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Invariant regions and existence of global solutions to a generalized m-component reaction-diffusion system with tridiagonal symmetric Toeplitz diffusion matrix

Régions invariantes et existence globale de solutions pour un système de réaction-diffusion généralisé à m-composants avec une matrice de diffusion tridiagonale de Toeplitz symétrique

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ABSTRACT. The aim of this paper is to construct invariant regions of a generalized m-component reaction-diffusion system with tridiagonal symmetric Toeplitz diffusion matrix and nonhomogeneous boundary conditions and polynomial growth for the nonlinear reaction terms. Using the eigenvalues and eigenvectors of the diffusion matrix and the parabolicity conditions. So we prove the global existence of solutions using Lyapunov functional.
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1. Introduction

Reaction-diffusion systems are systems involving constituents locally transformed into each other by chemical reactions and transported in space by diffusion. They arise, in many applications, in chemistry, chemical engineering and biology. They have been the subject of countless studies in the past few decades. One of the most important aspects of this broad eld is proving the global existence of solutions under certain assumptions and restrictions.

In 2001, Kouachi [11] prove that solutions of 2-component reaction-diffusion systems with a diagonal diffusion matrix exist globaly via a Lyapunov fonctional. Later in [13] he generalized his result to the case of 2×2 Toeplitz diffusion matrix and in [12] he showed the global existence of solutions assuming the reaction terms of a 3×3 diagonal system exhibit a polynomial growth. In 2007, Abdelmalek [3] proved the global existence of solutions for reaction-diffusion systems of three equations with a tridiagonal matrix of diffusion coefficients. And too one of the main results of these studies was obtained in 1989 by Morgan [16], where all the components satisfy the same boundary conditions (Neumann or Dirichlet), and the reaction terms are polynomially bounded and satisfy certain inequalities. In 1993, Hollis [8] completed the work of Morgan and established the global existence in the presence of mixed boundary conditions if certain structure requirements are placed on the system. In 2007, Abdelmalek and Kouachi [4] represented the proof of global existence of solutions of m-component reaction-diffusion systems $(m \geq 2)$ with a diagonal diffusion matrix and nonhomogeneous Neumann, Dirichlet or Robin conditions, where the reaction terms are polynomially growth. Later in 2014, Abdelmalek [2] generalized the result of [4] to the case of $m \times m$ tridiagonal symmetric Toeplitz diffusion matrix and to the case of $m \times m$ arbitrary tridiagonal Toeplitz matrix in [1].

In this paper we shall generalize the results obtained in [2]. We prove the existence of global solutions of m-component reaction-diffusion systems with tridiagonal symmetric Toeplitz diffusion matrix and nonhomogeneous Neumann, Dirichlet or Robin conditions. The reaction terms are assumed to be of polynomial growth.

We consider the following m-equations of reaction-diffusion system, with $m \geq 2$:

$$\frac{\partial U}{\partial t} - A_m \Delta U = F(U) \quad \text{in } (0, +\infty) \times \Omega, \tag{1.1}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, the vectors U and F and the matrix A_m are defined as:

$$U = (u_1, ..., u_m)^T = ((u_s)_{s=1}^m)^T,$$
$$F = (F_1, ..., F_m)^T = ((F_s)_{s=1}^m)^T,$$

$$A_m = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{m-1} \\ 0 & \cdots & 0 & b_{m-1} & a_m \end{pmatrix}. \tag{1.2}$$

The constants $(a_i)_{i=1}^m, (b_i)_{i=1}^{m-1}$ are supposed to be strictly positive and satisfy the condition

$$4b_i^2 \cos^2\left(\frac{\pi}{m+1}\right) < a_i a_{i+1}, \tag{1.3}$$

which reflects the parabolicity of the system and implies at the same time that the diffusion matrix A_m is positive defnite. That means the eigenvalues $(\lambda_i)_{i=1}^m$ $(\lambda_1 > \lambda_2 > \ldots > \lambda_m)$ of A_m are positive. The boundary conditions and initial data (respectively) for the proposed system are assumed to satisfy:

$$\alpha U + (1 - \alpha)\partial_n U = B \quad \text{on} \quad (0, +\infty) \times \partial\Omega, \tag{1.4}$$

and

$$U(0, x) = U_0(x) \quad \text{on} \quad \Omega, \tag{1.5}$$

where $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on the boundary $\partial\Omega$, the vectors B and U_0 are defined as:

$$B = ((\beta_s)_{s=1}^m)^T,$$
$$U_0 = ((u_s^0)_{s=1}^m)^T.$$

Here will consider three type of boundary conditions:

(i) Nonhomogeneous Robin boundary conditions, corresponding to

$$0 < \alpha < 1 \text{ and } B \in \mathbb{R}^m;$$

(ii) Homogeneous Neumann boundary conditions, corresponding to

$$\alpha = 0 \text{ and } B \equiv 0;$$

(iii) Homogeneous Dirichlet boundary conditions, corresponding to

$$1-\alpha=0 \text{ and } B\equiv 0.$$

The initial data are assumed to be in the regions:

$$\sum\nolimits_{\mathfrak{S},\mathfrak{Z}}=\left\{U_0\in\mathbb{R}^m \ : \ \begin{cases} w_z^0=\langle V_z,U_0\rangle\leq 0 & \text{if } \langle V_z,B\rangle\leq 0, \ z\in\mathfrak{Z} \\ w_s^0=\langle V_s,U_0\rangle\geq 0 & \text{if } \langle V_s,B\rangle\geq 0, \ s\in\mathfrak{S} \end{cases}\right\}, \tag{1.6}$$

where

$$\mathfrak{S}\cap\mathfrak{Z}=\phi, \mathfrak{S}\cup\mathfrak{Z}=\{1,2,\dots,m\}.$$

The notation $\langle .,.\rangle$ denotes the inner product in $\mathbb{R}^m$ and $V_s=(v_{s1},\dots,v_{sm})^T$ the eigenvector of the diffusion matrix A_m associated with the eigenvalue $(\lambda_s)_{s=1}^m$. Hence, we can see that there are 2^m regions.

Remark 1.1. *If we substitude* $(b_i)_{i=1}^{m-1}=b$ *and* $(a_i)_{i=1}^m=a$ *in* (1.2) *and in* (1.3)*, we get the matrix of diffusion and the parabolicity condition used in [2].*

The following assumptions are also made on the function Ψ defined by:

$$\Psi=((\Psi_s)_{s=1}^m)^T, \quad \Psi_s=\left\langle(-1)^{i_s}V_s,F\right\rangle, \quad i_s=1,2.$$

(A1) Ψ_s are continuously differentiable on $\mathbb{R}_+^m$ and Ψ is quasipositive, i.e. $\Psi_s(W)\geq 0$ for all $W=(w_1,\dots,w_{s-1},w_s,w_{s+1},\dots,w_m)\geq 0$ with $w_s=0$. These conditions on Ψ guarantee local existence of unique, nonnegative and classical solutions on a maximal time interval $[0,T_{max})$, see Hollis and Morgan [9].

(A2) The inequality

$$\langle S,\Psi(W)\rangle\leq C_2(1+\langle W,1\rangle), \tag{1.7}$$

such that

$$S=(d_1,d_2,\dots,d_{m-1},1),$$

for all $w_s\geq 0, s=1,\dots,m$ and all constants d_s satisfy $d_s\geq\overline{d}_s$, $s=1,\dots,m-1$, where $C_2\geq 0$ and $\overline{d}_s$ are positive constants sufficiently large.

(A3) Ψ_s has polynomial growth, see Hollis and Morgan [9], which means there are constants $C_1\geq 0$ and $N\geq 1$, such that

$$|\Psi_s(W)|\leq C_1(1+\langle W,1\rangle)^N, \tag{1.8}$$

for all $W\in\mathbb{R}_+^m$ and $s=1,\dots,m$.

2. Existence of local solutions

The usual norms in the spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are denoted respectively by:

$$\|u\|_p^p=\frac{1}{|\Omega|}\int_\Omega |u(x)|^p\,dx,$$

$$\|u\|_\infty=ess\sup_{x\in\Omega}|u(x)| \quad\text{and}\quad \|u\|_{C(\overline{\Omega})}=\max_{x\in\overline{\Omega}}|u(x)|.$$

For any initial data in $C\left(\overline{\Omega}\right)$ or $L^{\infty}\left(\Omega\right)$ local existence and uniqueness of solutions to the initial values problem (1.1) follow from the basic existence theory for abstract semi-linear differential equations (see Friedman [6], Henry [7], Pazzy [17]). As a consequence, the system (1.1) with the conditions (1.4)-(1.5) admits a unique classical solution U on $[0, T_{max}) \times \Omega$, where $T_{max}\left(\left\|w_1^0\right\|_{\infty}, \left\|w_2^0\right\|_{\infty}, \ldots, \left\|w_m^0\right\|_{\infty}\right)$ denotes the eventual blow-up time. Furthermore, if $T_{max} < +\infty$, then

$$\lim_{t\uparrow T_{max}} \sum_{s=1}^{m} \left\|u_s\left(t,.\right)\right\|_{\infty} = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\sum_{s=1}^{m} \left\|u_s\left(t,.\right)\right\|_{\infty} \leq C \quad \text{for all} \quad t \in [0, T_{max}),$$

then, $T_{max} = +\infty$.

3. Eigenvalues and eigenvectors

Our aim in this section is to get a three term reccurence relation of characteristic polynomial of the matrix A of dimension $m \times m$ in terms of matrices of dimensions $(m-1) \times (m-1)$ and $(m-2) \times (m-2)$ so the eigenvectors of this matrix. The eigenvalues of the matrix A_m are λ along with the solutions of characteristic polynomial $\det\left(A_m - \lambda I_m\right) = 0$. We denote the characteristic polynomial of A_m, A_{m-1}, A_{m-2} by $\phi_m\left(\lambda\right), \phi_{m-1}\left(\lambda\right), \phi_{m-2}\left(\lambda\right)$ respectively.

Lemma 3.1. *(See [15]) Let A_m be the tridiagonal matrix defined in* (1.2). *The eigenvalues of A_m are distinct and interlace strictly with the eigenvalues of A_{m-1} and A_{m-2}, for $m \geq 2$.*

$$\phi_0\left(\lambda\right) = 1,\ \phi_1\left(\lambda\right) = a_1 - \lambda\ \phi_m\left(\lambda\right) = a_m \phi_{m-1}\left(\lambda\right) - b_{m-1}^2 \phi_{m-2}\left(\lambda\right). \tag{3.1}$$

Lemma 3.2. *(See [5]) Let A_m be the real symmetric tridiagonal matrix definied in* (1.2)*, with diagonal entries positive. If*

$$4b_i^2 \cos^2\left(\frac{\pi}{m+1}\right) < a_i a_{i+1}, \quad i = 1, \ldots, m-1,$$

then A_m is positive definite.

Lemma 3.3. *(See [5]) The real symmetric tridiagonal matrix A_m, defined in* (1.2)*, is positive definite if and only if its principal minors* $\det A_s$*, for $s = 1, ..., m$, are positive.*

Lemma 3.4. *The eigenvector of the matrix A_m given in* (1.2) *associated with eigenvalue λ_s, for $s = 1, \ldots, m$ is given by $V_s = \left(v_{s1}, \ldots, v_{sm}\right)^T$, where $v_{s\ell}$ ($\ell = 1, \ldots, m$) are given by the following expressions:,*

$$\begin{cases} v_{sm} = 1, \\ v_{s(m-1)} = \frac{\lambda_s - a_m}{b_{m-1}}, \\ v_{s(\ell-1)} = -\frac{b_\ell v_{s(\ell+1)} + (a_\ell - \lambda_s) v_{s\ell}}{b_{\ell-1}}, \quad \ell = m-1, \ldots, 2. \end{cases} \tag{3.2}$$

Proof. Recall that the diffusion matrix is positive definite, hence its eigenvalues are necessarily positive, the eigenvectors of the diffusion matrix associated with the eigenvalues λ_ℓ are defind as $V = (v_1, \ldots, v_m)^T$. For an eigenpair (λ, V), the components in $A_m V = \lambda V$ are

$$\begin{cases} a_1 v_1 + b_1 v_2 = \lambda v_1, \\ b_{\ell-1} v_{\ell-1} + a_\ell v_\ell + b_\ell v_{\ell+1} = \lambda v_\ell, \quad 2 \leq \ell \leq m-1, \\ b_{m-1} v_{m-1} + a_m v_m = \lambda v_m. \end{cases}$$

If $v_m = 0$, then under the assumption $b_\ell \neq 0$ for all $\ell = 1, \ldots, m-1$ we have that all v_ℓ are zero. We can therefore take $v_m = 1$ and $V = (v_1, \ldots, v_{m-1})$ is a solution of upper triangular system

$$\begin{cases} b_{\ell-1} v_{\ell-1} + (a_\ell - \lambda) v_\ell + b_\ell v_{\ell+1} = 0, \quad 2 \leq \ell \leq m-2, \\ b_{m-2} v_{m-2} + (a_{m-1} - \lambda) v_{m-1} = -b_{m-1}, \\ b_{m-1} v_{m-1} = \lambda - a_m. \end{cases}$$

The solution of this system is given by

$$\begin{cases} v_{m-1} = \frac{\lambda - a_m}{b_{m-1}}, \\ v_{\ell-1} = -\frac{b_\ell v_{\ell+1} + (a_\ell - \lambda) v_\ell}{b_{\ell-1}}, \quad \ell = m-1, \ldots, 2. \end{cases}$$

□

4. Invariant Regions

Usually to construct an invariant regions for systems such (1.1) we make the linear change of variables u_i defined by the formula (4.4) to obtain a new equivalent system with diagonal diffusion matrix for which standard techniques can be applied to deduce global existence (see, e.g., Kouachi [14]).

Since the initial conditions are in $\sum_{\mathfrak{S},3}$, then under the assumptions (**A1**)-(**A2**), the next proposition says that the classical solution of the system (1.1) with the boundary conditions (1.4) and initial data (1.5) on $[0, T_{max}) \times \Omega$ remains in $\sum_{\mathfrak{S},3}$ for all t in $[0, T_{max})$.

Proposition 4.1. *Suppose that the assumptions (**A1**)-(**A2**) are satisfied. Then for any U_0 in $\sum_{\mathfrak{S},3}$ the classical solution U of the system* (1.1) *with the conditions* (1.4)-(1.5) *on* $[0, T_{max}) \times \Omega$ *remains in* $\sum_{\mathfrak{S},3}$ *for all t in* $[0, T_{max})$.

Proof. Let $V_s = (v_{s1}, \ldots, v_{sm})^T$ be an eigenvector of the matrix A_m associated with its eigenvalue $(\lambda_s)_{s=1}^m$ where $\lambda_1 > \lambda_2 > \ldots > \lambda_m$. Multiplying the k^{th} equation of (1.1) by $(-1)^{i_s} V_{sk}$, $i_s = 1, 2$ and $k = 1, \ldots, m$, and adding the resulting equations, we get

$$W_t - diag(\lambda_1, \lambda_2, \ldots, \lambda_m) \Delta W = \Psi(W) \quad \text{in} \quad (0, +\infty) \times \Omega, \tag{4.1}$$

with the boundary conditions and initial data are satisfy

$$\alpha W + (1 - \alpha) \partial_n W = \Gamma \quad \text{on} \quad (0, +\infty) \times \partial\Omega, \tag{4.2}$$

and

$$W(0, x) = W_0 \quad \text{in} \quad \Omega, \tag{4.3}$$

where

$$W = ((w_s)_{s=1}^m)^T, \quad w_s = \left\langle (-1)^{i_s} V_s, U \right\rangle, \tag{4.4}$$

$$\Psi = ((\Psi_s)_{s=1}^m)^T, \quad \Psi_s = \left\langle (-1)^{i_s} V_s, F \right\rangle,$$

$$\Gamma = ((\rho_s^0)_{s=1}^m)^T, \qquad \rho_s^0 = \left\langle (-1)^{i_s} V_s, B \right\rangle,$$

$$W_0 = ((w_s^0)_{s=1}^m)^T, \qquad w_s^0 = \left\langle (-1)^{i_s} V_s, U_0 \right\rangle.$$

Note that condition (1.3) guarantees the parabolicity of the proposed system (1.1), which with the conditions (1.4)-(1.5), is equivalent to (4.1)-(4.3) in the region:

$$\sum\nolimits_{\mathfrak{S},\mathfrak{Z}} = \left\{ U_0 \in \mathbb{R}^m : w_s^0 = \left\langle (-1)^{i_s} V_s, U_0 \right\rangle \geq 0 \ \text{ if } \ \rho_s^0 = \left\langle (-1)^{i_s} V_s, B \right\rangle \geq 0 \right\},$$

where $s = 1, \ldots, m$ and $i_s = 1, 2$. This region can be written as:

$$\sum\nolimits_{\mathfrak{S},\mathfrak{Z}} = \left\{ U_0 \in \mathbb{R}^m : \left\{ \begin{array}{ll} w_s^0 = \langle V_s, U_0 \rangle \geq 0 & \text{if } \rho_s^0 = \langle V_s, B \rangle \geq 0, s \in \mathfrak{S} \\ w_z^0 = \langle V_z, U_0 \rangle \leq 0 & \text{if } \rho_z^0 = \langle V_z, B \rangle \leq 0, z \in \mathfrak{Z} \end{array} \right\} \right.$$

where

$$\mathfrak{S} \cap \mathfrak{Z} = \emptyset, \mathfrak{S} \cup \mathfrak{Z} = \{1, 2, \ldots, m\}.$$

This implies that the components w_s are necessarily positive. □

Once the invariant regions are constructed, one can apply the Lyapunov technique and establish the global existence.

5. Existence of global solutions

It is well-known that in order to prove the global existence of solutions to a reaction-diffusion system (see, e.g., Henry [7] and Rothe [18]) it suffices to derive a uniform estimate of the associated reaction term on $[0, T_{\max})$ in the space $L^p(\Omega)$ for some $p > \frac{n}{2}$. Our aim is to construct Lyapunov polynomial functionals allowing us to obtain L^p-bounds on the components, which leads to global existence.

Since the reaction terms Ψ_s, $s = 1, \ldots, m$ are continuously differentiable on $\mathbb{R}_+^m$, it follows that for any initial data in $C(\overline{\Omega})$, it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\mathfrak{D} = -\begin{pmatrix} \lambda_1 \Delta & 0 & \ldots & 0 \\ 0 & \lambda_2 \Delta & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_m \Delta \end{pmatrix}. \tag{5.1}$$

Under these assumptions, the local existence result is well known (see Friedman [6], Henry [7], Pazzy [17]).

Assumption (**A1**) contains smoothness and quasi-positivity conditions that guarantee local existence and nonnegativity of solutions as long as they exist, via the maximum principle (see Smoller [19]). Assumption (**A3**) is the usual polynomial growth condition necessary to obtain uniform bounds from p-dependent L^P estimates, (see Abdelmalek and Kouachi [4], and Hollis and Morgan [9]).

Before we present the main results of this paper, let us define

$$K_l^r = K_{r-1}^{r-1} K_l^{r-1} - \left[H_l^{r-1}\right]^2, \; r = 3, \ldots, l, \tag{5.2}$$

where

$$H_l^r = \det_{1\le s,k\le l}\left((a_{s,k}) \, \underset{k\neq l-1,\ldots r}{s \neq l, \ldots r+1} \right) \prod_{k=1}^{k=r-2} (\det[k])^{2^{(r-k-2)}} \quad , r = 3, \ldots, l-1$$

$$K_l^2 = \underbrace{\lambda_1 \lambda_l \prod_{k=1}^{l-1} \theta_k^{2(p_k+1)^2} \prod_{k=l}^{m-1} \theta_k^{2(p_k+2)^2}}_{\text{positive}\, value} \left[\prod_{k=1}^{l-1} \theta_k^2 - A_{1l}^2\right]$$

and

$$H_l^2 = \underbrace{\lambda_1 \sqrt{\lambda_2 \lambda_l} \theta_1^{2(p_1+1)^2} \prod_{k=2}^{l-1} \theta_k^{(p_k+2)^2+(p_k+1)^2}}_{\text{positive}\, value} \prod_{k=l}^{m-1} \theta_k^{2(p_k+2)^2} \left[\theta_1^2 A_{2l} - A_{12} A_{1l}\right],$$

the exprission $\det_{1\le s,k\le l}\left((a_{s,k})_{\substack{s\neq l,\ldots r+1 \\ k\neq l-1,\ldots r}} \right)$ denotes the determinant of the r-square symmetric matrix obtained from $(a_{s,k})_{1\le s,k\le m}$ by removing the $(r+1)^{th}$, $(r+2)^{th}$,..., l^{th} rows and the r^{th}, $(r+1)^{th}, \ldots, (l-1)^{th}$ columns, and $\det[1], \ldots, \det[m]$ are the minors of the matrix $(a_{s,k})_{1\le s,k\le l}$. The elements of the matrix are:

$$a_{sk} = \frac{\lambda_s + \lambda_k}{2} \theta_1^{p_1^2} \ldots \theta_{(s-1)}^{p_{(s-1)}^2} \theta_s^{(p_s+1)^2} \ldots \theta_{k-1}^{\left(p_{(k-1)}+1\right)^2} \theta_k^{(p_k+2)^2} \ldots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2}, \tag{5.3}$$

where λ_s is defined in (3.1). Note that $A_{sk} = \frac{\lambda_s+\lambda_k}{2\sqrt{\lambda_s \lambda_k}}$ for all $s, k = 1, \ldots, m$, and $\theta_s, s = 1, \ldots, (m-1)$ are positive constants.
The main results are based on the following key proposition.

Proposition 5.1. *Suppose that the functions $\Psi_s, s = 1, \ldots, m$, are of polynomial growth and satisfy conditions* (1.7) *and* (1.8). *Let $(w_1(t,.), w_2(t,.), \ldots, w_m(t,.))$ be the solution of* (4.1)-(4.3) *and*

$$L(t) = \int_\Omega H_{p_m}(w_1(t,x), w_2(t,x), \ldots, w_m(t,x))\, dx, \tag{5.4}$$

where

$$H_{pm}(w_1, w_2, \ldots, w_m) = \sum_{p_{m-1}=0}^{p_m} \ldots \sum_{p_1=0}^{p_2} C_{p_m}^{p_{m-1}} \ldots C_{p_2}^{p_1} \theta_1^{p_1^2} \ldots \theta_{(m-1)}^{p_{(m-1)}^2} w_1^{p_1} w_2^{p_2-p_1} \ldots w_m^{p_m-p_{m-1}},$$

with p_m is a positive integer and $C_{p_k}^{p_s} = \frac{p_k!}{p_s!(p_k-p_s)!}$. Also suppose that the following condition is satisfied

$$K_l^l > 0, l = 2, \ldots, m. \tag{5.5}$$

where K_l^l was defined in (5.2). *Then the functional L is uniformly bounded on the interval $[0, T^*], T^* < T_{max}$.*

Before proving this proposition, we first need the following preparatory Lemmas, see [11] and [4].

Lemma 5.2. *Let H_{p_m} be the homogeneous polynomial defined by* (5.4). *Then*

$$\partial_{w_1} H_{p_m} = p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{(p_1+1)^2} \cdots \theta_{(m-1)}^{\left(p_{(m-1)}+1\right)^2} \times w_1^{p_1} w_2^{p_2-p_1} w_3^{p_3-p_2} \cdots w_m^{(p_m-1)-p_{m-1}}, \tag{5.6}$$

$$\partial_{w_s} H_{p_m} = p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{p_1^2} \cdots \theta_{s-1}^{p_{(s-1)}^2} \theta_s^{(p_s+1)^2} \cdots \theta_{(m-1)}^{\left(p_{(m-1)}+1\right)^2} \times w_1^{p_1} w_2^{p_2-p_1} w_3^{p_3-p_2} \cdots w_m^{(p_m-1)-p_{m-1}}, \tag{5.7}$$

for all $s = 2, \ldots, m-1$, and

$$\partial_{w_m} H_{p_m} = p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \cdots C_{p_3}^{p_2} C_{p_2}^{p_1} \theta_1^{p_1^2} \theta_2^{p_2^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2} \times w_1^{p_1} w_2^{p_2-p_1} w_3^{p_3-p_2} \cdots w_m^{(p_m-1)-p_{m-1}}. \tag{5.8}$$

Lemma 5.3. *The second partial derivatives of H_{p_m} are given by*

$$\partial_{w_1^2} H_n = p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_2=0}^{p_3} \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \times \theta_1^{(p_1+2)^2} \cdots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2} w_1^{p_1} w_2^{p_2-p_1} \cdots w_m^{(p_m-2)-p_{m-1}}, \tag{5.9}$$

$$\partial_{w_s^2} H_n = p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{p_1^2} \theta_2^{p_2^2} \cdots \theta_{(s-1)}^{p_{(s-1)}^2} \theta_s^{(p_s+2)^2} \cdots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2} w_1^{p_1} w_2^{p_2-p_1} \cdots w_m^{(p_m-2)-p_{m-1}}, \tag{5.10}$$

for all $s = 2, \ldots, m-1$, and

$$\partial_{w_s w_k} H_n = p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \times \theta_1^{p_1^2} \cdots \theta_{s-1}^{p_{s-1}^2} \theta_s^{(p_s+1)^2} \cdots \theta_{k-1}^{(p_{k-1}+1)^2} \theta_k^{(p_k+2)^2} \cdots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2} \times w_1^{p_1} w_2^{p_2-p_1} \cdots w_m^{(p_m-2)-p_{m-1}}, \tag{5.11}$$

for all $1 \le s < k \le m$. Finally, the second derivative in w_m is

$$\partial_{w_m^2} H_n = p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{p_1^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2} \times w_1^{p_1} w_2^{p_2-p_1} \cdots w_m^{(p_m-2)-p_{m-1}}. \tag{5.12}$$

Lemma 5.4. *Let A_m be the m-square symmetric matrix defined by $A_m = (a_{s,k})$ with $1 \le s, k \le m$. Then*

$$\begin{cases} K_m^m = \det[m] \prod_{k=1}^{k=m-2} (\det[k])^{2(m-k-2)}, m > 2, \\ K_2^2 = \det[2], \end{cases} \tag{5.13}$$

where the expressions of K_m^l and H_m^l are already defined in (5.2).

Proof of Proposition 5.1. The aim is to prove that $L(t)$ is uniformly bounded on the interval $[0,T^*], T^* < T_{max}$. Let us start by differentiating L with respect to t.

$$\begin{aligned} L^{'}(t) &= \int_{\Omega} \partial_t H_{p_m} dx = \int_{\Omega} \sum_{s=1}^{m} \partial_{w_s} H_{p_m} \frac{\partial w_s}{\partial t} dx \\ &= \int_{\Omega} \sum_{s=1}^{m} \partial_{w_s} H_{p_m} \left(\lambda_s \Delta w_s + \Psi_s\right) dx \\ &= I + J, \end{aligned}$$

where

$$I = \int_{\Omega} \sum_{s=1}^{m} \lambda_s \partial_{w_s} H_{p_m} \Delta w_s dx, \tag{5.14}$$

and

$$J = \int_{\Omega} \sum_{s=1}^{m} \partial_{w_s} H_{p_m} \Psi_s dx. \tag{5.15}$$

Using Green's formula, we can divide I into two parts I_1 and I_2, where

$$I_1 = \int_{\partial\Omega} \sum_{s=1}^{m} \lambda_s \partial_{w_s} H_{p_m} \partial_\eta w_s dx \tag{5.16}$$

and

$$I_2 = -\int_{\Omega} \left\langle T, \left(\left(\frac{\lambda_s + \lambda_k}{2} \partial_{w_k w_s} H_{p_m} \right)_{1 \le s,k \le m} \right) T \right\rangle dx, \tag{5.17}$$

where

$$T = (\nabla w_1, \nabla w_2, \ldots, \nabla w_m)^T.$$

Applying Lemma 5.2 and Lemma 5.3, yields

$$\begin{aligned} \left(\frac{\lambda_s + \lambda_k}{2} \partial_{w_k w_s} H_{p_m} \right)_{1 \le s,k \le m} &= p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m - 2} \ldots \sum_{p_1=0}^{p_2} C_{p_m - 2}^{p_{m-1}} \ldots \\ & C_{p_2}^{p_1} \left((a_{s,k})_{1 \le s,k \le m} \right) w_1^{p_1} \ldots w_m^{(p_m-2)-p_{m-1}} \end{aligned} \tag{5.18}$$

where $(a_{s,k})_{1 \le s,k \le m}$ is the matrix defined in (5.3).
Now, in order to prove that I is bounded, we will show that there exists a positive constant C_4 independent of $t \in [0, T_{max})$ such that

$$I_1 \le C_4 \quad \text{for all} \quad t \in [0, T_{max}), \tag{5.19}$$

and that

$$I_2 \le 0, \tag{5.20}$$

for several boundary conditions. First let us prove (5.19):

(i) If $0 < \alpha < 1$, then using the boundary conditions (4.2) we obtain

$$I_1 = \int_{\partial\Omega} \sum_{s=1}^{m} \lambda_s \partial_{w_s} H_{p_m} \left(\gamma_s - \sigma_s w_s\right) ds,$$

where

$$\sigma_s = \frac{\alpha}{\alpha - 1} \text{ and } \gamma_s = \frac{\rho_s^0}{1 - \alpha}, \; s = 1, \dots, m.$$

Since

$$H(W) = \sum_{s=1}^{m} \lambda_s \partial_{w_s} H_{p_m}(\gamma_s - \sigma_s w_s) = P_{n-1}(W) - Q_n(W),$$

where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n, and since the solution is positive it follows that

$$\limsup_{\sum_{s=1}^{m} |w_s| \to +\infty} H(W) = -\infty, \tag{5.21}$$

which proves that H is uniformly bounded on $\mathbb{R}_+^m$ and consequently proves (5.19).

(ii) If $\forall s = 1, \dots, m : \alpha = \beta_s = 0$, then the boundary conditions (4.2) become $\partial_\eta w_s = 0, \forall s = 1, \dots, m$ on $[0, T_{max}) \times \partial\Omega$. Consequently, from (5.16), it follows that $I_1 = 0$ on $[0, T_{max})$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0, T_{max}) \times \Omega$ implies $\partial_\eta w_s \leq 0, \forall s = 1, \dots, m$ on $[0, T_{max}) \times \partial\Omega$. Consequently, one obtains the same result in (5.19) with $C_4 = 0$. Hence the proof of (5.19) is complete.

Now, we prove (5.20). The quadratic form (with respect to $\nabla w_s, s = 1, \dots, m$) associated with the matrix $(a_{s,k})_{1 \leq s,k \leq m}$ which we defined in (5.3) is positive definite since its minors $det[1], det[2], ..., det[m]$ are all positive. Let us prove their positivity by induction. The first minor

$$\det[1] = \lambda_1 \theta_1^{(p_1+2)^2} \theta_2^{(p_2+2)^2} \dots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2} > 0$$

for $p_1 = 0, \dots, p_2, \; p_2 = 0, \dots, p_3, \dots, \; p_{m-1} = 0, \dots, p_m - 2$. For the second minor $\det[2]$, and according to Lemma 5.4, we have:

$$\det[2] = K_2^2 = \lambda_1 \lambda_2 \theta_1^{2(p_1+1)^2} \prod_{k=2}^{m-1} \theta_k^{2(p_k+2)^2} \left[\theta_1^2 - A_{12}^2\right].$$

Using (5.5) for $l = 2$ we get $\det[2] > 0$. Similarly, for the third minor $\det[3]$, and again using Lemma 5.4, we have:

$$K_3^3 = \det[3] \det[1].$$

Since $\det[1] > 0$, we conclude that

$$sign\left(K_3^3\right) = sign(\det[3]).$$

Again, using (5.5) for $l = 3$, we obtain $\det[3] > 0$. To conclude the proof, let us suppose $\det[k] > 0$ for $k = 1, 2, \dots, l-1$, and show that $\det[l]$ is necessarily positive. We have

$$\det[k] > 0, k = 1, ..., (l-1) \Rightarrow \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}} > 0. \tag{5.22}$$

From Lemma 5.4 we obtain

$$K_l^l = \det[l] \prod_{k=1}^{k=l-2} (det[k])^{2(l-k-2)},$$

and from (5.22) , we obtain $sign(K_l^l) = sign(\det[l])$.
Since $K_l^l > 0$ according to (5.5), then $\det[l] > 0$ and the proof of (5.20) is finished. It follows from(5.19) and (5.20) that I is bounded. No let us prove that J in (5.15) is bounded. Substituting the expressions of the partial derivatives given by Lemma 5.2 in the second integral of (5.15) yields

$$J = \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \right]$$

$$\left(\prod_{s=1}^{m-1} \theta_s^{(p_s+1)^2} \Psi_1 + \sum_{k=2}^{m-1} \prod_{k=1}^{k-1} \theta_k^{p_k^2} \prod_{s=k}^{m-1} \theta_s^{(p_s+1)^2} \Psi_k + \prod_{s=1}^{m-1} \theta_s^{p_s^2} \Psi_m \right) dx$$

$$= \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \right]$$

$$\left(\prod_{s=1}^{m-1} \frac{\theta_s^{(p_s+1)^2}}{\theta_s^{p_s^2}} \Psi_1 + \sum_{k=2}^{m-1} \prod_{k=1}^{k-1} \theta_k^{p_k^2} \prod_{s=k}^{m-1} \frac{\theta_s^{(p_s+1)^2}}{\theta_s^{p_s^2}} \Psi_k + \Psi_m \right) \prod_{s=1}^{m-1} \theta_s^{p_s^2} dx$$

$$= \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \right]$$

$$\left\langle \left(\prod_{s=1}^{m-1} \frac{\theta_s^{(p_s+1)^2}}{\theta_s^{p_s^2}}, \theta_1^{p_1^2} \prod_{s=2}^{m-1} \frac{\theta_s^{(p_s+1)^2}}{\theta_s^{p_s^2}}, \dots, \prod_{k=1}^{m-2} \theta_k^{p_k^2} \frac{\theta_{m-1}^{(p_{m-1}+1)^2}}{\theta_{m-1}^{p_{m-1}^2}}, 1 \right), \Psi \right\rangle \prod_{s=1}^{m-1} \theta_s^{p_s^2} dx.$$

Hence using the condition (1.7), we deduce that

$$J \le C_5 \int_\Omega \left[\sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_2}^{p_1} \dots C_{p_m-1}^{p_{m-1}} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \left(1 + \langle W, 1 \rangle\right) \right] dx.$$

To prove that the functional L is uniformly bounded on the interval $[0, T^*]$, let us first write

$$\sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_2}^{p_1} \dots C_{p_m-1}^{p_{m-1}} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \left(1 + \langle W, 1 \rangle\right)$$
$$= R_{p_m}(W) + S_{p_m-1}(W),$$

where $R_{p_m}(W)$ and $S_{p_m-1}(W)$ are two homogeneous polynomials of degrees p_m and p_{m-1}, respectively. Since all the polynomials H_{p_m} and R_{p_m} are of degree p_m, then there exists a positive constant C_6 such that

$$\int_\Omega R_{p_m}(W)\, dx \le C_6 \int_\Omega H_{p_m}(W)\, dx. \tag{5.23}$$

Applying Hölder's inequality to the following integral one obtains

$$\int_\Omega S_{p_m-1}(W)\, dx \le (meas\Omega)^{\frac{1}{p_m}} \left(\int_\Omega \left(S_{p_m-1}(W)\right)^{\frac{p_m}{p_m-1}} dx \right)^{\frac{p_m-1}{p_m}}.$$

Since for all $w_1, w_2, ..., w_{m-1} \ge 0$ and $w_m > 0$

$$\frac{\left(S_{p_m-1}(W)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}(W)} = \frac{\left(S_{p_m-1}(x_1, x_2, \dots, x_{m-1}, 1)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}(x_1, x_2, \dots, x_{m-1}, 1)},$$

where for all $s \in \{1, 2, \dots, m-1\} : x_s = \frac{w_s}{w_{s+1}}$ and

$$\lim_{x_s \to +\infty} \frac{\left(S_{p_m-1}(x_1, x_2, \dots, x_{m-1}, 1)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}(x_1, x_2, \dots, x_{m-1}, 1)} < +\infty,$$

one asserts that there exists a positive constant C_7 such that

$$\frac{\left(S_{p_m-1}\left(W\right)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}\left(W\right)} \leq C_7, \text{ for all } w_1, w_2, \ldots, w_{m-1} \geq 0. \tag{5.24}$$

Hence the functional L satisfies the differential inequality

$$L^{'}\left(t\right) \leq C_6 L\left(t\right) + C_8 L^{\frac{p_m-1}{p_m}}\left(t\right),$$

which for $Z = L^{\frac{1}{p_m}}$ can be written as

$$p_m Z^{'} \leq C_6 Z + C_8. \tag{5.25}$$

A simple integration gives the uniform bound of the functional L on the interval $[0, T^{*}]$. This ends the proof. □

We can now establish the main results of this paper.

Theorem 5.5. *Under the assumptions **(A1)**-**(A3)**, all solutions of* (4.1)-(4.3) *with positive initial data in* $L^{\infty}\left(\Omega\right)$ *are in* $L^{\infty}\left(0, T^{*}; L^{p}\left(\Omega\right)\right)$ *for some* $p \geq 1$.

Corollary 5.6. *Under the assumptions of Theorem 5.5 and assuming moreover that the condition* (1.3) *is satisfied, all solutions of* (4.1)-(4.3) *with positive initial data in* $L^{\infty}\left(\Omega\right)$ *are global for some* $p > \frac{Nn}{2}$.

Proof of Theorem 5.5*.* The proof is an immediate consequence of Proposition 5.1 and the inequality

$$\int_{\Omega} \left\langle W, 1\right\rangle^{p} dx \leq C_9 L\left(t\right) \quad on\ [0, T^{*}], \text{ for some } p \geq 1. \tag{5.26}$$

□

Proof of Corollary 5.6*.* From Theorem 5.5, it follows that there exists a positive constant C_{10} such that

$$\int_{\Omega} \left\langle W, 1\right\rangle^{p} dx \leq C_{10}. \tag{5.27}$$

From (1.8), we have that for all $s \in \{1, 2, \ldots, m\}$

$$\left|\Psi_s\left(W\right)\right|^{\frac{p}{N}} \leq C_{11}\left(1 + \left\langle W, 1\right\rangle\right)^{p}. \tag{5.28}$$

Since $w_1, w_2, \ldots, w_m$ are in $L^{\infty}\left(0, T^{*}; L^{p}\left(\Omega\right)\right)$ and $\frac{p}{N} > \frac{n}{2}$, then the solution is global. □

6. Final remarks

Recall that if $V = (v_{s1}, v_{s2}, \ldots, v_{sm})^{T}$ is an eigenvector of diffusion matrix A^{T} associated with eigenvalue λ_s, then $(-1)V$ is also. Let us consider the diagonalizing matrix of eigenvectors

$$P = ((-1)^{i_1} V_1 \mid (-1)^{i_2} V_2 \mid \ldots \mid (-1)^{i_m} V_m),$$

with the powers i_s

$$i_s \in \{1, 2\}, s = 1, \ldots, m.$$

Pre-multiplying the system (1.1) by P^T yields

$$\begin{aligned} P^T(U_t - A_m \Delta U) &= P^T F \\ P^T U_t - \Delta P^T A_m U &= P^T F \\ P^T U_t - \Delta P^T A_m (P^T)^{-1} P^T U &= P^T F. \end{aligned} \tag{6.1}$$

Now, let us simplify the expression $P^T A_m (P^T)^{-1}$, $P^T U$ and $P^T F$

$$\begin{aligned} P^T A_m (P^T)^{-1} &= P^T (A_m^T)^T (P^{-1})^T \\ &= (A_m^T P)^T (P^{-1})^T \\ &= (P^{-1} A_m^T P)^T \\ &= (diag(\lambda_1, \lambda_2, \ldots, \lambda_m))^T \\ &= diag(\lambda_1, \lambda_2, \ldots, \lambda_m). \end{aligned} \tag{6.2}$$

$$\begin{aligned} P^T U &= ((-1)^{i_1} V_1 \mid (-1)^{i_2} V_2 \mid \ldots \mid (-1)^{i_m} V_m)^T U \\ &= (\left\langle (-1)^{i_s} V_1, U \right\rangle, \left\langle (-1)^{i_s} V_2, U \right\rangle, \ldots, \left\langle (-1)^{i_s} V_m, U \right\rangle)^T \\ &= (w_1, w_2, \ldots, w_m)^T = W. \end{aligned} \tag{6.3}$$

Hence, $P^T U_t = W_t$. Similarly, we get

$$\begin{aligned} P^T F &= ((-1)^{i_1} V_1 \mid (-1)^{i_2} V_2 \mid \ldots \mid (-1)^{i_m} V_m)^T F \\ &= \left(\left\langle (-1)^{i_1} V_1, F \right\rangle, \left\langle (-1)^{i_2} V_2, F \right\rangle, \ldots, \left\langle (-1)^{i_m} V_m, F \right\rangle\right)^T \\ &= (\Psi_1, \Psi_2, \ldots, \Psi_m)^T = \Psi. \end{aligned} \tag{6.4}$$

Substituting (6.2)-(6.4), in (6.1) results in the equivalent system (4.1). Pre-multiplying (1.4) by P^T, we get the boundary condition (4.2)

$$\alpha P^T U + (1-\alpha)\, \partial_\eta P^T U = P^T B, \tag{6.5}$$

where

$$\begin{aligned} P^T B &= ((-1)^{i_1} V_1 \mid (-1)^{i_2} V_2 \mid \ldots \mid (-1)^{i_m} V_m)^T B \\ &= \left(\left\langle (-1)^{i_1} V_1, B \right\rangle, \left\langle (-1)^{i_2} V_2, B \right\rangle, \ldots, \left\langle (-1)^{i_m} V_m, B \right\rangle\right)^T \\ &= \left(\rho_1^0, \rho_2^0, \ldots, \rho_m^0\right) = \Gamma \end{aligned} \tag{6.6}$$

Substituting (6.3) and (6.6) in (6.5) gives the boundary condition (4.2) for the equivalent system (4.1). Note that condition (1.3) guarantees the parabolicity of the proposed reaction-diffusion system in (1.1) with the conditions (1.4)-(1.5), which implies it is equivalent to (4.1)-(4.3) in the regions:

$$\textstyle\sum_s = \left\{ U_0 \in \mathbb{R}^m : w_s^0 = \left\langle (-1)^{i_s} V_s, U_0 \right\rangle \geq 0 \text{ if } \rho_s^0 = \left\langle (-1)^{i_s} V_s, B \right\rangle \geq 0 \right\},$$

where $i_s \in \{1, 2\}$, $s = 1, \ldots, m$.

- If $i_s = 1$, we have

  $$\textstyle\sum_s = \left\{ U_0 \in \mathbb{R}^m : w_s^0 = \langle (-1) V_s, U_0 \rangle \geq 0 \text{ if } \rho_s^0 = \langle (-1) V_s, B \rangle \geq 0 \right\},$$

  tantamount to

  $$\textstyle\sum_{\mathfrak{S},\mathfrak{Z}} = \left\{ U_0 \in \mathbb{R}^m : w_z^0 = \langle V_z, U_0 \rangle \leq 0 \text{ if } \rho_z^0 = \langle V_z, B \rangle \leq 0, z \in \mathfrak{Z} \right\}.$$

- If $i_s = 2$, we have

$$\sum_s = \left\{U_0 \in \mathbb{R}^m : w_s^0 = \langle V_s, U_0\rangle \geq 0 \text{ if } \rho_s^0 = \langle V_s, B\rangle \geq 0\right\},$$

  equivalent to

$$\sum_{\mathfrak{S},\mathfrak{Z}} = \left\{U_0 \in \mathbb{R}^m : w_s^0 = \langle V_z, U_0\rangle \geq 0 \text{ if } \rho_s^0 = \langle V_s, B\rangle \leq 0, s \in \mathfrak{S}\right\}.$$

Then for $s = 1, \ldots, m$ the regions $\sum_s$ are equivalent to following regions

$$\sum_{\mathfrak{S},\mathfrak{Z}} = \left\{U_0 \in \mathbb{R}^m \text{such that} \left\{ \begin{array}{ll} w_s^0 = \langle V_s, U_0\rangle \geq 0 & \text{if } \rho_s^0 = \langle V_s, B\rangle \geq 0, s \in \mathfrak{S} \\ w_z^0 = \langle V_z, U_0\rangle \leq 0 & \text{if } \rho_z^0 = \langle V_z, B\rangle \leq 0, z \in \mathfrak{Z} \end{array} \right\}\right.$$

where

$$\mathfrak{S} \cap \mathfrak{Z} = \emptyset, \mathfrak{S} \cup \mathfrak{Z} = \{1, 2, \ldots, m\}.$$

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References

[1] S. Abdelmalek, Existence of global solutions via invariant regions for a generalized reaction diffusion system with a tridiagonal Toeplitz matrix of diffusion coefficients, Functional Analysis: Theory, Methods and applications 2(2016), 12 − 27.

[2] S. Abdelmalek, Invariant regions and global solutions for reaction-diffusion systems with a tridiagonal symmetric Toeplitz matrix of diffusion coefficients, Electron. J. Diff. Equ., Vol. 2014(2014), No. 247, pp. 1 − 14.

[3] S. Abdelmalek, Invariant regions and global existence of solutions for reaction-diffusion systems with a tri-diagonal matrix of diffusion coefficients and nonhomogeneous boundary conditions, Journal of Applied Mathematics, Vol. (2007), 1 − 15.

[4] S. Abdelmalek and S. Kouachi, Proof of existence of global solutions for m-component reaction-diffusion systems with mixed boundary conditions via the Lyapunov functional method, Journal of Physics A, vol. 40 (2007), 12335 − 12350.

[5] M. Andelic and C. M. da Fonseca, Sufficient conditions for positive definiteness of tridiagonal matrices revisited, Positivity, Vol. 15, No. 1(2011), pp. 155 − 159.

[6] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, New Jersey, 1964.

[7] D. Henry, Geometric theory of semi-linear parabolic equations, Lecture Notes in Mathematics 840, Springer-Verlag, New-York, 1984.

[8] S. L. Hollis, On the question of global existence for reaction-diffusion systems with mixed boundary conditions Quarterly of Appl. Math., Vol. 51, No. 2 (1993), pp. 241-250.

[9] S. L. Hollis and J. J. Morgan, On the blow-up of solutions to some semilinear and quasilinear reaction-diffusion systems, Rocky Moutain J. Math., vol 14 (1994), 1447-1465.

[10] S. L.Hollis and J. J. Morgan, Partly dissipative reaction-di usion systems and a model of phosphorus diffusion in silicon, Nonlinear Anal. T. M. A., 19 (1992), 427-440.

[11] S. Kouachi, Existence of global solutions to reaction-diffusion systems via a Lyapunov functional, Electron. J. Diff. Equ., 2001(2001), No. 68, pp. 1 − 10.

[12] S. Kouachi, Existence of global solutions to reaction-diffusion systems with nonhomogeneous boundary conditions via a Lyapunov functional, Electron. J. Diff. Equ., Vol. $2002(2002)$, No. 88, pp. $1-13$.

[13] S. Kouachi, Global existence of solutions for reaction-diffusion systems with a full matrix of diffusion coefficients and nonhomogeneous boundary conditions. Electron. J. Qual. Theory Differ. Equ., 4 (2002), pp. 1–10.

[14] S. Kouachi, Invariant regions and global existence of solutions for reaction-diffusion systems with full matrix of diffusion coefficients and nonhomogeneous boundary conditions. Georgian Math. J. 11 (2004), 349-359.

[15] D. Kulkarni, D. Schmidh and S. Tsui, Eigenvalues of tridiagonal pseudo-Toeplitz matrices, Linear Algebra and its Applications, $297(1999)$, $63-80$.

[16] J. Morgan, Global existence for semilinear parabolic systems, SIAM J. Math. Anal., SIAM J. Math. Anal. 20 (1989), 1128-1144.

[17] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

[18] F. Rothe, Global solutions of reaction-diffusion systems, Lecture Notes in Math. 1072, Springer-Verlag, Berlin, 1984.

[19] J. Smoller, Shock waves and reaction-diffusion equations, Springer-Verlag, New York, 1983.

