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Explicit formulas of the heat kernel on the quaternionic projective spaces

Formules explicites du noyau de la chaleur sur les espaces projectifs des quaternions

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ABSTRACT. We consider the heat equation on the quaternionic projective space $P_n(\mathbb{H})$, and we establish two formulas for the heat kernel, a series expansion involving Jacobi polynomials, and an integral representation involving a θ-function. More precisely, using the quaternonic Hopf fibration and the explicit integral representation of the heat kernel on the complex projective space $P_{2n+1}(\mathbb{C})$, as well as an integral representation of Jacobi polynomials in terms of Gegenbauer polynomials, we give an explicit integral representation of the heat kernel on the n-quaternionic projective space. We also establish an explicit series expansion of the heat kernel in terms of the Jacobi polynomials. Moreover, we derive an explicit formula of the heat kernel on the sphere S^4.
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1. Introduction

The heat kernel has long been a fundamental tool in solving many problems in different branches of mathematics, mathematical physics and engineering, see for example [7, 12, 15] and the references cited therein. It can be defined as the fundamental solution of the heat equation $\frac{\partial}{\partial t} = \Delta_M$, for the Laplace–Beltrami operator Δ_M on a given rank one Riemannian symmetric space $M = G/K$, and then solves the initial value (Cauchy) problem

$$\begin{cases} \Delta_M E_M(t;x,y) = \dfrac{\partial}{\partial t} E_M(t;x,y); \quad t>0, x,y\in M, \\ \lim\limits_{t\to 0} \displaystyle\int_M E_M(t;x,y) f(y)dy = f(x), \end{cases}$$

for given $f \in \mathbb{C}^\infty(M)$. Moreover, it is unique and depends only in t and the geodesic distance $r := d(x,y)$ by the symmetry of M. This fact is used in determining and studying $E_M(t;x,y) = E_M(t,r)$. The explicit expression of $E_M(t,r)$ plays a central role in several fields of research such as index theory, geometric analysis, probability, analysis of Sobolev-type inequalities, vacuum polarization, Casimir effect, string theory and quantum anomalies, among others. See e.g. [1, 3, 6, 11, 12, 13, 15, 18] and references therein for striking applications. For M of noncompact type, such as the real $H^n(\mathbb{R})$, complex $H^n(\mathbb{C})$ and quaternionic $H^n(\mathbb{H})$ hyperbolic spaces, the explicit integral representation of the heat kernel is well-known in the literature. It is given by many authors in slightly different forms (see e.g.

*To the memory of Ahmed Intissar.

[2, 8, 14, 19, 20, 22]). For example the heat kernel $H^*_{n,2}(t,r)$ of the complex hyperbolic space $H^n(\mathbb{C})$ is given in [2, 22] by

$$H^*_{n,2}(t,r) = \frac{e^{-n^2 t}}{(2\pi)^n\sqrt{\pi t}} \int_r^{+\infty} \frac{\sinh u}{\sqrt{\cosh^2 u - \cosh^2 r}} \left(\frac{-1}{\sinh u}\frac{d}{du}\right)^n \left(e^{-u^2/4t}\right) du. \tag{1.1}$$

Expansion formulas of the heat kernel for the classical rank one compact symmetric spaces G/K (i.e., spheres, complex projective spaces or quaternionic projective spaces) may be given in terms of zonal functions with respect to K or as integrals over K of the heat kernel of the group G (see [4]). An explicit expansion for the heat kernel $H_{n,2}(t,r)$ on the complex projective space $P_n(\mathbb{C})$ in terms of the specific Jacobi polynomials $P^{(n,0)}_\ell$ as well as its integral representation are obtained in [16]. More exactly, we have

$$H_{n,2}(t,r) = \frac{1}{\pi^n}\sum_{l=0}^{+\infty}(2l+n)\frac{(l+n-1)!}{l!}e^{-4l(l+n)t}P^{(n-1,0)}_l(\cos(2r)) \tag{1.2}$$

and

$$H_{n,2}(t,r) = \frac{e^{n^2 t}}{2^{n-2}\pi^{n+1}} \int_r^{\pi/2} \frac{\sin u}{\sqrt{\cos^2 r - \cos^2 u}} D^n\Big(\theta_{n+1}(t,u)\Big) du, \tag{1.3}$$

where we have set $D = \frac{-1}{\sin u}\frac{d}{du}$ for convenience of notation. The involved theta function θ_{n+1} is given by

$$\theta_{n+1}(t,u) = \sum_{\ell=0}^{+\infty} e^{-4(\ell+n/2)^2 t}\cos((2\ell+n)u). \tag{1.4}$$

The formula in (1.2) generalizes, somehow, the result given by Fisher, Jungster and Williams in [10] for the heat kernel of the two sphere S^2 by taking $n=1$ (here S^2 is identified with the complex projective space $P_1(\mathbb{C})$). In fact, we have

$$H_{1,2}(t,r) = \frac{1}{\pi}\sum_{\ell=0}^{+\infty}(2\ell+1)e^{-4\ell(\ell+1)t}P_\ell(\cos(2r)). \tag{1.5}$$

The aim of this paper is to provide an explicit integral representation of the heat kernel $H_{n,4}$ for the initial value problem associated with the heat equation for the Laplace–Beltrami operator on the quaternionic projective space $P_n(\mathbb{H})$, viewed as dual space of the quaternionic hyperbolic space. We also derive an explicit expansion in terms of Jacobi polynomials of type $P^{(2n-1,1)}_\ell$ as well as an explicit formula for the heat kernel on the sphere S^4. The fundamental tool in proving our main results will be played by the quaternionic Hopf fibration and the identities (1.2) and (1.3) combined with two key lemmas 3.1 and 3.2.

The main result is stated in the next section (Section 2). Its proof is given in Section 3, while the proof of the key lemmas will be postponed in Section 4.

2. Statement of main theorem

Associated to the homogeneous coordinates

$$x = [z_0 : z_1 : \cdots : z_n], \quad y = [w_0 : w_1 : \cdots : w_n]$$

of x, y in the complex projective space $P_n(\mathbb{C})$ of $\mathbb{C}^{n+1}$, one defines

$$\overline{x}.y = \sum_{i=0}^{n} \overline{z_i}.w_i, \quad |x| = \sqrt{\overline{x}.x}. \tag{2.1}$$

The space $P_n(\mathbb{C})$ is a Kählerian manifold when equipped with the Fubini–Study metric ds^2_{FS}, namely the metric associated to the Kähler form

$$\omega_{FS} = \frac{i}{2\pi}\partial\overline{\partial}\log\left(1 + \sum_{k=1}^{n}|z_k|^2\right) \tag{2.2}$$

and given by

$$ds^2_{FS} = \left(1 + \sum_{k=1}^{n}|z_k|^2\right)^{-2}\left\{\sum_{i,j=1}^{n}\left(1 + \sum_{k=1}^{n}|z_k|^2\right)\delta_{ij} - \overline{z}_i z_j\right\} dz_i \otimes d\overline{z}_j, \tag{2.3}$$

in the local coordinates of the chart $\mathbb{C}^n = \{[1 : z_1 : \cdots : z_n]\}$. The associated distance can be shown to be $SU(n+1)$-invariant and given through

$$\cos d(x, y) = \frac{|\overline{x}.y|}{|x||y|}; \quad x, y \in P_n(\mathbb{C}), \tag{2.4}$$

while the corresponding Laplace–Beltarmi operator reads

$$\Delta_{P_n(\mathbb{C})} := \left(1 + \sum_{k=1}^{n}|z_k|^2\right)\sum_{i,j=1}^{n}\left(\delta_{ij} + \overline{z}_i z_j\right)\frac{\partial^2}{\partial z_i \partial \overline{z}_j}.$$

Now, let $\mathbf{j}$ be another imaginary unit, $\mathbf{j}^2 = -1$, independent of $\mathbf{i} = \sqrt{-1}$ and such that $\mathbf{ij} = -\mathbf{ji}$. Thus, we denote by $\mathbb{H} = \mathbb{C} + \mathbb{C}\mathbf{j}$ the division algebra of quaternionic numbers and by

$$P_n(\mathbb{H}) = Sp(1+n)/Sp(1)Sp(n)$$

the right-quaternionic projective space of $\mathbb{H}^{n+1}$ (with the right-action) defined in a similar way as $P_n(\mathbb{C})$, where $Sp(n)$ is the compact symplectic group. It carries the Fubini–Study metric ds^2_{FSq} analogous to the one for $P_n(\mathbb{C})$. The corresponding distance is given by the same formula (2.4) with $x, y \in P_n(\mathbb{H})$. This can be handled by identifying $\mathbb{H}^{n+1}$ to $\mathbb{C}^{2n+2}$, so that one can consider the projection map $\pi : P_{2n+1}(\mathbb{C}) \longrightarrow P_n(\mathbb{H})$,

$$\pi([z_0 : z_1 : \cdots : z_n : z'_0 : z'_1 : \cdots : z'_n]) := [q_0 : q_1 : \cdots : q_n] \tag{2.5}$$

with

$$q_\ell = z_\ell + z'_\ell j; \qquad \ell = 0, 1, 2, \cdots, n,$$

defining a Riemannian submersion for the compact quaternion-Kähler symmetric space $(P_n(\mathbb{H}), ds^2_{FSq})$.

The main result of this paper provides the explicit integral and series representation of the heat kernel $H_{n,4}(t,r)$ solving the heat equation

$$\Delta_{P_n(\mathbb{H})} H_{n,4}(t,x,y) = \frac{\partial}{\partial t} H_{n,4}(t,x,y), \quad t>0, x,y \in P_n(\mathbb{H}), \tag{2.6}$$

with

$$\lim_{t\to 0} \int\limits_{P_n(\mathbb{H})} H_{n,4}(t,x,y) f(y) dy = f(x), \tag{2.7}$$

for the Laplace–Beltrami operator $\Delta_{P_n(\mathbb{H})}$ on $P_n(\mathbb{H})$ for given $f \in \mathbb{C}^\infty(P_n(\mathbb{H}))$. Namely, we assert the following.

Theorem 2.1. *For every $t>0$ and $0 \le r < \pi/2$, we have*

$$H_{n,4}(t,r) = \frac{e^{(2n+1)^2 t}}{2^{2n-2}\pi^{2n+1}} \int\limits_r^{\pi/2} \frac{\sin u \sqrt{\cos^2 r - \cos^2 u}}{\cos^2 r} D^{2n+1}\Big(\theta_{2n+2}(t,u)\Big) du \tag{2.8}$$

and

$$H_{n,4}(t,r) = \frac{1}{\pi^{2n}} \sum_{\ell=0}^{+\infty} (2\ell+2n+1) \frac{(\ell+2n)!}{(\ell+1)!} e^{-4\ell(\ell+2n+1)t} P_\ell^{(2n-1,1)}(\cos(2r)). \tag{2.9}$$

As immediate consequence, we recover the integral representation of the associated heat kernel $E_{S^4}(t,r)$ solving the heat problem on the unit sphere (S^4, ds^2_{FS}) as well as its expansion series in terms of the Jacobi polynomials. This follows by considering the particular case of $n=1$ and identifying $P_1(\mathbb{H})$ to the unit sphere S^4.

Corollary 2.2. *We have*

$$E_{S^4}(t,r) = \frac{e^{9t}}{2\pi^4} \int\limits_r^{\pi/2} \frac{\sin u \sqrt{\cos^2 r - \cos^2 u}}{\cos^2 r} D^3\Big(\theta_4(t,u)\Big) du \tag{2.10}$$

and

$$E_{S^4}(t,r) = \frac{1}{\pi^2} \sum_{\ell=0}^{+\infty} (2\ell+3)(\ell+2) e^{-4\ell(\ell+3)t} P_\ell^{(1,1)}(\cos(2r)). \tag{2.11}$$

Remark 2.3. *By identity* (2.11)*, we recover the explicit formula for the series expansion for the heat kernel on the unit sphere S^4 obtained in [4].*

Remark 2.4. *The obtained formulas for $H_{n,2}(t,r)$ and $H_{n,4}(t,r)$ can be unified. In fact, if we let $\mathbb{F}_k$ denote the field $\mathbb{C} = \mathbb{F}_1$ and $\mathbb{H} = \mathbb{F}_2$ with $\dim_{\mathbb{R}}(\mathbb{F}) = 2k$ for $k=1,2$, then*

$$H_{n,2k}(t,r) = \frac{e^{(k(n+1)-1))^2 t}}{2^{kn-2}\pi^{kn+1}\cos^{2(k-1)} r} \int\limits_r^{\pi/2} \frac{\sin u}{(\cos^2 r - \cos^2 u)^{3/2-k}} \times D^{k(n+1)-1}\Big(\theta_{k(n+1)}(t,u)\Big) du \tag{2.12}$$

and

$$H_{n,2k}(t,r) = \frac{1}{\pi^{kn}} \sum_{\ell=0}^{+\infty} (2\ell + k(n+1) - 1) \frac{(\ell + k(n+1) - 2)!}{(\ell + k - 1)!} \times e^{-4\ell(\ell+k(n+1)-1)t} P_\ell^{(kn-1,k-1)}(\cos(2r)). \tag{2.13}$$

3. Proof of the main theorem

Notice first that the identification of $\mathbb{H}^{n+1}$ with $\mathbb{C}^{2n+2}$ leads to the quaternionic Hopf fibration

$$(P^1(\mathbb{C}), ds^2_{FS}) \hookrightarrow (P_{2n+1}(\mathbb{C}), ds^2_{FS}) \xrightarrow{\pi} (P_n(\mathbb{H}), ds^2_{FSq}). \tag{3.1}$$

This follows since the projection π is a Riemannian submersion with totally geodesic fibers $\pi^{-1}(q)$ (totally geodesic submanifold) for every $q \in P_n(\mathbb{H})$ [17, Chapter IV-7]. Notice also that to any smooth complex-valued mapping (section) $s : P_n(\mathbb{H}) \longrightarrow \mathbb{C}$, we associate $s^* : P_{2n+1}(\mathbb{C}) \longrightarrow \mathbb{C}$ defined by

$$s^*(x) := s(\pi(x)) = \pi^* s(x).$$

Therefore, the following fundamental intertwining relation

$$\pi^* \circ \Delta_{P_n(\mathbb{H})} = \Delta_{P_{2n+1}(\mathbb{C})} \circ \pi^* \tag{3.2}$$

holds true for the Laplace–Beltrami operators $\Delta_{P_{2n+1}(\mathbb{C})}$ and $\Delta_{P_n(\mathbb{H})}$ on the Riemannian spaces $(P_{2n+1}(\mathbb{C}), ds^2_{FS})$ and $(P_n(\mathbb{H}), ds^2_{FSq})$ respectively. A direct computation using (3.2) shows that if $S : \mathbb{R}^> \times P_n(\mathbb{H}) \longrightarrow \mathbb{C}$ is solution of the heat equation

$$\begin{cases} \Delta_{P_n(\mathbb{H})} S(t;q) = \frac{\partial}{\partial t} S(t;q) \\ \lim\limits_{t\to 0} S(t;q) = f(q) \end{cases},$$

for fixed smooth function $f \in \mathbb{C}^\infty(P_n(\mathbb{H}))$, and moreover the function

$$S^*(t,x) := S(t,\pi(x)) = \pi^* \left(S(t,\cdot)\right)(x)$$

satisfies the heat equation for the Laplace–Beltrami operator on the complex projective space $P_{2n+1}(\mathbb{C})$, to wit

$$\begin{cases} \Delta_{P_{2n+1}(\mathbb{C})} S^*(t,x) = \frac{\partial}{\partial t} S^*(t,x) \\ \lim\limits_{t\to 0} S^*(t,x) = f^*(x) \end{cases}.$$

Subsequently, $S^*(t;x)$ is given by the integral formula

$$S^*(t;x) = \int\limits_{P_{2n+1}(\mathbb{C})} H_{2n+1,2}(t, d(x,y)) f^*(y) dy,$$

where the heat kernel $H_{2n+1,2}(t; d(x,y))$ is given by the integral representation [16, Theorem 1.1]

$$H_{2n+1,2}(t, d(x,y)) = \frac{e^{(2n+1)^2 t}}{2^{2n-1}\pi^{2n+2}} \int\limits_{d(x,y)}^{\pi/2} \frac{\sin u D^{2n+1}\left(\theta_{2n+2}(t,u)\right)}{\sqrt{\cos^2(d(x,y)) - \cos^2 u}} du. \tag{3.3}$$

Accordingly, by setting

$$\Upsilon_{2n+1}(t;u) := \sin u D^{2n+1}\left(\theta_{2n+2}(t,u)\right) \tag{3.4}$$

and making appeal to the Heaviside step function $H_s(u)$ (defined as being zero when $u < 0$ and one when $u > 0$), we get

$$S^*(t;x) = \frac{e^{(2n+1)^2 t}}{2^{2n-1}\pi^{2n+2}} \int\limits_{P_{2n+1}(\mathbb{C})} \left(\int\limits_0^{\pi/2} \frac{H_s\left(u - d(x,y)\right) \Upsilon_{2n+1}(t;u)}{\sqrt{\cos^2 d(x,y) - \cos^2 u}} du \right) f^*(y) dy$$

$$= \frac{e^{(2n+1)^2 t}}{2^{2n-1}\pi^{2n+2}} \int\limits_0^{\pi/2} \left(\int\limits_{P_{2n+1}(\mathbb{C})} \frac{H_s\left(u - d(x,y)\right) f^*(y)}{\sqrt{\cos^2 d(x,y) - \cos^2 u}} dy \right) \Upsilon_{2n+1}(t;u) du.$$

The last equality readily follows by Fubini's Theorem. Thus, since the solution $S(t;\cdot)$ at the origin $O_{\mathbb{H}} := O_{P_n(\mathbb{H})} = [1:0:0:\cdots:0] \in P_n(\mathbb{H})$ coincides with the solution $S^*(t,\cdot)$ at the origin $O_{\mathbb{C}} := O_{P_{2n+1}(\mathbb{C})} = [1:0:0:\cdots:0] \in P_{2n+1}(\mathbb{C})$, we get

$$S\left(t;O_{\mathbb{H}}\right) = S^*\left(t;O_{\mathbb{C}}\right) = \frac{e^{(2n+1)^2 t}}{2^{2n-1}\pi^{2n+2}} \int\limits_0^{\pi/2} T_f\left(u;O_{\mathbb{C}}\right) \Upsilon_{2n+1}(t;u) du,$$

where $T_f\left(u;O_{\mathbb{C}}\right)$ stands for

$$T_f\left(u;O_{\mathbb{C}}\right) := \int\limits_{P_{2n+1}(\mathbb{C})} \frac{H_s\left(u - d\left(O_{\mathbb{C}},y\right)\right) f^*(y)}{\sqrt{\cos^2 d\left(O_{\mathbb{C}},y\right) - \cos^2 u}} dy. \tag{3.5}$$

The key observation here is the following Lemma.

Lemma 3.1 ([5]). *The quantity $T_f\left(u;O_{\mathbb{C}}\right)$ can be rewritten as*

$$T_f\left(u;O_{\mathbb{C}}\right) = 2\pi \int\limits_{q\in P_n(\mathbb{H})} \frac{H_s\left(u - d\left(O_{\mathbb{H}},q\right)\right) \sqrt{\cos^2 d(O_{\mathbb{H}},q) - \cos^2 u}}{\cos^2 d(O_{\mathbb{H}},q)} f(q) dq. \tag{3.6}$$

Before embarking on its proof (see Section 4), we begin by giving a skeleton key explaining the integration formulas over the fibers of the map π. Thus, every $q \in P_n(\mathbb{H})$ can be represented as $q = [1 : q'_1 : q'_2 : \cdots : q'_n]$ with $(q'_1, q'_2, \cdots, q'_n) \in \mathbb{H}^n$. Thus, the geodesic spherical coordinates can be used to rewrite q as

$$q = [1, \tan(\varphi)p] = [1 : \tan(\varphi)p_1 : \tan(\varphi)p_2 : \cdots : \tan(\varphi)p_n],$$

with $0 \leq \varphi \leq \pi/2$ and $p = (p_1, p_2, \cdots, p_n) \in S(\mathbb{H}^n) = S^{4n-1}$. This is to say that the fiber $\pi^{-1}(q)$ at $q \in P_n(\mathbb{H})$ can be identified to $P_1(\mathbb{C})$. The angle φ is given by $\varphi = d(O_{\mathbb{C}}, q)$. This is immediate from the fact that

$$\cos^2 d(O_{\mathbb{H}}, q) = \frac{1}{\tan^2\varphi + 1} = \cos^2\varphi.$$

Now, for arbitrary $y = [y_0 : y_1 : \cdots : y_n : y'_0 : y'_1 : \cdots : y'_n]$ in $P_{2n+1}(\mathbb{C})$ belonging to the fiber at q; i.e., such that $\pi(y) = q$, there exists $\lambda \in \mathbb{H}$ such that $|\lambda| = 1$, $y_0 + jy'_0 = \lambda$ and

$$y_\ell + jy_\ell = \lambda\tan(\varphi)p_\ell;\ \ \ell = 1, 2, \cdots, n.$$

Therefore, since $S(\mathbb{H})$ is embedded in $\mathbb{R}^3$, we can make use of the spherical coordinates to represent λ as

$$\lambda = \cos v + j \sin v \cos\theta + ji \sin v \sin\theta = \cos v + j \sin v e^{i\theta}$$

for some $0 \leq \theta \leq 2\pi$ and $0 \leq v \leq \pi/2$. This gives rise to the special parameterization of the fiber $\pi^{-1}(q)$ by the the radial and angular coordinates (v, θ), to wit

$$y = [\cos v + j \sin v e^{i\theta} : tan(\varphi)p(\cos v + j \sin v e^{i\theta})]. \tag{3.7}$$

Moreover, since

$$\cos^2 d(O_{\mathbb{C}}, y) = \frac{\cos^2 v}{\tan^2\varphi + 1} = \cos^2\varphi \cos^2 v,$$

it is clear that the induced volume measure on that fiber is given by

$$dvol(\pi^{-1}(q))(y) = -\frac{1}{2} d(\cos^2 v) d\theta = \cos v \sin(v) dv d\theta.$$

Subsequently, the integral formula bridging $P_{2n+1}(\mathbb{C})$ to $P_n(\mathbb{H})$ is the following

$$\begin{aligned} \int_{P_{2n+1}(\mathbb{C})} g(y)dy &= \int_{q\in P_n(\mathbb{H})} \int_{\pi^{-1}(q)} g(y) dvol(\pi^{-1}(q))(y) dq \\ &= \int_{q\in P_n(\mathbb{H})} \mathcal{R}(g)(q) dq, \end{aligned}$$

where we have set

$$\begin{aligned} \mathcal{R}(g)(q) &= \int_{\pi^{-1}(q)} g(y) dvol(\pi^{-1}(q))(y) \\ &= \int_0^{2\pi} \int_0^{\pi/2} g(y) \cos v \sin v dv\theta \end{aligned}$$

for a given function g defined on $P_{2n+1}(\mathbb{C})$. The consideration of the radial function $g(y) = G(\cos(r)) = G(\cos\varphi \cos v)$, with $r = d(O_{\mathbb{C}}, y)$, leads to

$$\mathcal{R}(g)(q) = \frac{2\pi}{\cos^2\varphi} \int_0^{\cos\varphi} tG(t)dt,$$

where we have made the change of variable $t := \cos\varphi \cos v$. Accordingly, for a pair of functions (f, g), with f defined on $P_n(\mathbb{H})$ and g on $P_{2n+1}(\mathbb{C})$, one obtains

$$\int_{P_{2n+1}(\mathbb{C})} f^*(y) g(y) dy = \int_{P_n(\mathbb{H})} f(q) \mathcal{R}(g)(q) dq. \tag{3.8}$$

By applying the formula (3.8) with g being the heat kernel on $P_{2n+1}(\mathbb{C})$,the heat kernel for $P_n(\mathbb{H})$ becomes the image under the map $\mathcal{R}$ of the heat kernel for $P_{2n+1}(\mathbb{C})$. Notice also that, under the map $\mathcal{R}$, the image of a spherical function for $P_{2n+1}(\mathbb{C})$ is again a spherical function for $P_n(\mathbb{H})$.

Therefore, the solution at the origin reduces further to

$$S\left(t;O_{\mathbb{H}}\right)=\frac{e^{(2n+1)^2t}}{2^{2n-2}\pi^{2n+1}}\int\limits_{q\in P_n(\mathbb{H})}\left(\int\limits_{d(O_{\mathbb{H}},q)}^{\pi/2}\frac{\sqrt{\cos^2 d(O_{\mathbb{H}},q)-\cos^2 u}}{\cos^2 d(O_{\mathbb{H}},q)}\Upsilon_{2n+1}(t;u)du\right)f(q)dq.$$

This shows in particular that the heat kernel $H_{n,4}$ for the quaternionic projective space at the origin is given by

$$H_{n,4}(t,O_{\mathbb{H}},q)=\frac{e^{(2n+1)^2t}}{2^{2n-2}\pi^{2n+1}}\int\limits_{d(O_{\mathbb{H}},q)}^{\pi/2}\frac{\sqrt{\cos^2 d(O_{\mathbb{H}},q)-\cos^2 u}}{\cos^2 d(O_{\mathbb{H}},q)}\Upsilon_{2n+1}(t;u)du.$$

But, since the $H_{n,4}(t,x,y)=H_{n,4}(t,d(x,y))$ depends only of the geodesic distance $r=d(p,q)$, we conclude the following

$$H_{n,4}(t;x,y)=\frac{e^{(2n+1)^2t}}{2^{2n-2}\pi^{2n+1}}\int\limits_{r}^{\pi/2}\frac{\sqrt{\cos^2 r-\cos^2 u}}{\cos^2 r}\Upsilon_{2n+1}(t;u)\sin u du.$$

This completes the proof of assertion (i) in Theorem 2.1.

The proof of (ii) in Theorem 2.1 lies essentially on the following lemma providing an appropriate integral representation for Jacobi polynomials in terms of the Gegenbauer polynomials C_ℓ^λ [23],

$$C_\ell^\lambda(\xi)=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\ell+2\lambda\right)}{\Gamma\left(2\lambda\right)\Gamma\left(\ell+\lambda+\frac{1}{2}\right)}P^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(\xi). \tag{3.9}$$

Lemma 3.2. *We have*

$$P_\ell^{(2n-1,1)}(\cos(2r))=\frac{(\ell+1)!}{2^{2n-2}\pi(\ell+2n)!}\int\limits_{r}^{\pi/2}\frac{\sin u\sqrt{\cos^2 r-\cos^2 u}}{\cos^2 r}\times\left(\frac{-1}{\sin u}\frac{d}{du}\right)^{2n}\left(C^1_{2\ell+2n}(\cos u)\right)du. \tag{3.10}$$

In fact, starting from (2.8), we , can rewrite the quantity $D^{2n+1}\left(\theta_{2n+2}(t,u)\right)$ as

$$\begin{aligned}D^{2n+1}\left(\theta_{2n+2}(t,u)\right)&=D^{2n}\left(D(\theta_{2n+2}(t,u))\right)\\&=\sum_{\ell=0}^{+\infty}(2\ell+2n+1)e^{-(2\ell+2n+1)^2t}D^{2n}\left(\frac{\sin((2\ell+2n+1)u)}{\sin u}\right).\end{aligned}$$

Therefore, the integral formula in (2.8) can be rewritten as

$$\begin{aligned}H_{n,4}(t,r)=&\frac{e^{(2n+1)^2t}}{2^{2n-2}\pi^{2n+1}}\sum_{\ell=0}^{+\infty}(2\ell+2n+1)e^{-(2\ell+2n+1)^2t}\times\\&\times\left(\int\limits_{r}^{\pi/2}\frac{\sin u\sqrt{\cos^2 r-\cos^2 u}}{\cos^2 r}D^{2n}\left(C^1_{2\ell+2n}(\cos u)\right)du\right).\end{aligned}$$

Finally, making use of Lemma 3.2, we see that the heat kernel $H_{n,4}(t,r)$ admits the following explicit series expansion in terms of the Jacobi polynomials

$$H_{n,4}(t,r) = \frac{1}{\pi^{2n}} \sum_{\ell=0}^{+\infty} (2\ell+2n+1) \frac{(\ell+2n)!}{(\ell+1)!} e^{-4\ell(\ell+2n+1)t} P_\ell^{(2n-1,1)}(\cos(2r)).$$

This completes the proof of (ii) and therefore of Theorem 2.1.

4. Proofs of key lemmas

For the proof of Lemma 3.1 we follow in spirit the ideas presented by Bunke and Olbrich in [5].

Proof of Lemma 3.1. Let $B_{P_{2n+1}(\mathbb{C})}(O_{\mathbb{C}}, u) = \{y \in P_{2n+1}(\mathbb{C}), \quad d(O_{\mathbb{C}}, y) < u\}$ denote the ball in $P_{2n+1}(\mathbb{C})$ of radius u centred at the origin O. By means of Fubini's Theorem, we get

$$\begin{aligned}
T_f(u; O_{\mathbb{C}}) &:= \int_{P_{2n+1}(\mathbb{C})} \frac{H_s(u - d(O_{\mathbb{C}}, y))}{\sqrt{\cos^2 d(O_{\mathbb{C}}, y) - \cos^2 u}} f^*(y) dy \\
&= \int_{y \in B_{P_{2n+1}(\mathbb{C})}(O_{\mathbb{C}}, u)} \frac{H_s(u - d(O_{\mathbb{C}}, y))}{\sqrt{\cos^2 d(O_{\mathbb{C}}, y) - \cos^2 u}} f^*(y) dy \\
&= \int_{q \in B_{P_n(\mathbb{H})}(O_{\mathbb{H}}, u)} \left(\int_{y \in \pi^{-1}(q)} \frac{dvol(\pi^{-1}(q))(y)}{\sqrt{\cos^2 d(O_{\mathbb{C}}, y) - \cos^2 u}} \right) f(q) dq. \qquad (4.1)
\end{aligned}$$

Now, by a straightforward computation, we can reduce the right hand-side of (4.1) to

$$\begin{aligned}
T_f(u; O_{\mathbb{C}}) &= -\frac{1}{2} \int_{q \in B_{P_n(\mathbb{H})}(O_{\mathbb{H}}, u)} \left(\int_0^{2\pi} \int_1^{\frac{\cos^2 u}{\cos^2 d(O_{\mathbb{H}}, q)}} \frac{dt_v d\theta}{\sqrt{\cos^2 d(O_{\mathbb{H}}, q) t_v - \cos^2 u}} \right) f(q) dq \\
&= 2\pi \int_{q \in B_{P_n(\mathbb{H})}(O_{\mathbb{H}}, u)} \left(\frac{1}{\cos^2 d(O_{\mathbb{H}}, q)} \int_{\cos^2 u}^{\cos^2 d(O_{\mathbb{H}}, q)} \frac{ds_v}{2\sqrt{s_v - \cos^2 u}} \right) f(q) dq \\
&= 2\pi \int_{q \in B_{P_n(\mathbb{H})}(O_{\mathbb{H}}, u)} \frac{\sqrt{\cos^2 d(O_{\mathbb{H}}, q) - \cos^2 u}}{\cos^2 d(O_{\mathbb{H}}, q)} f(q) dq \\
&= 2\pi \int_{q \in P_n(\mathbb{H})} \frac{H_s(u - d(O_{\mathbb{H}}, q)) \sqrt{\cos^2 d(O_{\mathbb{H}}, q) - \cos^2 u}}{\cos^2 d(O_{\mathbb{H}}, q)} f(q) dq.
\end{aligned}$$

∎

Proof of Lemma 3.2. Starting from the identity [9, Theorem 2.2], giving the integral representation for the Jacobi polynomials in terms of the Gegenbauer polynomials $C_l^\lambda(\cos(\theta))$, and the fact that $P_l^{(\alpha,\beta)}(x) =$

$(-1)^l P_l^{(\beta,\alpha)}(x)$, we can rewrite $P_l^{(\alpha,\beta)}(2t^2-1)$ as

$$P_l^{(\alpha,\beta)}(2t^2-1) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha+\beta+1)\Gamma(l+\beta+1)}{\Gamma\left(\beta+\frac{1}{2}\right)\Gamma(l+\alpha+\beta+1)} \int_0^1 C_{2l}^{\alpha+\beta+1}(tu)(1-u^2)^{\beta-1/2} du$$

valid for $\Re e\alpha > -\frac{1}{2}$ and $\Re e\beta > -\frac{1}{2}$. Thus, using the known facts $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1/2) = \sqrt{\pi}$, and setting $t = \cos d$, we see that the particular of $\alpha = 2n-1, \beta = 1$ reduces further to

$$\begin{aligned} P_l^{(2n-1,1)}(\cos(2r)) &= \frac{4(2n)!(l+1)!}{\pi(l+2n)!} \int_0^1 C_{2l}^{2n+1}(u\cos r)(1-u^2)^{1/2} du \\ &= \frac{4(2n)!(l+1)!}{\pi(l+2n)!} \int_0^{\cos r} C_{2l}^{2n+1}(v) \frac{(\cos^2 r - v^2)^{1/2}}{\cos^2 r} dv \\ &= \frac{4(2n)!(l+1)!}{\pi(l+2n)!} \int_r^{\pi/2} C_{2l}^{2n+1}(\cos u) \frac{\sqrt{\cos^2 r - \cos^2 u}}{\cos^2 r} \sin u du. \end{aligned} \quad (4.2)$$

The second and the third equalities follow by making use of the change of variables $u = \frac{v}{\cos r}$ and next $v = \cos u$, respectively. Now, since

$$\frac{d^n}{dx^n} C_{l+n}^{\lambda}(x) = 2^n (\lambda)_n C_l^{\lambda+n}(x),$$

$(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$ being the Pochhammer symbol, which can be obtained by induction from $\frac{d}{dx} C_{l+1}^{\lambda}(x) = 2\lambda C_l^{\lambda+1}(x)$, we get

$$\frac{1}{2^{2n}(2n)!} \left(\frac{-1}{\sin u} \frac{d}{du}\right)^{2n} \left(C_{2l+2n}^1(\cos u)\right) = C_{2l}^{2n+1}(\cos u). \quad (4.3)$$

Therefore, by inserting this in (4.2) infers the identity (3.10). ∎

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