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On the control of a nonlinear system of viscoelastic equations

Sur le contrôle d'un système non-linéaire d'équations viscoélastiques

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ABSTRACT. In this paper we consider a nonlinear system of two coupled viscoelastic equations, prove the well posedness, and investigate the asymptotic behaviour of this system. We use minimal and general conditions on the relaxation functions and establish explicit energy decay formula which gives the best decay rates expected under this level of generality. Our new result generalizes the earlier related results in the literature.
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1 Introduction

In this paper, we are concerned with the following coupled system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + f_1(u,v) = 0, & \text{in } \Omega \times (0,\infty) \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + f_2(u,v) = 0, & \text{in } \Omega \times (0,\infty) \\ u = v = 0, \quad \text{on } \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1,\ v(\cdot,0) = v_0,\ v_t(\cdot,0) = v_1, \text{ in } \Omega, \end{cases} \tag{1.1}$$

where u and v denote the transverse displacements of waves and Ω is an open bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The relaxation functions g_1, g_2 and the nonlinearities f_1, f_2 will be specified later. System (1.1) arises in the theory of viscoelasticity and describes the interaction of two scalar fields. In [3], Andrade and Mognon considered a similar problem with the nonlinearities

$$f_1(u,v) = |u|^{p-2}\, u\, |v|^p \qquad \text{and} \qquad f_2(u,v) = |v|^{p-2}\, v\, |u|^p$$

where $p > 1$ if $n = 1,2$ and $1 < p \leq \frac{n-1}{n-2}$ if $n \geq 3$. They proved the well posedness for the problem under restrictive assumptions on the relaxation functions. In [37], Santos considered the coupling

$$f_1(u,v) = a(u-v) \qquad \text{and} \qquad f_2(u,v) = -a(u-v),$$

where a is a positive constant, and assumed that

$$-a_1 g_i^p(t) \leq g_i'(t) \leq -a_2 g_i^p(t) \qquad \text{and} \qquad 0 \leq g_i''(t) \leq \gamma g_i^p(t), \qquad i = 1,2,$$

for some $1 \leq p < 2$. He proved that when the kernels decay exponentially (resp. polynomially) the first- and the second-order energy of the solution decay exponentially (resp. polynomially). In [25], Messaoudi and Tatar used the following weaker conditions on the relaxation functions

$$g_i'(t) \leq -c_i g_i^{p_i}(t), \qquad i = 1,2,$$

for some $1 \leq p_1, p_2 < 3/2$, and more general forms of nonlinearities. They proved an exponential decay for $(p_1, p_2) = (1, 1)$ and a polynomial decay for $(p_1, p_2) \neq (1, 1)$. Liu [17] used the same hypothesis for a quasilinear system and established uniform decay results.

The problem, with a single viscoelastic equation, has been extensively discussed by many authors. We refer to [5,24,26,27,28] for subsequent results which proved that the energy decays exponentially (resp. polynomially) if the relaxation function g decays exponentially (resp.polynomially). The same results were obtained by Alabau-Boussouira et al. [2], Cannarsa et al. [6] and Rivera et al. [29,30] for more general abstract equations and by Cavalcanti and Oquendo [9] for equations with both viscoelastic and frictional damping terms..

Then, a natural question was raised: how does the energy behave as the kernel function does not necessarily decay polynomially or exponentially? Messaoudi [20,21] studied

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b\,|u|^{\gamma}\,u$$

for $b = 0$ and $b = 1$ and considered relaxation functions satisfying

$$g'(t) \leq -\xi(t)g(t),\ t \geq 0 \tag{1.2}$$

where $\xi(t) : \mathbb{R}_+ \to \mathbb{R}_+$is a nonincreasing differentiable function with

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \leq k \tag{1.3}$$

for some constant k. He proved that the decay rate of the solution energy is similar to that of the relaxation function which is not necessarily of exponential or polynomial type. After that a series of papers using (1.2) and (1.3) has appeared, see for instance [11,16,23,32,35,38]. On the other hand, a condition of the form

$$g'(t) \leq -\chi\left(g(t)\right), \tag{1.4}$$

where χ is a convex function satisfying **some smoothness properties,** was introduced by Alabau-Boussouira and Cannarsa [1] and used then by several authors with different approaches. We refer to [7,10,13,34] where decay results in terms of χ were obtained, to Lasiecka and Wang [14] where not only general but also optimal result was established, and to Cavalcanti et al. [8] where new methodology and tricks were used to treat the interaction between the nonlinearity and viscoelasticity.

Motivated by these works, Liu [15] imposed the conditions (1.2) and (1.3) on g_1, g_2 in the coupled system (1.1) and improved the earlier result in [25]. Later on, Mustafa [33] treated (1.1) using only (1.2). The same was done by Hao and Cai [12], Messaoudi and Al-Gharabli [22], and Said-Houari et al. [36] for similar systems.

Our aim in this work is to investigate (1.1) for relaxation functions of more general type than the ones in (1.2) and (1.4), namely $g'(t) \leq -\xi(t)\chi\left(g(t)\right)$, where χ is increasing and convex **without any additional constraints**. Such a condition on g was first used by Mustafa [31] for a single linear viscoelastic wave equation, and then used by Liu et al. [18,19] for a Moore-Gibson-Thompson equation with memory.

Here, we use this condition, overcome the difficulty brought by the nonlinearities, and establish explicit energy decay formulas which give the best optimal decay rates expected under this level of generality and from which the usual exponential and polynomial decay rate estimates are only special cases. Our results improve and generalize the earlier related results in the literature. The proof is based on the multiplier method and makes use of some properties of convex functions. The paper is organized as follows. In section 2, we present our results. Then, in section 3, the well posedness of the problem is established. Some technical lemmas are provided in section 4. The proof of the main result is given in section 5.

2 The main results

We consider the following hypotheses

(H1) $g_i : [0, \infty) \to (0, \infty)$ (for $i = 1, 2$) are differentiable functions satisfying

$$1 - \int_0^\infty g_i(s)ds = l_i > 0, \tag{2.1}$$

and there exists a C^1 function $H : (0, \infty) \to (0, \infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0, r]$, $r < m_0 = \min\{g_1(0), g_2(0)\}$, with $H(0) = H'(0) = 0$, such that

$$g_i'(t) \leq -\xi_i(t)H(g_i(t)), \quad (i = 1, 2) \quad \forall t \geq 0 \tag{2.2}$$

where ξ_1, ξ_2 are positive nonincreasing differentiable functions.

(H2) $f_i : \mathbb{R}^2 \to \mathbb{R}$ (for $i = 1, 2$) are C^1 functions, with $f_i(0, 0) = 0$, and there exists a function F such that

$$f_1(x, y) = \frac{\partial F}{\partial x}, f_2(x, y) = \frac{\partial F}{\partial y},$$

$$F \geq 0, \qquad xf_1(x, y) + yf_2(x, y) - F(x, y) \geq 0,$$

and

$$\left|\frac{\partial f_i}{\partial x}(x, y)\right| + \left|\frac{\partial f_i}{\partial y}(x, y)\right| \leq d(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1}) \qquad \forall (x, y) \in \mathbb{R}^2 \tag{2.3}$$

for some constant $d > 0$ and $\beta_{ij} \geq 1, (n-2)\beta_{ij} \leq n$ for $i, j = 1, 2$.

We observe that assumption (H2) gives, for some positive constant k, that

$$|f_i(x, y)| \leq k(|x| + |y| + |x|^{\beta_{i1}} + |y|^{\beta_{i2}}) \tag{2.4}$$

and

$$|f_i(x, y) - f_i(r, s)| \leq k(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1} + |r|^{\beta_{i1}-1} + |s|^{\beta_{i2}-1})(|x - r| + |y - s|) \tag{2.5}$$

for all $(x, y), (r, s) \in \mathbb{R}^2$ and $i = 1, 2$. Throughout this paper, c is used to denote a generic positive constant. We will also be using the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for $2 \leq s \leq 2n/(n-2)$ if $n \geq 3$ or $s \geq 2$ if $n = 1, 2$; i.e., for any $\phi \in H_0^1(\Omega)$,

$$\|\phi\|_s \leq c\,\|\nabla\phi\|_2 . \tag{2.6}$$

Now, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_{\Omega}\left(u_t^2 + (1 - \int_0^t g_1(s)ds)\left|\nabla u\right|^2\right)dx + \frac{1}{2}(g_1 \circ \nabla u)(t)$$

$$+\frac{1}{2}\int_{\Omega}\left(v_t^2 + (1 - \int_0^t g_2(s)ds)\left|\nabla v\right|^2\right)dx + \frac{1}{2}(g_2 \circ \nabla v)(t) + \int_{\Omega} F(u,v)dx,$$

where

$$(g_i \circ w)(t) = \int_{\Omega}\int_0^t g_i(t-\tau)\left|w(t) - w(\tau)\right|^2 d\tau dx.$$

For the well-posedness, we prove, in section 3, the following result.

Theorem 2.1. *Let* $(u_0, u_1), (v_0, v_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ *be given. Assume that (H1) and (H2) are satisfied, then problem* (1.1) *has a unique strong solution*

$$u, v \in L^\infty(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}_+; H_0^1(\Omega)) \cap W^{2,\infty}(\mathbb{R}_+; L^2(\Omega)).$$

Our main stability result is the following.

Theorem 2.2. *Assume that* $($*H1*$)$ *and (H2) hold. Then there exist positive constants* $k_1 \leq 1$ *and* k_2 *such that, along the solution of (1.1), the energy functional satisfies*

$$E(t) \leq k_2 H_1^{-1}\left(k_1\int_{t_1}^t \xi(s)ds\right) \tag{2.7}$$

where

$$H_1(t) = \int_t^r \frac{1}{sH'(s)}ds \quad , \quad \xi(t) = \min\{\xi_1(t), \xi_2(t)\}$$

and $t_1 = \max\{g_1^{-1}(r), g_2^{-1}(r)\}$*. Here,* H_1 *is strictly decreasing and convex on* $(0, r]$*, with* $\lim_{t\to 0} H_1(t) = +\infty$.

The proof of this Theorem will be given in section 5. As a diret application, the following corollary shows that the optimal exponential and polynomial decay rate estimates, already proved for g_i satisfying (2.1) and $g_i' \leq -a_i g_i^{p_i}, 1 \leq p_1, p_2 < 2$, are special cases of our result.

Corollary 2.3. *Assume that* $($*H1*$-$*H2*$)$ *hold where, for* $i = 1, 2$*, (2.2) is given by*

$$g_i'(t) \leq -\xi_i(t) g_i^{p_i}(t), \quad 1 \leq p_i < 2$$

Then there are positive constants $k, \overline{k}, k_1$ *such that the decay rate of* E *is given by*

$$E(t) \leq \begin{cases} k e^{-k_1\int_0^t \xi(s)ds}, & \text{if } p = 1 \\ \overline{k}\left(1 + \int_0^t \xi(s)ds\right)^{\frac{-1}{p-1}}, & \text{if } 1 < p < 2 \end{cases}. \tag{2.8}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ *and* $p = \max\{p_1, p_2\}$.

Proof. Here, with $M_0 = \max\{g_1(0), g_2(0)\}$, we see that $g_i'(t) \leq -M_0^p \xi_i(t) H(g_i(t))$ where $H(s) = \left(\frac{s}{M_0}\right)^p$ is linear for $p = 1$, and strictly increasing and strictly convex on $(0, m_0]$ for $1 < p < 2$. Then (2.7) and simple calculations lead to (2.8).

Remarks

(1) It has to be noted that, in case $\int_0^\infty \xi(s)ds = \infty$, Theorem 2.2 ensures $\lim_{t\to\infty} E(t) = 0$. Moreover, the decay rate of $E(t)$ driven by (2.7) is optimal in the sense that it is consistent with the decay rates of $g_1(t)$ and $g_2(t)$ driven by (2.2). In fact, making use of (2.2) for $i = 1, 2$ yields

$$g_i'(t) \leq -\xi_i(t) H(g_i(t)) \Longrightarrow \int\limits_{g_i^{-1}(r)}^{t} \frac{-g_i'(s)}{H(g_i(s))} ds = \int\limits_{g_i(t)}^{r} \frac{ds}{H(s)} \geq \int\limits_{g_i^{-1}(r)}^{t} \xi_i(s) ds$$

so, if we define $H_0(t) = \int_t^r \frac{1}{H(s)} ds$, then H_0 is strictly decreasing and convex on $(0, r]$, with $\lim_{t\to 0} H_0(t) = +\infty$, and $H_0(g_i(t)) \geq \int_{g_i^{-1}(r)}^t \xi_i(s)ds \geq \int_{t_1}^t \xi(s)ds$ which means

$$g_i(t) \leq H_0^{-1}\left(\int\limits_{t_1}^{t} \xi(s) ds\right), \qquad \forall t \geq t_1.$$

Also, it is evident, by the properties of H, H_0 and H_1, that

$$H_1(t) = \int\limits_t^r \frac{1}{sH'(s)} ds \leq \int\limits_t^r \frac{1}{H(s)} ds = H_0(t) \Longrightarrow H_1^{-1}(t) \leq H_0^{-1}(t).$$

This shows that (2.7) provides the best decay rates expected under the very general assumption (H1).

(2) Also, it is easy to notice that we can start the integration inside at zero where if $\overline{k}_1 < k_1$ is chosen so that $\overline{k}_1 \int_0^{2t_1} \xi(s)ds = k_1 \int_{t_1}^{2t_1} \xi(s)ds$, then, as H_1^{-1} is decreasing,

$$E(t) \leq k_2 H_1^{-1}\left(\overline{k}_1 \int\limits_0^t \xi(s) ds\right), \qquad \forall t \geq 2t_1.$$

(3) The above results allow relaxation functions which are not necessarily of exponential or polynomial decay and they are obtained under general hypotheses that allow to deal with a much larger class of functions g_1, g_2 that guarantee the uniform stability of (1.1) with an explicit formula for the decay rates of the energy. We give additional examples of application.

Examples. A- If $g_1(t) = g_2(t) = a\exp(-t^q)$, for $0 < q < 1$ and a chosen so that g_i, for $i = 1, 2$, satisfy (2.1), then $g_i'(t) = -H(g_i(t))$ where $H(t) = \frac{qt}{[\ln(a/t)]^{\frac{1}{q}-1}}$. Since

$$H'(t) = \frac{(1-q) + q\ln\left(\frac{a}{t}\right)}{\left[\ln\left(\frac{a}{t}\right)\right]^{\frac{1}{q}}} \qquad \text{and} \qquad H''(t) = \frac{(1-q)\left[\ln\left(\frac{a}{t}\right) + \frac{1}{q}\right]}{\left[\ln\left(\frac{a}{t}\right)\right]^{\frac{1}{q}+1}},$$

then the function H satisfies hypothesis (H1) on the interval $(0, r]$ for any $0 < r < a$. Therefore, we can use (2.7) to get

$$\begin{aligned} H_1(t) &= \int_t^r \frac{1}{sH'(s)}ds = \int_t^r \frac{\left[\ln\frac{a}{s}\right]^{\frac{1}{q}}}{s\left[1-q+q\ln\frac{a}{s}\right]}ds = \int_{\ln\frac{a}{r}}^{\ln\frac{a}{t}} \frac{u^{\frac{1}{q}}}{1-q+qu}du \\ &= \frac{1}{q}\int_{\ln\frac{a}{r}}^{\ln\frac{a}{t}} u^{\frac{1}{q}-1}\left[\frac{u}{\frac{1-q}{q}+u}\right]du \le \frac{1}{q}\int_{\ln\frac{a}{r}}^{\ln\frac{a}{t}} u^{\frac{1}{q}-1}du \le \left[\ln\frac{a}{t}\right]^{\frac{1}{q}} \end{aligned}$$

$$\Longrightarrow H_1^{-1}(t) \le a\exp(-t^q) \Longrightarrow E(t) \le ak_2\exp(-k_3t^q).$$

B- Considering the functions $g_1(t) = g_2(t) = \frac{a}{(t+e)[\ln(t+e)]^p}$ where $p > 1$ and a chosen so that (2.1) is satisfied. We can see that $g_i'(t) = \frac{-a[\ln(t+e)+p]}{(t+e)^2[\ln(t+e)]^{p+1}}$ can be written as

$$\begin{aligned} g_i'(t) &= \frac{-\left[\ln(t+e)+p\right]}{(t+e)\ln(t+e)}g(t) \\ \underset{\text{by } (2.8)-case1}{\Longrightarrow} E(t) &\le ke^{-k_1\int_0^t \frac{[\ln(t+e)+p]}{(t+e)\ln(t+e)}ds} = \frac{k}{\left((t+e)\left[\ln(t+e)\right]^p\right)^{k_1}} \end{aligned}$$

which is slower rate, as $k_1 \le 1$, than $g_1(t)$ and $g_2(t)$. But, it can also be written as

$$\begin{aligned} g_i'(t) &= \frac{-\left[\ln(t+e)+p\right]}{a^{\frac{1}{p}}(t+e)^{1-\frac{1}{p}}}(g(t))^{1+\frac{1}{p}} \\ \underset{\text{by } (2.8)-case2}{\Longrightarrow} E(t) &\le \overline{k}\left(1+\int_0^t \frac{\ln(t+e)+p}{a^{\frac{1}{p}}(t+e)^{1-\frac{1}{p}}})ds\right)^{-p} \underset{\text{for large } t}{\le} \frac{\overline{k}}{(t+e)\left[\ln(t+e)\right]^p} \end{aligned}$$

which is the same rate as the relaxation functions.

(4) The well-known Jensen's inequality will be of essential use in establishing our main result. If F is a convex function on $[a, b]$, $f : \Omega \to [a, b]$ and h are integrable functions on Ω, $h(x) \ge 0$, and $\int_\Omega h(x)dx = k > 0$, then Jensen's inequality states that

$$F\left[\frac{1}{k}\int_\Omega f(x)h(x)dx\right] \le \frac{1}{k}\int_\Omega F[f(x)]h(x)dx.$$

(5) We easily deduce, by (H1), that $\lim_{t\to+\infty} g_i(t) = 0$. Hence, $t_1 = \max\{g_1^{-1}(r), g_2^{-1}(r)\}$ is well-defined and, for $i = 1, 2$,

$$g_i(t) \le r, \qquad \forall\, t \ge t_1. \tag{2.9}$$

As g_i and ξ_i are positive nonincreasing continuous functions and H is a positive continuous function, then, for all $t \in [0, t_1]$,

$$\begin{cases} 0 < g_i(t_1) \le g_i(t) \le g_i(0) \\ 0 < \xi_i(t_1) \le \xi_i(t) \le \xi_i(0) \end{cases} \Longrightarrow a \le \xi_i(t)H(g_i(t)) \le b$$

for some positive constants a and b. Consequently, for all $t \in [0, t_1]$,

$$g_i'(t) \leq -\xi_i(t)H(g_i(t)) \leq -\frac{a}{M_0}M_0 \leq -\frac{a}{M_0}g_i(t). \tag{2.10}$$

(6) If different functions H_1 and H_2 have the properties mentioned in (H1) such that $g_1'(t) \leq -\xi_2(t)H_1(g_1(t))$ and $g_2'(t) \leq -\xi_2(t)H_2(g_2(t))$, then there is $r \leq \min\{r_1, r_2\}$ small enough so that, say, $H_1(t) \leq H_2(t)$ on the interval $(0, r]$. Thus, the function $H(t) = H_1(t)$ satisfies (H1) for both functions g_1 and $g_2, \forall\, t \geq t_1$.

(7) If H is a strictly increasing and strictly convex C^2 function on $(0, r]$, with $H(0) = H'(0) = 0$, then it has an extension $\overline{H}$ which is strictly increasing and strictly convex C^2 function on $(0, \infty)$. For instance, if $H(r) = a, H'(r) = b, H''(r) = c$, we can define $\overline{H}$, for $t > r$, by

$$\overline{H}(t) = \frac{c}{2}t^2 + (b - cr)t + (a + \frac{c}{2}r^2 - br). \tag{2.11}$$

3 Proof of Theorem 2.1

The existence is proved using Galerkin method. In order to do so, we take $\{w_i\}_{i=1}^{\infty}$ to be the eigenfunctions of the Laplacian operator subject to Dirichlet boundary conditions. Then $\{w_i\}_{i=1}^{\infty}$ is orthogonal basis of $H_0^1(\Omega)$ and of $H^2(\Omega) \cap H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let $V_m = span\,\{w_1, w_2, ..., w_m\}$ and the projection of the initial data on the finite dimensional subspace V_m is given by

$$u_0^m = \sum_{j=1}^{m} a_j w_j, \qquad v_0^m = \sum_{j=1}^{m} b_j w_j, \qquad u_1^m = \sum_{j=1}^{m} c_j w_j, \qquad v_1^m = \sum_{j=1}^{m} d_j w_j$$

where, $(u_0^m, v_0^m) \to (u_0, v_0)$ in $(H^2(\Omega) \cap H_0^1(\Omega))^2$ and $(u_1^m, v_1^m) \to (u_1, v_1)$ in $(H_0^1(\Omega))^2$ as $m \to \infty$. We search the approximate solutions

$$u^m(x,t) = \sum_{j=1}^{m} h_{j,m}(t) w_j(x), \qquad v^m(x,t) = \sum_{j=1}^{m} k_{j,m}(t) w_j(x)$$

of the approximate problem in V_m

$$\begin{cases} \int_\Omega \left(u_{tt}^m w + \nabla u^m.\nabla w - \int_0^t g_1(t-\tau)\nabla u^m(\tau).\nabla w d\tau + f_1(u^m, v^m)w \right) dx = 0 \\ \int_\Omega \left(v_{tt}^m w + \nabla v^m.\nabla w - \int_0^t g_2(t-\tau)\nabla v^m(\tau).\nabla w d\tau + f_2(u^m, v^m)w \right) dx = 0 \\ \quad u^m(0) = u_0^m, \qquad u_t^m(0) = u_1^m, \qquad v^m(0) = v_0^m, \qquad v_t^m(0) = v_1^m. \end{cases} \tag{3.1}$$

This system leads to a system of ODE for uknown functions $h_{j,m}(t), k_{j,m}(t)$. Based on standard existence theory for ODE, one can conclude the existence of a solution (u^m, v^m) of (3.1) on a maximal time interval $[0, t_m)$, for each $m \in \mathbb{N}$. The a priori estimate that follows implies that in fact $t_m = +\infty$.

• (A priori estimate 1): In (3.1), let $w = u_t^m$ in the first equation and $w = v_t^m$ in the second equation, add the resulting equations, and integrate by parts to obtain

$$\frac{d}{dt}E^m(t) = \frac{1}{2}(g_1' \circ \nabla u^m) - \frac{1}{2}g_1(t)\left\|\nabla u^m\right\|_2^2 + \frac{1}{2}(g_2' \circ \nabla v^m) - \frac{1}{2}g_2(t)\left\|\nabla v^m\right\|_2^2.$$

This means, using (H1), that, for some positive constant C independent of t and m,

$$E^m(t) \leq E^m(0) \leq C. \tag{3.2}$$

• (A priori estimate 2): In (3.1), let $w = -\Delta u_t^m$ in the first equation and $w = -\Delta v_t^m$ in the second equation, add the resulting equations, integrate by parts, and use (H1) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\begin{array}{c} \|\nabla u_t^m\|_2^2 + (1-\int_0^t g_1(s)ds)\|\Delta u^m\|_2^2 + \|\nabla v_t^m\|_2^2 \\ +(1-\int_0^t g_2(s)ds)\|\Delta v^m\|_2^2 + g_1\circ\Delta u^m + g_2\circ\Delta v^m \end{array}\right] \\
= & \frac{1}{2}(g_1'\circ\Delta u^m) - \frac{1}{2}g_1(t)\|\Delta u^m\|_2^2 + \frac{1}{2}(g_2'\circ\Delta v^m) - \frac{1}{2}g_2(t)\|\Delta v^m\|_2^2 \\
& + \int_\Omega (f_1(u^m,v^m)\Delta u_t^m + f_2(u^m,v^m)\Delta v_t^m)\,dx \\
\leq & \int_\Omega (f_1(u^m,v^m)\Delta u_t^m + f_2(u^m,v^m)\Delta v_t^m)\,dx
\end{aligned}$$

Then, integrating over $(0,t)$ yields

$$\begin{aligned}
&\frac{1}{2}\left[\begin{array}{c} \|\nabla u_t^m\|_2^2 + (1-\int_0^t g_1(s)ds)\|\Delta u^m\|_2^2 + \|\nabla v_t^m\|_2^2 \\ +(1-\int_0^t g_2(s)ds)\|\Delta v^m\|_2^2 + g_1\circ\Delta u^m + g_2\circ\Delta v^m \end{array}\right] \\
\leq & \frac{1}{2}\left[\|\nabla u_1^m\|_2^2 + \|\Delta u_0^m\|_2^2 + \|\nabla v_1^m\|_2^2 + \|\Delta v_0^m\|_2^2\right] \\
& + \int_\Omega \left(\begin{array}{c} f_1(u^m,v^m)\Delta u^m - f_1(u_0^m,v_0^m)\Delta u_0^m \\ +f_2(u^m,v^m)\Delta v^m - f_2(u_0^m,v_0^m)\Delta v_0^m \end{array}\right) dx \\
& - \int_0^t\int_\Omega \left(\begin{array}{c} \frac{\partial f_1}{\partial u}(u^m,v^m)u_t^m\Delta u^m + \frac{\partial f_1}{\partial v}(u^m,v^m)v_t^m\Delta u^m \\ +\frac{\partial f_2}{\partial u}(u^m,v^m)u_t^m\Delta v^m + \frac{\partial f_2}{\partial v}(u^m,v^m)v_t^m\Delta v^m \end{array}\right) dxds \qquad (3.1)
\end{aligned}$$

To estimate the terms in the right hand side of (3.3), we use (2.4), Young's inequality, and (2.6) and take (3.2) into account to get

$$\begin{aligned}
\int_\Omega f_1(u^m,v^m)\Delta u^m dx \leq & \; k\int_\Omega (|u^m| + |v^m| + |u^m|^{\beta_{11}} + |v^m|^{\beta_{12}})\,|\Delta u^m|\,dx \\
\leq & \; \delta\|\Delta u^m\|_2^2 + \frac{c}{\delta}\int_\Omega (|u^m|^2 + |v^m|^2 + |u^m|^{2\beta_{11}} + |v^m|^{2\beta_{12}})dx \\
\leq & \; \delta\|\Delta u^m\|_2^2 + \frac{c}{\delta}(\|\nabla u^m\|_2^2 + \|\nabla v^m\|_2^2 + \|\nabla u^m\|_2^{2\beta_{11}} + \|\nabla v^m\|_2^{2\beta_{12}}) \\
\leq & \; \delta\|\Delta u^m\|_2^2 + \frac{c}{\delta}. \qquad (3.2)
\end{aligned}$$

Now, we estimate $J := -\int_\Omega \frac{\partial f_1}{\partial u}(u^m,v^m)u_t^m\Delta u^m dx$ as follows. First, we observe that $\frac{\beta_{1j}-1}{2\beta_{1j}} + \frac{1}{2\beta_{1j}} + \frac{1}{2} = 1$ and use (H2) and the generalized Hölder's inequality to infer

$$\begin{aligned}
|J| \leq & \; d\int_\Omega (1 + |u^m|^{\beta_{11}-1} + |v^m|^{\beta_{12}-1})\,|u_t^m|\,|\Delta u^m|\,dx \\
\leq & \; d(\|u_t^m\|_2 + \|u^m\|_{2\beta_{11}}^{\beta_{11}-1}\|u_t^m\|_{2\beta_{11}} + \|v^m\|_{2\beta_{12}}^{\beta_{12}-1}\|u_t^m\|_{2\beta_{12}})\,\|\Delta u^m\|_2
\end{aligned}$$

Then, by (2.6), (3.2), and Young's inequality, we arrive at

$$
\begin{aligned}
|J| &\leq c(1 + \|\nabla u^m\|_2^{\beta_{11}-1} + \|\nabla v^m\|_2^{\beta_{12}-1}) \|\nabla u_t^m\|_2 \|\Delta u^m\|_2 \\
&\leq c \|\nabla u_t^m\|_2 \|\Delta u^m\|_2 \leq c \|\nabla u_t^m\|_2^2 + c \|\Delta u^m\|_2^2 . \qquad (3.3)
\end{aligned}
$$

Since the other terms in (3.3) can be similarly treated and the norms of the initial data are uniformly bounded, we combine (3.3-3.5), use (H1), and take δ small enough to end up with

$$
\begin{aligned}
&\|\nabla u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|\nabla v_t^m\|_2^2 + \|\Delta v^m\|_2^2 \\
\leq\ & c + c \int_0^t \left(\|\nabla u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|\nabla v_t^m\|_2^2 + \|\Delta v^m\|_2^2 \right) ds.
\end{aligned}
$$

Using Gronwall's inequality, this implies that

$$
\|\nabla u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|\nabla v_t^m\|_2^2 + \|\Delta v^m\|_2^2 \leq C, \qquad \forall t \in [0, T] \text{ and } m \in \mathbb{N}. \qquad (3.6)
$$

• (A priori estimate 3): In (3.1), let $w = u_{tt}^m$ in the first equation and $w = v_{tt}^m$ in the second equation. Then, by exploiting the previous estimates and using similar arguments, we find

$$
\|u_{tt}^m\|_2^2 + \|v_{tt}^m\|_2^2 \leq C, \qquad \forall t \in [0, T] \text{ and } m \in \mathbb{N}. \qquad (3.7)
$$

From (3.2), (3.6), and (3.7), we conclude that

$$
\begin{aligned}
&u^m, v^m \text{ are uniformly bounded in } L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \\
&u_t^m, v_t^m \text{ are uniformly bounded in } L^\infty(0, T; H_0^1(\Omega)), \\
&u_{tt}^m, v_{tt}^m \text{ are uniformly bounded in } L^\infty(0, T; L^2(\Omega)), \qquad (3.4)
\end{aligned}
$$

which implies that there exist subsequences of u^m, v^m, which we still denote in the same way, such that

$$
\begin{aligned}
&u^m \overset{*}{\rightharpoonup} u \text{ and } v^m \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \\
&u_t^m \overset{*}{\rightharpoonup} u_t \text{ and } v_t^m \overset{*}{\rightharpoonup} v_t \quad \text{in } L^\infty(0, T; H_0^1(\Omega)), \\
&u_{tt}^m \overset{*}{\rightharpoonup} u_{tt} \text{ and } v_{tt}^m \overset{*}{\rightharpoonup} v_{tt} \quad \text{in } L^\infty(0, T; L^2(\Omega)).
\end{aligned}
$$

In particular, making use of Aubin-Lions Theorem, we find, up to a subsequence, that

$$
u^m \to u \text{ and } v^m \to v \quad \text{strongly in } L^2(0, T; L^2(\Omega)).
$$

Then,

$$
u^m \to u \text{ and } v^m \to v \quad \text{a.e in } (0, T) \times \Omega
$$

and therefore, from (H2),

$$
f_i(u^m, v^m) \to f_i(u, v) \quad \text{a.e in } (0, T) \times \Omega, \text{ for } i = 1, 2. \qquad (3.9)
$$

Also, as $u^m$and v^m are bounded in $L^\infty(0, T; L^2(\Omega))$, then the use of (2.4) and (2.6) gives that $f_i(u^m, v^m)$ is bounded in $L^\infty(0, T; L^2(\Omega))$. From this and (3.9), we can deduce that

$$
f_i(u^m, v^m) \rightharpoonup f_i(u, v) \quad \text{in } L^2(0, T; L^2(\Omega)), \text{ for } i = 1, 2.
$$

Regarding the initial conditions, we can also use (3.8) in standard way to verify that

$$u(0) = u_0, \quad u_t(0) = u_1, \quad v(0) = v_0, \quad v_t(0) = v_1.$$

Combining the results obtained above, we can pass to the limit and conclude that (u, v) is a strong solution of system (1.1). For uniqueness, let us assume that $(u_1, v_1), (u_2, v_2)$ are two strong solutions of (1.1). Then, $(z, q) = (u_1 - u_2, v_1 - v_2)$ satisfies, for all $w \in H_0^1(\Omega)$,

$$\begin{cases} \int_\Omega \left(z_{tt}w + \nabla z.\nabla w - \int_0^t g_1(t-\tau)\nabla z(\tau).\nabla w d\tau \right) dx = \int_\Omega (f_1(u_2, v_2) - f_1(u_1, v_1))w dx \\ \int_\Omega \left(q_{tt}w + \nabla q.\nabla w - \int_0^t g_2(t-\tau)\nabla q(\tau).\nabla w d\tau \right) dx = \int_\Omega (f_2(u_2, v_2) - f_2(u_1, v_1))w dx. \end{cases}$$

Substituting $w = z_t$ in the first equation and $w = q_t$ in the second equation, adding the resulting equations, integrating by parts, and using (H1) yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\begin{array}{c} \|z_t\|_2^2 + (1 - \int_0^t g_1(s)ds)\,\|\nabla z\|_2^2 + \|q_t\|_2^2 \\ +(1 - \int_0^t g_2(s)ds)\,\|\nabla q\|_2^2 + g_1 \circ \nabla z + g_2 \circ \nabla q \end{array} \right] \\ \leq\ & \int_\Omega (f_1(u_2, v_2) - f_1(u_1, v_1))z_t dx + \int_\Omega (f_2(u_2, v_2) - f_2(u_1, v_1))q_t dx. \end{aligned} \tag{3.5}$$

Making use of (2.5) and following similar arguments used to obtain (3.5), we find

$$\begin{aligned} & \int_\Omega (f_1(u_2, v_2) - f_1(u_1, v_1))z_t dx + \int_\Omega (f_2(u_2, v_2) - f_2(u_1, v_1))q_t dx \\ \leq\ & k \int_\Omega (1 + |u_1|^{\beta_{11}-1} + |u_2|^{\beta_{11}-1} + |v_1|^{\beta_{12}-1} + |v_2|^{\beta_{12}-1})(|z| + |q|)\,|z_t|\, dx \\ & +k \int_\Omega (1 + |u_1|^{\beta_{21}-1} + |u_2|^{\beta_{21}-1} + |v_1|^{\beta_{22}-1} + |v_2|^{\beta_{22}-1})(|z| + |q|)\,|q_t|\, dx \\ \leq\ & c(\|z_t\|_2^2 + \|\nabla z\|_2^2 + \|q_t\|_2^2 + \|\nabla q\|_2^2). \end{aligned} \tag{3.6}$$

Combining (3.10) and (3.11), integrating over $(0, t)$, and using Gronwall's Lemma, then we deduce that

$$\|z_t\|_2^2 + \|\nabla z\|_2^2 + \|q_t\|_2^2 + \|\nabla q\|_2^2 = 0$$

which means that $(u_1, v_1) = (u_2, v_2)$. This completes the proof.

Now, if $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and considering standard density arguments, we can prove that problem (1.1) has a unique weak solution

$$u, v \in C(\mathbb{R}_+; H_0^1(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega)).$$

4 Technical lemmas

In this section, we establish several lemmas needed to prove Theorem 2.2.

Lemma 4.1. *Let (u,v) be the solution of* (1.1). *Then the energy functional satisfies*

$$E'(t) = \frac{1}{2}(g_1' \circ \nabla u) - \frac{1}{2}g_1(t)\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}(g_2' \circ \nabla v) - \frac{1}{2}g_2(t)\int_\Omega |\nabla v|^2\, dx \leq 0. \tag{4.1}$$

Proof. By multiplying the first equation in (1.1) by u_t and the second by v_t, integrating over Ω, using integration by parts, hypotheses (H1) and (H2), and some manipulations, we obtain (4.1) for any regular solution. This equality remains valid for weak solutions by a simple density argument.

Now we are going to construct a Lyapunov functional $\mathcal{L}$ equivalent to E, with which we can show the desired result.

Lemma 4.2. *Under the assumptions (H1) and (H2), the functional I defined by*

$$I(t) := \int\limits_{\Omega} u u_t dx + \int\limits_{\Omega} v v_t dx$$

satisfies, along the solution, with $l = \min\{l_1, l_2\}$, the estimate

$$\begin{aligned} I'(t) \;\leq\; & -\frac{l}{2}\int\limits_{\Omega}\left(|\nabla u|^2 + |\nabla v|^2\right)dx + \int\limits_{\Omega}\left(u_t^2 + v_t^2\right)dx + cC_{\alpha_1}(h_1 \circ \nabla u)(t) \\ & + cC_{\alpha_2}(h_2 \circ \nabla v)(t) - \int\limits_{\Omega} F(u,v)dx. \end{aligned} \tag{4.1}$$

for any $0 < \alpha_i < 1$ $(i = 1, 2)$, where

$$C_{\alpha_i} = \int\limits_0^{\infty} \frac{g_i^2(s)}{\alpha_i g_i(s) - g_i'(s)} ds \qquad \text{and} \qquad h_i(t) = \alpha_i g_i(t) - g_i'(t) \tag{4.3}$$

Proof. Direct computations, using (1.1), (H1-H2), and Young's inequality, yield

$$\begin{aligned} I'(t) \;=\; & \int\limits_{\Omega} u_t^2 dx - \int\limits_{\Omega} |\nabla u|^2 dx + \int\limits_{\Omega} \nabla u \cdot \int\limits_0^t g_1(t-\tau)\nabla u(\tau)d\tau dx \\ & + \int\limits_{\Omega} v_t^2 dx - \int\limits_{\Omega} |\nabla v|^2 dx + \int\limits_{\Omega} \nabla v \cdot \int\limits_0^t g_2(t-\tau)\nabla v(\tau)d\tau dx \\ & - \int\limits_{\Omega}[u f_1(u,v) + v f_2(u,v)]dx \end{aligned}$$

$$\begin{aligned} = & \int\limits_{\Omega} u_t^2 dx - \left(1 - \int\limits_0^t g_1(s)ds\right)\int\limits_{\Omega}|\nabla u|^2 dx + \int\limits_{\Omega}\nabla u \cdot \int\limits_0^t g_1(t-\tau)(\nabla u(\tau) - \nabla u(t))d\tau dx \\ & + \int\limits_{\Omega} v_t^2 dx - \left(1 - \int\limits_0^t g_1(s)ds\right)\int\limits_{\Omega}|\nabla v|^2 dx + \int\limits_{\Omega}\nabla v \cdot \int\limits_0^t g_2(t-\tau)(\nabla v(\tau) - \nabla v(t))d\tau dx \\ & - \int\limits_{\Omega}[u f_1(u,v) + v f_2(u,v)]dx \end{aligned}$$

$$\leq \int_\Omega u_t^2 dx - l \int_\Omega |\nabla u|^2 dx + \delta \int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta} \int_\Omega \left(\int_0^t g_1(t-\tau) |\nabla u(\tau) - \nabla u(t)| d\tau \right)^2 dx$$

$$+ \int_\Omega v_t^2 dx - l \int_\Omega |\nabla v|^2 dx + \delta \int_\Omega |\nabla v|^2 dx + \frac{1}{4\delta} \int_\Omega \left(\int_0^t g_2(t-\tau) |\nabla v(\tau) - \nabla v(t)| d\tau \right)^2 dx$$

$$- \int_\Omega F(u,v) dx \tag{4.4}$$

for any $\delta > 0$. Now, the use of Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g_1(t-\tau) |\nabla u(\tau) - \nabla u(t)| d\tau \right)^2 dx \\
= &\int_\Omega \left(\int_0^t \frac{g_1(t-\tau)}{\sqrt{\alpha_1 g_1(t-\tau) - g_1'(t-\tau)}} \sqrt{\alpha_1 g_1(t-\tau) - g_1'(t-\tau)} |\nabla u(\tau) - \nabla u(t)| ds \right)^2 dx \\
\leq &\left(\int_0^t \frac{g_1^2(s)}{\alpha_1 g_1(s) - g_1'(s)} ds \right) \int_\Omega \int_0^t [\alpha_1 g_1(t-\tau) - g_1'(t-\tau)] |\nabla u(\tau) - \nabla u(t)|^2 d\tau dx \\
\leq &\ C_{\alpha_1} (h_1 \circ \nabla u)(t).
\end{aligned} \tag{4.2}$$

Similar calculations also yield

$$\int_\Omega \left(\int_0^t g_2(t-\tau) |\nabla v(\tau) - \nabla v(t)| d\tau \right)^2 dx \leq C_{\alpha_2} (h_2 \circ \nabla v)(t). \tag{4.3}$$

Combining (4.4)-(4.6) and choosing δ small enough give (4.2).

Lemma 4.3. *Under the assumptions (H1) and (H2), the functional K defined by*

$$K(t) = K_1(t) + K_2(t),$$

with

$$\begin{aligned}
K_1(t) \ &: \ = - \int_\Omega u_t \int_0^t g_1(t-\tau)(u(t) - u(\tau)) d\tau dx \\
K_2(t) \ &: \ = - \int_\Omega v_t \int_0^t g_2(t-\tau)(v(t) - v(\tau)) d\tau dx,
\end{aligned}$$

satisfies, for any $0 < \delta < 1$ *and all* $t \geq t_1$*, the estimate*

$$K'(t) \;\leq\; -(g_0-\delta)\int_\Omega \left(u_t^2+v_t^2\right)dx + \delta c\int_\Omega \left(|\nabla u|^2+|\nabla v|^2\right)dx + \frac{c\left[C_{\alpha_1}+1\right]}{\delta}(h_1\circ\nabla u)(t) + \frac{c\left[C_{\alpha_2}+1\right]}{\delta}(h_2\circ\nabla v)(t) \tag{4.4}$$

where $g_0 = \min\{\int_0^{t_1} g_1(s)ds, \int_0^{t_1} g_2(s)ds\}$.

Proof. By exploiting equations (1.1) and integrating by parts, we have

$$K_1'(t) = \left(1-\int_0^t g_1(s)ds\right)\int_\Omega \nabla u\cdot\int_0^t g_1(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx$$
$$+\int_\Omega\left(\int_0^t g_1(t-\tau)\left|\nabla u(t)-\nabla u(\tau)\right|d\tau\right)^2 dx + \int_\Omega f_1(u,v)\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx$$
$$-\int_\Omega u_t\int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx - \left(\int_0^t g_1(s)ds\right)\int_\Omega u_t^2 dx. \tag{4.8}$$

Using Young's inequality and similar calculations in (4.5), we obtain

$$\left(1-\int_0^t g_1(s)ds\right)\int_\Omega \nabla u\cdot\int_0^t g_1(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau dx \leq \delta\int_\Omega |\nabla u|^2 dx + \frac{cC_{\alpha_1}}{\delta}(h_1\circ\nabla u)(t) \tag{4.9}$$

and

$$\begin{aligned}
&-\int_\Omega u_t\int_0^t g_1'(t-\tau)(u(t)-u(\tau))d\tau dx\\
=\;& \int_\Omega u_t\int_0^t h_1(t-\tau)(u(t)-u(\tau))d\tau dx - \int_\Omega u_t\int_0^t \alpha_1 g_1(t-\tau)(u(t)-u(\tau))d\tau dx\\
\leq\;& \frac{c}{\delta}\int_\Omega\left(\int_0^t \sqrt{h_1(t-\tau)}\sqrt{h_1(t-\tau)}\left|u(t)-u(\tau)\right|d\tau\right)^2 dx\\
&+\delta\int_\Omega u_t^2 dx + \frac{c\alpha_1^2}{\delta}\int_\Omega\left(\int_0^t g_1(t-\tau)\left|u(t)-u(\tau)\right|d\tau\right)^2 dx\\
\leq\;& \delta\int_\Omega u_t^2 dx + \frac{c\left(\int_0^t h_1(s)ds\right)}{\delta}(h_1\circ u) + \frac{c\alpha_1^2 C_{\alpha_1}}{\delta}(h_1\circ u)\\
\leq\;& \delta\int_\Omega u_t^2 dx + \frac{c}{\delta}(h_1\circ\nabla u) + \frac{cC_{\alpha_1}}{\delta}(h_1\circ\nabla u).
\end{aligned} \tag{4.5}$$

To estimate the third term in the right hand side of (4.8), we use (2.4), (2.6) and the fact that $\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq 2E(t) \leq 2E(0)$ to get

$$
\begin{aligned}
& \int_\Omega f_1(u,v) \int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau dx \\
\leq\ & \delta c \int_\Omega (|u|^2 + |v|^2 + |u|^{2\beta_{11}} + |v|^{2\beta_{12}})dx + \frac{c}{\delta}\int_\Omega \left(\int_0^t g_1(t-\tau)(u(t)-u(\tau))d\tau \right)^2 dx \\
\leq\ & \delta c(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\nabla u\|_2^{2\beta_{11}} + \|\nabla v\|_2^{2\beta_{12}}) + \frac{c}{\delta} C_{\alpha_1}(h_1 \circ \nabla u)(t) \\
=\ & \delta c(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\nabla u\|_2^{2(\beta_{11}-1)} \|\nabla u\|_2^2 + \|\nabla v\|_2^{2(\beta_{12}-1)} \|\nabla v\|_2^2) + \frac{c}{\delta}(h_1 \circ \nabla u)(t) \\
\leq\ & \delta c(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + [2E(0)]^{(\beta_{11}-1)} \|\nabla u\|_2^2 + [2E(0)]^{(\beta_{12}-1)} \|\nabla v\|_2^2) + \frac{c}{\delta}(h_1 \circ \nabla u)(t) \\
\leq\ & \delta c \|\nabla u\|_2^2 + \delta c \|\nabla v\|_2^2 + \frac{c}{\delta}(h_1 \circ \nabla u)(t).
\end{aligned} \tag{4.6}
$$

By combining (4.8)-(4.11), we obtain

$$
\begin{aligned}
K_1'(t) \ \leq\ & -(\int_0^t g_1(s)ds - \delta) \int_\Omega u_t^2 dx + \delta c \int_\Omega |\nabla u|^2 dx \\
& + \frac{c\,[C_{\alpha_1}+1]}{\delta}(h_1 \circ \nabla u)(t) + \delta c \int_\Omega |\nabla v|^2 dx.
\end{aligned}
$$

Since K_2 can be dealt with similarly, (4.7) is established.

Next, we use the functional

$$
M(t) = \int_\Omega \int_0^t f_1(t-\tau)\,|\nabla u(\tau)|^2\, d\tau dx + \int_\Omega \int_0^t f_2(t-\tau)\,|\nabla v(\tau)|^2\, d\tau dx \tag{4.12}
$$

where $f_i(t) = \int_t^\infty g_i(s)ds$.

Lemma 4.4. *Assume that (H1) and (H2) hold. The functional M satisfies, along the solution of (1.1), the estimate*

$$
M'(t) \leq -\frac{1}{2}(g_1 \circ \nabla u)(t) - \frac{1}{2}(g_2 \circ \nabla v)(t) + 3(1-l)\int_\Omega \left(|\nabla u(t)|^2 + |\nabla v(t)|^2 \right) dx. \tag{4.13}
$$

Proof. By Young's inequality and the fact $f_i'(t) = -g_i(t)$, we see that

$$
\begin{aligned}
M'(t) \ =\ & f_1(0)\int_\Omega |\nabla u(t)|^2 dx - \int_\Omega \int_0^t g_1(t-\tau)\,|\nabla u(\tau)|^2\, d\tau dx \\
& + f_2(0)\int_\Omega |\nabla v(t)|^2 dx - \int_\Omega \int_0^t g_2(t-\tau)\,|\nabla v(\tau)|^2\, d\tau dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_\Omega \int_0^t g_1(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|^2 d\tau dx \\
&\quad -2\int_\Omega \nabla u(t)\cdot \int_0^t g_1(t-\tau)(\nabla u(\tau)-\nabla u(t))d\tau dx + f_1(t)\int_\Omega \left|\nabla u(t)\right|^2 dx \\
&\quad -\int_\Omega \int_0^t g_2(t-\tau)\left|\nabla v(\tau)-\nabla v(t)\right|^2 d\tau dx \\
&\quad -2\int_\Omega \nabla v(t)\cdot \int_0^t g_2(t-\tau)(\nabla v(\tau)-\nabla v(t))d\tau dx + f_2(t)\int_\Omega \left|\nabla v(t)\right|^2 dx.
\end{aligned}
$$

But

$$
\begin{aligned}
&-2\int_\Omega \nabla w(t)\cdot \int_0^t g_i(t-\tau)(\nabla w(\tau)-\nabla w(t))d\tau dx \\
&\leq 2(1-l_i)\int_\Omega \left|\nabla w(t)\right|^2 dx + \frac{\int_0^t g_i(s)ds}{2(1-l_i)}\int_\Omega \int_0^t g_i(t-\tau)\left|\nabla w(\tau)-\nabla w(t)\right|^2 d\tau dx.
\end{aligned}
$$

Then, as $f_i(t) \leq f_i(0) = (1-l_i)$ and $\int_0^t g_i(s)ds \leq (1-l_i)$, we get (4.13).

Lemma 4.5. *The functional $\mathcal{L}$ defined by*

$$\mathcal{L}(t) := NE(t) + N_1 I(t) + N_2 K(t)$$

satisfies, for suitable choice of $N, N_1, N_2 > 0$, that

$$
\begin{aligned}
\mathcal{L}'(t) &\leq -4(1-l)\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \int_\Omega (u_t^2+v_t^2)dx - \int_\Omega F(u,v)dx \\
&\quad +\frac{1}{4}(g_1\circ\nabla u)(t) + \frac{1}{4}(g_2\circ \nabla v)(t), \qquad \forall\, t \geq t_1 \qquad (4.7)
\end{aligned}
$$

and

$$\mathcal{L}(t) \sim E(t) \qquad (4.15)$$

Proof. By combining (4.1), (4.2), (4.7), recalling that $g_i' = (\alpha_i g_i - h_i)$, and taking $\delta = 1/N_2$ in (4.7), we obtain, for all $t \geq t_1$,

$$
\begin{aligned}
\mathcal{L}'(t) &\leq -\left(\frac{l}{2}N_1 - c\right)\int_\Omega (|\nabla u|^2+|\nabla v|^2)dx - (N_2 g_0 - 1 - N_1)\int_\Omega (u_t^2+v_t^2)dx - \int_\Omega F(u,v)dx \\
&\quad +\frac{\alpha_1}{2}N(g_1\circ\nabla u)(t) - \left(\frac{1}{2}N - cN_2^2 - C_{\alpha_1}[cN_2^2 + cN_1]\right)(h_1\circ\nabla u)(t) \\
&\quad +\frac{\alpha_2}{2}N(g_2\circ\nabla v)(t) - \left(\frac{1}{2}N - cN_2^2 - C_{\alpha_2}[cN_2^2 + cN_1]\right)(h_2\circ\nabla v)(t)
\end{aligned}
$$

At this point, we choose N_1 large enough so that

$$\frac{l}{2}N_1 - c > 4(1-l),$$

then N_2 large enough so that

$$N_2 g_0 - 1 - N_1 > 1.$$

Now, as g_i and $-g_i'$ are positive functions and $\alpha_i > 0$ we find that

$$0 < \frac{\alpha_i g_i^2(s)}{\alpha_i g_i(s) - g_i'(s)} < \frac{\alpha_i g_i^2(s)}{-g_i'(s)} \Longrightarrow \lim_{\alpha_i \to 0} \frac{\alpha_i g_i^2(s)}{\alpha_i g_i(s) - g_i'(s)} = 0 \qquad \text{for each } s \in [0, \infty)$$

and also

$$\frac{\alpha_i g_i^2(s)}{\alpha_i g_i(s) - g_i'(s)} < g_i(s)$$

where g_i is integrable on $[0, \infty)$, then, by the Lebesgue dominated convergence theorem, we deduce, for $i = 1, 2$, that

$$\alpha_i C_{\alpha_i} = \int_0^\infty \frac{\alpha_i g_i^2(s)}{\alpha_i g_i(s) - g_i'(s)} ds \longrightarrow 0 \qquad \text{as} \qquad \alpha_i \longrightarrow 0.$$

Hence, there is $0 < \gamma < 1$ such that if $\alpha_i < \gamma$ then

$$\alpha_i C_{\alpha_i} < \frac{1}{8[cN_2^2 + cN_1]}.$$

Let us choose N large enough and choose α_1, α_2 satisfying

$$\frac{1}{4}N - cN_2^2 > 0 \qquad \text{and} \qquad \alpha_1 = \alpha_2 = \frac{1}{2N} < \gamma$$

which means

$$\frac{1}{2}N - cN_2^2 - C_{\alpha_i}[cN_2^2 + cN_1] > 0.$$

So, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \ \le\ & -4(1-l)\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \int_\Omega (u_t^2 + v_t^2)dx - \int_\Omega F(u,v)dx \\ & + \frac{1}{4}(g_1 \circ \nabla u)(t) + \frac{1}{4}(g_2 \circ \nabla v)(t). \end{aligned}$$

On the other hand, we can easily find, using Young's and Poincaré's inequalities, that

$$|\mathcal{L}(t) - NE(t)| \le N_1 |I(t)| + N_2 |K(t)| \le cE(t).$$

Hence, we can choose N even larger (if needed) so that, for some constants $a_1, a_2 > 0$,

$$a_1 E(t) \le \mathcal{L}(t) \le a_2 E(t),$$

that is, (4.15) is satisfied.

5 Proof of Theorem 2.2

We start using (2.10) and (4.1) to conclude that, for any $t \geq t_1$,

$$\int_0^{t_1} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds + \int_0^{t_1} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds$$
$$\leq -\frac{M_0}{a} \int_0^{t_1} \left(g_1'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx - g_2'(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dx \right) ds$$
$$\leq -cE'(t)$$

which can be used in (4.14) and then take $F(t) = \mathcal{L}(t) + cE(t)$, which is clearly equivalent to $E(t)$, to get, for some constant $m > 0$ and for all $t \geq t_1$,

$$\begin{aligned}
\mathcal{L}'(t) \;&\leq\; -4(1-l) \int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx - \int_\Omega (u_t^2 + v_t^2) dx - \int_\Omega F(u,v) dx \\
&\quad + \frac{1}{4}(g_1 \circ \nabla u)(t) + \frac{1}{4}(g_2 \circ \nabla v)(t) \\
&\leq\; -mE(t) + c(g_1 \circ \nabla u)(t) + c(g_2 \circ \nabla v)(t) \\
&\leq\; -mE(t) - cE'(t) + c \int_{t_1}^{t} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\
&\quad + c \int_{t_1}^{t} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad F'(t) \leq -mE(t) + c \int_{t_1}^{t} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\
+ c \int_{t_1}^{t} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds
\end{aligned} \tag{5.8}$$

We consider the following two cases.

(I) $H(t)$ **is linear**: By Multiplying (5.1) by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ and using (H1) and (4.1), we obtain

$$\begin{aligned}
\xi(t)F'(t) \;&\leq\; -m\xi(t)E(t) + c\xi(t) \int_{t_1}^{t} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\
&\quad + c\xi(t) \int_{t_1}^{t} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 \, dxds
\end{aligned}$$

$$
\begin{aligned}
&\leq -m\xi(t)E(t) + c\int_{t_1}^{t} \xi_1(s)g_1(s)\int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2\,dxds \\
&\quad + c\int_{t_1}^{t} \xi_2(s)g_2(s)\int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2\,dxds \\
&\leq -m\xi(t)E(t) - c\int_{t_1}^{t} g_1'(s)\int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2\,dxds \\
&\quad - c\int_{t_1}^{t} g_2'(s)\int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2\,dxds \\
&\leq -m\xi(t)E(t) - cE'(t).
\end{aligned}
$$

Using the fact that ξ is a nonincreasing continuous function as ξ_1 and ξ_2 are nonincreasing, and so ξ is differentiable, with $\xi'(t) \leq 0$, for a.e. t, then we infer that

$$
(\xi F + cE)'(t) \leq \xi(t)F'(t) + cE'(t) \leq -m\xi(t)E(t), \qquad \text{a.e. } t \geq t_1.
$$

Hence, using the fact that $\xi F + cE \sim E$, we easily obtain

$$
E(t) \leq c' e^{-\overline{c}\int_{t_1}^{t}\xi(s)ds}.
$$

(II) $H(t)$ **is nonlinear**: First, we use Lemma 4.4 and Lemma 4.5 to deduce that

$$
L(t) = \mathcal{L}(t) + M(t)
$$

is nonnegative and it satisfies, for all $t \geq t_1$,

$$
\begin{aligned}
L'(t) &\leq -(1-l)\int_{\Omega} (|\nabla u|^2 + |\nabla v|^2)dx - \int_{\Omega} (u_t^2 + v_t^2)dx - \int_{\Omega} F(u,v)dx \\
&\quad -\frac{1}{4}(g_1 \circ \nabla u)(t) - \frac{1}{4}(g_2 \circ \nabla v)(t) \\
&\leq -BE(t)
\end{aligned}
$$

where B is some positive constant. Therefore

$$
B\int_{t_1}^{t} E(s)ds \leq L(t_1) - L(t) \leq L(t_1)
$$

which implies that

$$
\int_{0}^{\infty} E(s)ds < \infty. \tag{5.2}
$$

Hence, by noting that

$$\begin{aligned} I_1(t)+I_2(t) &= q\int_{t_1}^{t}\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds+q\int_{t_1}^{t}\int_{\Omega}|\nabla v(t)-\nabla v(t-s)|^2\,dxds \\ &\le cq\int_0^t E(s)ds, \end{aligned}$$

(5.2) allows for a constant $0<q<1$ chosen so that, , for all $t\ge t_1$,

$$I_i(t)<1. \tag{5.3}$$

We also assume, without loss of generality, t_1 large enough so that $I_i(t_1)>0$ which mean $I_i(t)>0$ for all $t\ge t_1$. Also, we define $\lambda(t)$ by

$$\lambda(t):=-\int_{t_1}^{t}g_1'(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds,$$

and observe that $\lambda(t)\le -cE'(t)$. Since H is strictly convex on $(0,r]$ and $H(0)=0$, then

$$H(\theta x)\le \theta H(x)$$

provided $0\le\theta\le 1$ and $x\in(0,r]$. The use of this fact, hypothesis (H1), (5.3), and Jensen's inequality leads to

$$\begin{aligned} \lambda(t) &= \frac{1}{qI_1(t)}\int_{t_1}^{t}I_1(t)(-g_1'(s))\int_{\Omega}q\,|\nabla u(t)-\nabla u(t-s)|^2\,dxds \\ &\ge \frac{1}{qI_1(t)}\int_{t_1}^{t}I_1(t)\xi_1(s)H(g_1(s))\int_{\Omega}q\,|\nabla u(t)-\nabla u(t-s)|^2\,dxds \\ &\ge \frac{\xi(t)}{qI_1(t)}\int_{t_1}^{t}H(I_1(t)g_1(s))\int_{\Omega}q\,|\nabla u(t)-\nabla u(t-s)|^2\,dxds \\ &\ge \frac{\xi(t)}{q}H\left(\frac{1}{I_1(t)}\int_{t_1}^{t}I_1(t)g_1(s)\int_{\Omega}q\,|\nabla u(t)-\nabla u(t-s)|^2\,dxds\right) \\ &= \frac{\xi(t)}{q}H\left(q\int_{t_1}^{t}g_1(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds\right) \\ &= \frac{\xi(t)}{q}\overline{H}\left(q\int_{t_1}^{t}g_1(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2\,dxds\right) \end{aligned}$$

where $\overline{H}$ is an extension of H such that $\overline{H}$ is strictly increasing and strictly convex C^2 function on $(0,\infty)$ [see (2.11)]. This implies that

$$\int_{t_1}^{t} g_1(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 \, dx ds \leq \frac{1}{q} \overline{H}^{-1}\left(\frac{q\lambda(t)}{\xi(t)}\right).$$

We also define

$$\chi(t) := -\int_{t_1}^{t} g_2'(s) \int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2 \, dx ds.$$

and repeat the above steps to get

$$\int_{t_1}^{t} g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2 \, dx ds \leq \frac{1}{q} \overline{H}^{-1}\left(\frac{q\chi(t)}{\xi(t)}\right).$$

Therefore, using the properties of $\overline{H}$, (5.1) becomes

$$\begin{aligned} F'(t) &\leq -mE(t) + c\overline{H}^{-1}\left(\frac{q\lambda(t)}{\xi(t)}\right) + c\overline{H}^{-1}\left(\frac{q\chi(t)}{\xi(t)}\right) \\ &\leq -mE(t) + c\overline{H}^{-1}\left(\frac{q\lambda(t)}{\xi(t)} + \frac{q\chi(t)}{\xi(t)}\right), \qquad \forall\, t \geq t_1. \end{aligned} \tag{5.9}$$

Now, for $\varepsilon_0 < r$, using (5.4), and the fact that $E' \leq 0$, $\overline{H}' > 0, \overline{H}'' > 0$, we find that the functional F_1, defined by

$$F_1(t) := \overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F(t) + E(t)$$

is equivalent to E and

$$\begin{aligned} F_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} \overline{H}''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F(t) + \overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F'(t) + E'(t) \\ &\leq -mE(t)\overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \overline{H}^{-1}\left(\frac{q\lambda(t)}{\xi(t)} + \frac{q\chi(t)}{\xi(t)}\right) + E'(t). \end{aligned} \tag{5.5}$$

Let $\overline{H}^*$ be the convex conjugate of $\overline{H}$ in the sense of Young (see [4] p. 61-64), then

$$\overline{H}^*(s) = s(\overline{H}')^{-1}(s) - \overline{H}[(\overline{H}')^{-1}(s)] \tag{5.6}$$

and $\overline{H}^*$ satisfies the following Young's inequality

$$AB \leq \overline{H}^*(A) + \overline{H}(B). \tag{5.7}$$

With $A = \overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = \overline{H}^{-1}\left(\frac{q\lambda(t)}{\xi(t)} + \frac{q\chi(t)}{\xi(t)}\right)$, using (4.1) and (5.5)-(5.7), we arrive at

$$\begin{aligned} F_1'(t) &\leq -mE(t)\overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\overline{H}^*\left(\overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\frac{q\lambda(t)}{\xi(t)} + c\frac{q\chi(t)}{\xi(t)} + E'(t) \\ &\leq -mE(t)\overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)}\overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\frac{q\lambda(t)}{\xi(t)} + c\frac{q\chi(t)}{\xi(t)} + E'(t). \end{aligned}$$

Then, we multiply by $\xi(t)$ and use the fact that, as $\varepsilon_0 \frac{E(t)}{E(0)} < r$, $\overline{H}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ to get

$$\begin{aligned} \xi(t)F_1'(t) &\leq -m\xi(t)E(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)}\xi(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ &\quad + cq\lambda(t) + cq\chi(t) + \xi(t)E'(t) \\ &\leq -m\xi(t)E(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)}\xi(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) \end{aligned}$$

Consequently, with $F_2 = \xi F_1 + cE$, which satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 F_2(t) \leq E(t) \leq \alpha_2 F_2(t), \tag{5.8}$$

and with a suitable choice of ε_0, we obtain, for some constant $k > 0$ and for all $t \geq t_1$,

$$F_2'(t) \leq -k\xi(t)\left(\frac{E(t)}{E(0)}\right)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -k\xi(t)H_2\left(\frac{E(t)}{E(0)}\right), \tag{5.9}$$

where $H_2(t) = tH'(\varepsilon_0 t)$.

Since $H_2'(t) = H'(\varepsilon_0 t) + \varepsilon_0 t H''(\varepsilon_0 t)$, then, using the strict convexity of H on $(0, r]$, we find that $H_2'(t)$, $H_2(t) > 0$ on $(0, 1]$. Thus, with

$$R(t) = \frac{\alpha_1 F_2(t)}{E(0)},$$

taking in account (5.8) and (5.9), we have

$$R(t) \sim E(t) \tag{5.10}$$

and, for some $k_1 > 0$,

$$R'(t) \leq -k_1 \xi(t) H_2(R(t)), \qquad \forall t \geq t_1.$$

Then, the integration over (t_1, t) yields

$$\int_{t_1}^{t} \frac{-R'(s)}{H_2(R(s))} ds \geq k_1 \int_{t_1}^{t} \xi(s) ds \Longrightarrow \int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_1)} \frac{1}{sH'(s)} ds \geq k_1 \int_{t_1}^{t} \xi(s) ds$$

$$\Longrightarrow R(t) \leq \frac{1}{\varepsilon_0} H_1^{-1}(k_1 \int_{t_1}^{t} \xi(s) ds), \tag{5.11}$$

where $H_1(t) = \int_t^r \frac{1}{sH'(s)} ds$. Here, we have used, based on the properties of H, the fact that H_1 is strictly decreasing function on $(0, r]$ and $\lim_{t\to 0} H_1(t) = +\infty$. A combination of (5.10) and (5.11), estimate (2.7) is established.

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Bibliography

[1] Alabau-Boussouira F. and Cannarsa P., A general method for proving sharp energy decay rates for memory dissipative evolution equations, C. R. Acad. Sci. Paris Ser. I 347 (2009), 867-872.

[2] Alabau-Boussouira F., Cannarsa P. and Sforza D., Decay estimates for the second order evolution equation with memory, J. Funct. Anal. 245 (2008), 1342-1372.

[3] Andrade D. and Mognon A., Global solutions for a system of Klein-Gordon equations with memory, Bol. Soc. Paran. Mat. 21 (1/2) (2003), 127-138.

[4] Arnold V.I., Mathematical Methods of Classical Mechanics, Springer-Verlag, New York,1989.

[5] Barreto R. and Munoz Rivera J.E., Uniform rates of decay in nonlinear viscoelasticity for polynomial decaying kernels, Appl. Anal. 60 (1996), 263–283.

[6] Cannarsa P. and Sforza D., Integro-differential equations of hyperbolic type with positive definite kernels, J. Differential Equations 250 (2011), 4289–4335.

[7] Cavalcanti M.M., Domingos Cavalcanti V.N., Lasiecka I. and Nascimento F.A., Intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative effects, Discrete Contin. Dyn. Syst. Ser. B 19 (7) (2014), 1987-2012.

[8] Cavalcanti M.M., Domingos Cavalcanti V.N., Lasiecka I. and Webler C.M., Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density, Adv. Nonlinear Anal. 6(2) (2017), 121-145.

[9] Cavalcanti M. M. and Oquendo H. P., Frictional versus viscoelastic damping in a semilinear wave equation, SIAM J. Control Optim. 42 (4) (2003), 1310-1324.

[10] Guesmia A., Asymptotic stability of abstract dissipative systems with infinite memory, J. Math. Anal. Appl. 382 (2011), 748–760.

[11] Han X. and Wang M., General decay of energy for a viscoelastic equation with nonlinear damping, Math. Methods Appl. Sci. 32 (3) (2009), 346–358.

[12] Hao J. and Cai L., Uniform decay of solutions for coupled viscoelastic wave equations, Electron. J. Differential Equations 2016 (72) (2016), 1–11.

[13] Lasiecka I., Messaoudi S.A., and Mustafa M.I., Note on intrinsic decay rates for abstract wave equations with memory, J. Math. Phys. 54, 031504 (2013); doi: 10.1063/1.4793988.

[14] Lasiecka I. and Wang X., Intrinsic decay rate estimates for semilinear abstract second order equations with memory, New prospects in direct, inverse and control problems for evolution equations, 271303, Springer INdAM Ser., 10, Springer, Cham, 2014.

[15] Liu W.J., General decay of solutions of a nonlinear system of viscoelastic equations, Acta Appl. Math. 110 (2010), 153-165.

[16] Liu W.J., General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms, J. Math. Phys 50 (11) (2009), art. No 113506.

[17] Liu W.J., Uniform decay of solutions for a quasilinear system of viscoelastic equations, Nonlinear Anal. 71 (2009), 2257-2267.

[18] Liu W.J., Chen Z.J. and Chen D.Q., New general decay results for a Moore-Gibson-Thompson equation with memory, Appl. Anal. (2020), 99:15, 2622-2640, DOI: 10.1080/00036811.2019.1577390.

[19] Liu W.J., Chen Z.J. and Tu Z., New general decay result for a fourth-order Moore-Gibson-Thompson equation with memory, Electronic Research Archive 28 (1) (2020), 433-457.

[20] Messaoudi S.A. General decay of solutions of a viscoelastic equation, J. Math. Anal. Appl. 341 (2008), 1457-1467.

[21] Messaoudi S.A., General decay of the solution energy in a viscoelastic equation with a nonlinear source, Nonlinear Anal. 69 (2008), 2589-2598.

[22] Messaoudi S.A. and Al-Gharabli M.M., A general decay result of a nonlinear system of wave equations with infinite memories, Appl. Math. Computations 259 (2015), 540–551.

[23] Messaoudi S.A. and Mustafa M.I., A stability result in a memory-type Timoshenko system, Dynam. Systems Appl. 18 (2009), 457-468.

[24] Messaoudi S.A. and Tatar N.-e,, Exponential and polynomial decay for a quasilinear viscoelastic equation, Nonlinear Anal. TMA 68 (2007), 785–793.

[25] Messaoudi S.A. and Tatar N.-e, Uniform stabilization of solutions of a nonlinear system of viscoelastic equations, Appl. Anal. 87 (3) (2008), 247-263.

[26] Munoz Rivera J.E., Asymptotic behavior in linear viscoelasticity, Quart. Appl. Math. 52 (4) (1994), 628-648.

[27] Munoz Rivera J.E. and Oquendo H. P., Exponential stability to a contact problem of partially viscoelastic materials, J. Elasticity 63 (2) (2001), 87-111.

[28] Munoz Rivera J.E. and Salvatierra A. P., Asymptotic behavior of the energy in partially viscoelastic materials, Quart. Appl. Math. 59 (3) (2001), 557-578.

[29] Munoz Rivera J.E., Naso M.G. and Vegni F.M., Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory, J. Math. Anal. Appl. 286 (2) (2003), 692-704.

[30] Munoz Rivera J.E. and Naso M.G., On the decay of the energy for systems with memory and indefinite dissipation, Asympt. Anal. 49 (3-4) (2006), 189-204.

[31] Mustafa M.I., Optimal decay rates for the viscoelastic wave equation, Math. Methods Appl. Sci. 41 (1) (2018), 192–204.

[32] Mustafa M.I., Uniform decay rates for viscoelastic dissipative systems, J. Dyn. Control Syst. 22 (1) (2016), 101–116.

[33] Mustafa M.I., Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations, Nonlinear Anal. RWA 13 (2012), 452–463.

[34] Mustafa M.I. and Messaoudi S.A., General stability result for viscoelastic wave equations, J. Math. Phys 53, 053702 (2012); doi: 10.1063/1.4711830.

[35] Park J. and Park S., General decay for quasilinear viscoelastic equations with nonlinear weak damping, J. Math. Physics 50 (8), art. No 083505 (2009).

[36] Said-Houari B., Messaoudi S.A. and Guesmia A., General decay of solutions of a nonlinear system of viscoelastic wave equations, Nonlinear Differ. Equ. Appl. 18 (6) (2011), 659–684.

[37] Santos M.L., Decay rates for solutions of a system of wave equations with memory, Electron. J. Differential Equations 2002 (38) (2002), 1-17.

[38] Wu S.T., General decay for a wave equation of Kirchhoff type with a boundary control of memory type, Boundary Value Problems 2011 (2011), 15pp.

