Omega \frac{1}{h_2(x)}|v|^{h_2(x)}dx\Big)\Big] \\
&-(\alpha^- + \beta^-)\int_{\partial\Omega} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx > 0. \qquad (3.6)
\end{aligned}$$

By definition of $J_{\lambda,\mu}(u,v)$ we can write

$$\begin{aligned}
J_{\lambda,\mu}(u,v) \;\le\; & \Big(\frac{1}{\min\{p^-,q^-\}} - \frac{1}{\alpha^+ + \beta^+}\Big)\Big[\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx \\
+\; & \int_\Omega (|\nabla v|^{q(x)} + |v|^{q(x)})dx\Big] \\
+\; & \Big(\frac{1}{\alpha^+ + \beta^+} - \frac{1}{\max\{r_1^+,r_2^+\}}\Big)\Big[\lambda\Big(\int_\Omega a(x)|u|^{r_1(x)} + \int_\Omega |u|^{h_1(x)}dx\Big) \\
+\; & \mu\Big(\int_\Omega b(x)|v|^{r_2(x)}dx + \int_\Omega |v|^{h_2(x)}dx\Big)\Big]. \qquad (3.7)
\end{aligned}$$

Now, if we multiply (3.1) by $-(\alpha^- + \beta^-)$ and add with (3.6), we get

$$Q(u,v) \le \frac{\max\{p^+,q^+\} - \alpha^- - \beta^-}{\min\{h_1^-,h_2^-\} - \alpha^- - \beta^-} P(u,v) \qquad (3.8)$$

and applying (3.8) in (3.7), it follows

$$\begin{aligned}
J_{\lambda,\mu}(u,v) \;&\le\; \Big[\frac{\alpha^+ + \beta^+ - \min\{p^-,q^-\}}{\min\{p^-,q^-\}(\alpha^+ + \beta^+)} + \frac{\max\{p^+,q^+\} - \alpha^- - \beta^-}{\max\{r_1^+,r_2^+\}(\alpha^+ + \beta^+)}\Big] P(u,v) \\
&\le\; -\Big[\frac{(\min\{p^-,q^-\} - \alpha^+ - \beta^+)(\max\{r_1^+,r_2^+\} - \min\{p^-,q^-\})}{\max\{r_1^+,r_2^+\}(\alpha^+ + \beta^+)\min\{p^-,q^-\}}\Big] P(u,v) < 0.
\end{aligned}$$

Thus $\alpha_0^+(\lambda,\mu) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v) < 0$. □

Lemma 3.6. *If $0 < \lambda + \mu < \delta$, there exists a minimizer of $J_{\lambda,\mu}(u,v)$ on $M^+_{\lambda,\mu}$.*

Proof. Since $J_{\lambda,\mu}$ is bounded below on $M_{\lambda,\mu}$ and so on $M^+_{\lambda,\mu}$. Then, there exists a minimizing sequence $\{(u^+_n, v^+_n)\} \subseteq M^+_{\lambda,\mu}$ such that

$$\lim_{n\to\infty} J_{\lambda,\mu}(u^+_n, v^+_n) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = \alpha^+_0(\lambda,\mu) < 0.$$

Since $J_{\lambda,\mu}$ is coercive, $\{(u^+_n, v^+_n)\}$ is bounded below in W. Thus, we may assume that, without loss of generality, $(u^+_n, v^+_n) \rightharpoonup (u^+_0, v^+_0)$ in W. Hence $u^+_n \rightharpoonup u^+_0$ in $W^{1,p(x)}(\Omega)$, $v^+_n \rightharpoonup v^+_0$ in $W^{1,q(x)}(\Omega)$ and by the compact embeddings we have

$$\begin{aligned} u^+_n \to u^+_0 \quad &\text{in} \quad L^{r_1(x)}(\Omega),\ L^{h_1(x)}(\Omega),\ L^{\alpha(x)+\beta(x)}(\Omega),\\ v^+_n \to v^+_0 \quad &\text{in} \quad L^{r_2(x)}(\Omega),\ L^{h_2(x)}(\Omega),\ L^{\alpha(x)+\beta(x)}(\Omega). \end{aligned}$$

This implies

$$\begin{aligned} Q(u^+_n, v^+_n) \to Q(u^+_0, v^+_0) \qquad &\text{as } n\to\infty,\\ R(u^+_n, v^+_n) \to R(u^+_0, v^+_0) \qquad &\text{as } n\to\infty. \end{aligned}$$

Now, we shall prove $u^+_n \to u^+_0$ in $W^{1,p(x)}(\Omega)$, $v^+_n \to v^+_0$ in $W^{1,q(x)}(\Omega)$. Suppose otherwise, then either

$$\|u^+_0\|_p < \liminf_{n\to\infty} \|u^+_n\|_p \qquad \text{or} \qquad \|v^+_0\|_q < \liminf_{n\to\infty} \|v^+_n\|_q.$$

Using the fact that $\langle J'_{\lambda,\mu}(u^+_n, v^+_n), (u^+_n, v^+_n)\rangle = 0$ and (3.5) we can write the followings

$$\begin{aligned} \lim_{n\to\infty} J_{\lambda,\mu}(u^+_n, v^+_n) \;&>\; \left(\frac{1}{\max\{p^+,q^+\}} - \frac{1}{\min\{h^-_1,h^-_2\}}\right)\lim_{n\to\infty} P(u_n,v_n)\\ &-\; \left(\frac{1}{\alpha^-+\beta^-} - \frac{1}{\min\{h^-_1,h^-_2\}}\right)\lim_{n\to\infty} R(u_n,v_n), \end{aligned}$$

$$\begin{aligned} \alpha^+_0(\lambda,\mu) \;&=\; \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v)\\ &>\; \left(\frac{1}{\max\{p^+,q^+\}} - \frac{1}{\min\{r^-_1,r^-_2\}}\right)\|(u^+_0,v^+_0)\|^{\min\{p^-,q^-\}}\\ &-\; \left(\frac{1}{\alpha^-+\beta^-} - \frac{1}{\min\{r^-_1,r^-_2\}}\right)\|(u^+_0,v^+_0)\|^{\alpha^++\beta^+}, \end{aligned}$$

since $\min\{p^-,q^-\} > \alpha^+ + \beta^+$, for $\|(u^+_0, v^+_0)\| > 1$, we have

$$\alpha^+_0(\lambda,\mu) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v) > 0.$$

So that is a contradiction. Hence

$$\begin{aligned} u^+_n &\to u^+_0 \text{ in } W^{1,p(x)}(\Omega),\\ v^+_n &\to v^+_0 \text{ in } W^{1,q(x)}(\Omega). \end{aligned}$$

This implies

$$J_{\lambda,\mu}(u_n^+, v_n^+) \to J_{\lambda,\mu}(u_0^+, v_0^+) = \inf_{u,v\in M_{\lambda,\mu}^+} J_{\lambda,\mu}(u,v) \qquad \text{as } n \to \infty.$$

Thus, (u_0^+, v_0^+) is a minimizer for $J_{\lambda,\mu}$ on $M_{\lambda,\mu}^+$. □

Lemma 3.7. *If $0 < \lambda + \mu < \delta$, then for all $(u,v) \in M_{\lambda,\mu}^-$, $J_{\lambda,\mu}(u,v) > 0$.*

Proof. Let $(u,v) \in M_{\lambda,\mu}^-(\Omega)$. We have

$$\begin{aligned}
&\min\{p^-, q^-\}\bigg(\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega (|\nabla v|^{q(x)} + |v|^{q(x)})dx\bigg) \\
&- \max\{h_1^+, h_2^+\}\bigg[\lambda\bigg(\int_\Omega a(x)|u|^{r_1(x)} + \int_\Omega |u|^{h_1(x)}dx\bigg) \\
&+\mu\bigg(\int_\Omega b(x)|v|^{r_2(x)}dx + \int_\Omega |v|^{h_2(x)}dx\bigg)\bigg] \\
&-(\alpha^- + \beta^-)\int_{\partial\Omega} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx < 0.
\end{aligned} \tag{3.9}$$

By definition of $J_{\lambda,\mu}(u,v)$ and (3.1), we have

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &> \left(\frac{1}{\max\{p^+, q^+\}} - \frac{1}{\min\{h_1^-, h_2^-\}}\right) P(u,v) \\
&- \left(\frac{1}{\alpha^- + \beta^-} - \frac{1}{\min\{h_1^-, h_2^-\}}\right) R(u,v).
\end{aligned} \tag{3.10}$$

Now, if we multiply (3.1) by $-\max\{r_1^+, r_2^+\}$ and add with (3.9), we get

$$R(u,v) \le \frac{(\max\{r_1^+, r_2^+\} - \min\{p^-, q^-\})}{\max\{r_1^+, r_2^+\} - \alpha^+ - \beta^+} P(u,v), \tag{3.11}$$

and applying (3.11) in (3.10), it follows

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &\ge \left(\frac{\min\{h_1^-, h_2^-\} - \max\{p^+, q^+\}}{\min\{h_1^-, h_2^-\}\max\{p^+, q^+\}}\right) P(u,v) \\
&+ \left(\frac{\max\{r_1^+, r_2^+\} - \min\{p^-, q^-\}}{\min\{h_1^-, h_2^-\}(\alpha^- + \beta^-)}\right) P(u,v) \\
&\ge \frac{(\min\{h_1^-, h_2^-\} - \max\{p^+, q^+\})(\alpha^- + \beta^- + \max\{p^+, q^+\})}{\min\{h_1^-, h_2^-\}\max\{p^+, q^+\}(\alpha^- + \beta^-)} P(u,v) > 0.
\end{aligned}$$

□

Theorem 3.8. *If $0 < \lambda + \mu < \delta$, there exists a minimizer of $J_{\lambda,\mu}$ on $M_{\lambda,\mu}^-$.*

Proof. Since $J_{\lambda,\mu}$ is bounded below on $M_{\lambda,\mu}$ and so on $M^-_{\lambda,\mu}$, then there exists a minimizing sequence$\{(u^-_n, v^-_n)$ $M^-_{\lambda,\mu}$ such that

$$\lim_{n\to\infty} J_{\lambda,\mu}(u^-_n, v^-_n) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v) = \alpha^-_0(\lambda, \mu).$$

Since $J_{\lambda,\mu}$ is coercive, $\{(u^-_n, v^-_n)\}$ is bounded below in W. Thus, we may assume that, without loss of generality, $(u^-_n, v^-_n) \rightharpoonup (u^-_0, v^-_0)$ in W. Hence $u^-_n \rightharpoonup u^-_0$ in $W^{1,p(x)}(\Omega)$, $v^-_n \rightharpoonup v^-_0$ in $W^{1,q(x)}(\Omega)$ and by the compact embeddings we have

$$\begin{aligned} u^-_n \to u^-_0 \quad &\text{in} \quad L^{r_1(x)}(\Omega),\ L^{h_1(x)}(\Omega),\ L^{\alpha(x)+\beta(x)}(\Omega), \\ v^-_n \to v^-_0 \quad &\text{in} \quad L^{r_2(x)}(\Omega),\ L^{h_2(x)}(\Omega),\ L^{\alpha(x)+\beta(x)}(\Omega). \end{aligned}$$

This implies

$$\begin{aligned} Q(u^-_n, v^-_n) \to Q(u^-_0, v^-_0) \qquad &\text{as } n \to \infty, \\ R(u^-_n, v^-_n) \to R(u^-_0, v^-_0) \qquad &\text{as } n \to \infty. \end{aligned}$$

Moreover, if $(u^-_0, v^-_0) \in M^-_{\lambda,\mu}$, then there is a constant $t > 0$ such that $(tu^-_0, tv^-_0) \in M^-_{\lambda,\mu}$ and $J_{\lambda,\mu}(u^-_0, v^-_0) \geq J_{\lambda,\mu}(tu^-_0, tv^-_0)$. Indeed, since

$$\begin{aligned} I'_{\lambda,\mu}(u, v) &= \int_\Omega p(x)(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_\Omega q(x)(|\nabla v|^{q(x)} + |v|^{q(x)})dx \\ &- \lambda\left(\int_\Omega a(x)r_1(x)|u|^{r_1(x)}dx + \int_\Omega h_1(x)|u|^{h_1(x)}dx\right) \\ &- \mu\left(\int_\Omega b(x)r_2(x)|v|^{r_2(x)}dx + \int_\Omega h_2(x)|v|^{h_2(x)}dx\right) \\ &- \int_{\partial\Omega}(\alpha(x) + \beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \end{aligned}$$

then,

$$\begin{aligned} I'_{\lambda,\mu}(tu^-_0, tv^-_0) &= \int_\Omega p(x)(|\nabla tu^-_0|^{p(x)} + |tu^-_0|^{p(x)})dx + \int_\Omega q(x)(|\nabla tv^-_0|^{q(x)} + |tv^-_0|^{q(x)})dx \\ &- \lambda\left(\int_\Omega a(x)r_1(x)|tu^-_0|^{r_1(x)}dx + \int_\Omega h_1(x)|tu^-_0|^{h_1(x)}dx\right) \\ &- \mu\left(\int_\Omega b(x)r_2(x)|tv^-_0|^{r_2(x)}dx + \int_\Omega h_2(x)|tv^-_0|^{h_2(x)}dx\right) \\ &- \int_{\partial\Omega}(\alpha(x) + \beta(x))c(x)|tu^-_0|^{\alpha(x)}|tv^-_0|^{\beta(x)}dx \\ &\leq t^{\max\{p^+,q^+\}} \max\{p^+, q^+\}\Bigg[\int_\Omega(|\nabla u^-_0|^{p(x)} + |u^-_0|^{p(x)})dx \end{aligned}$$

$$
\begin{aligned}
&+ \int_\Omega (|\nabla v_0^-|^{q(x)} + |v_0^-|^{q(x)})dx\Big] \\
&- t^{h_1^-} h_1^- \lambda\Big(\int_\Omega a(x)|u_0^-|^{r_1(x)} + \int_\Omega |u_0^-|^{h_1(x)}dx\Big) \\
&- t^{h_2^-} h_2^- \mu\Big(\int_\Omega b(x)|v_0^-|^{r_2(x)}dx + \int_\Omega |v_0^-|^{h_2(x)}dx\Big) \\
&- t^{\alpha^-+\beta^-}(\alpha^-+\beta^-)\int_{\partial\Omega} c(x)|u_0^-|^{\alpha(x)}|v_0^-|^{\beta(x)}dx \\
&\leq \Big(t^{\max\{p^+,q^+\}}\max\{p^+,q^+\} - t^{\min\{h_1^-,h_2^-\}}\min\{h_1^-,h_2^-\}\Big)P(u_0^-,v_0^-) \\
&+ \Big(t^{\min\{h_1^-,h_2^-\}}\min\{h_1^-,h_2^-\} - t^{\alpha^-+\beta^-}(\alpha^-+\beta^-)\Big)R(u_0^-,v_0^-) \\
&\leq 2\Big(t^{\max\{p^+,q^+\}}\max\{p^+,q^+\} - t^{\min\{h_1^-,h_2^-\}}\min\{h_1^-,h_2^-\}\Big)\|(u_0^-,v_0^-)\|^{\max\{p^+,q^+\}} \\
&+ C_7\Big(t^{\min\{h_1^-,h_2^-\}}\min\{h_1^-,h_2^-\} - t^{\alpha^-+\beta^-}(\alpha^-+\beta^-)\Big)\|(u_0^-,v_0^-)\|^{\alpha^++\beta^+}.
\end{aligned}
$$

By (H_2) it follows $I'_\lambda(tu_0^-,tv_0^-) < 0$. Hence by the definition of $M^-_{\lambda,\mu}$, $(tu_0^-,tv_0^-) \in M^-_{\lambda,\mu}$.

Now, we shall prove $u_n^- \to u_0^-$ in $W^{1,p(x)}(\Omega)$, $v_n^- \to v_0^-$ in $W^{1,q(x)}(\Omega)$. Suppose otherwise, then either

$$\|u_0^-\|_p < \liminf_{n\to\infty}\|u_n^-\|_p \qquad \text{or} \qquad \|v_0^-\|_q < \liminf_{n\to\infty}\|v_n^-\|_q.$$

We have

$$
\begin{aligned}
J_{\lambda,\mu}(tu_0^-,tv_0^-) &\leq \frac{t^{\max\{p^+,q^+\}}}{\min\{p^-,q^-\}}P(u_0^-,v_0^-) - \frac{t^{\min\{h_1^-,h_2^-\}}}{\max\{r_1^+,r_2^+\}}Q(u_0^-,v_0^-) - \frac{t^{\alpha^-+\beta^-}}{\alpha^++\beta^+}R(u_0^-,v_0^-) \\
&< \lim_{n\to\infty}\left[\frac{t^{\max\{p^+,q^+\}}}{\min\{p^-,q^-\}}P(u_n^-,v_n^-) - \frac{t^{\min\{h_1^-,h_2^-\}}}{\max\{r_1^+,r_2^+\}}Q(u_n^-,v_n^-) - \frac{t^{\alpha^-+\beta^-}}{\alpha^++\beta^+}R(u_n^-,v_n^-)\right] \\
&\leq \lim_{n\to\infty} J_{\lambda,\mu}(tu_n^-,tv_n^-) \leq \lim_{n\to\infty} J_{\lambda,\mu}(u_n^-,v_n^-) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = \alpha_0^-(\lambda,\mu).
\end{aligned}
$$

This implies that $J_{\lambda,\mu}(tu_0^-,tv_0^-) < \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = \alpha_0^-(\lambda,\mu)$, which is a contradiction. Hence

$$
\begin{aligned}
&u_n^- \to u_0^- \text{ in } W^{1,p(x)}(\Omega), \\
&v_n^- \to v_0^- \text{ in } W^{1,q(x)}(\Omega).
\end{aligned}
$$

This implies

$$J_{\lambda,\mu}(u_n^-,v_n^-) \to J_{\lambda,\mu}(u_0^-,v_0^-) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) \qquad \text{as } n\to\infty.$$

Thus, (u_0^-,v_0^-) is a minimizer for $J_{\lambda,\mu}$ on $M^-_{\lambda,\mu}$. □

Corollary 3.9. *By Theorems 3.2 and 3.8 we conclude that there exists* $(u_0^+,v_0^+) \in M^+_{\lambda,\mu}$ *and* $(u_0^-,v_0^-) \in M^-_{\lambda,\mu}$ *such that* $J_{\lambda,\mu}(u_0^+,v_0^+) = \inf_{(u,v)\in M^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v)$ *and* $J_{\lambda,\mu}(u_0^-,v_0^-) = \inf_{(u,v)\in M^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v)$. *Moreover,* *since* $J_{\lambda,\mu}(u_0^{\pm}, v_0^{\pm}) = J_{\lambda,\mu}(|u_0^{\pm}|, |v_0^{\pm}|)$ *and* $(|u_0^{\pm}|, |v_0^{\pm}|) \in M_{\lambda,\mu}^{\pm}$, *we may assume* $(u_0^{\pm}, v_0^{\pm}) \geq 0$. *By Theorem 3.2,* $(u_0^{\pm}, v_0^{\pm})$ *are critical points of* $J_{\lambda,\mu}$ *on* W *and hence are weak solutions. Finally, by the Harnack inequality due to [21, 23], we obtain that* $(u_0^{\pm}, v_0^{\pm})$ *are positive solutions of* (1.1).

Remark 3.10. *Our ideas can also be applied to the following elliptic system*

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda\left(a(x)|u|^{r_1(x)-2}u + |u|^{h_1(x)-2}u\right) + \frac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} & \text{in } \Omega \\ -\Delta_{q(x)}v + |v|^{q(x)-2}v = \mu\left(b(x)|v|^{r_2(x)-2}v + |v|^{q(x)-2}v\right) + \frac{\beta(x)}{\alpha(x)+\beta(x)}c(x)|v|^{\beta(x)-2}v|u|^{\alpha(x)} & \text{in } \Omega \\ u(x) = v(x) = 0 \qquad \text{on } \partial\Omega \end{cases}$$

where $p, q, r_1, r_2, \alpha, \beta, a, b$ *and* c *are as before. The results presented here have analogous statements for the latter problem. The proofs of the multiplicity results are similar to the ones performed for problem* (1.1)*, so we leave the details to the reader.*

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