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$*$-Lie higher derivable mappings on $*$-rings

Fonctions $*$-LIE supérieurement dérivables sur les $*$-anneaux

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ABSTRACT. Let $\mathcal{R}$ be a $*$-ring with the center $\mathcal{Z}(\mathcal{R})$ and $\mathbb{N}$ be the set of all non-negative integers. Let $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$ be the family of mappings $L_n : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) such that $L_0 = I_{\mathcal{R}}$, the identity mapping of $\mathcal{R}$. Then $\mathfrak{L}$ is said to be a $*$-Lie higher derivable mapping of $\mathcal{R}$ if $L_n([X^*, Y]) = \sum\limits_{i+j=n} [L_i(X)^*, L_j(Y)]$ holds for all $X, Y \in \mathcal{R}$. In this paper, it is shown that, if $\mathcal{R}$ is a $*$-ring containing a nontrivial self adjoint idempotent which admits a $*$-Lie higher derivable mapping $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$, then there exists an element $Z_{X,Y}$ (depending on X and Y) in the center $\mathcal{Z}(\mathcal{R})$ such that $L_n(X + Y) = L_n(X) + L_n(Y) + Z_{X,Y}$.
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1. Introduction

Throughout this paper $\mathcal{R}$ will denote an associative ring with the center $\mathcal{Z}(\mathcal{R})$. Recall that a ring $\mathcal{R}$ is said to be prime if for any $X, Y \in \mathcal{R}$, $X\mathcal{R}Y = \{0\}$ implies $X = 0$ or $Y = 0$. An additive mapping $X \mapsto X^*$ on a ring $\mathcal{R}$ is called involution in case $(XY)^* = Y^*X^*$ and $(X^*)^* = X$ hold for all $X, Y \in \mathcal{R}$. A ring equipped with an involution is called a ring with involution or $*$-ring (see [12]). An additive mapping $L : \mathcal{R} \to \mathcal{R}$ is said to be a derivation on $\mathcal{R}$ if $L(XY) = L(X)Y + XL(Y)$ for all $X, Y \in \mathcal{R}$. An additive mapping $L : \mathcal{R} \to \mathcal{R}$ is called a Lie derivation if $L([X, Y]) = [L(X), Y] + [X, L(Y)]$ holds for all $X, Y \in \mathcal{R}$, where $[X, Y] = XY - YX$ is the usual Lie product. If the condition of additivity is dropped from the above definition, then the corresponding Lie derivation is called a Lie derivable map. Obviously, every derivation is a Lie derivation. However, the converse statements is not true in general.

In an attempt to generalize the concept of derivation, the notion of higher derivation was introduced by Hasse and Schmidt [11]. Motivated by the existence of higher derivation, the notion of higher derivable mapping was studied by many authors. Let $\mathbb{N}$ be the set of all non-negative integers. A family $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$ of mappings $L_n : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) such that $L_0 = I_{\mathcal{R}}$, the identity mapping on $\mathcal{R}$, is said to be

(i) a *higher derivable* on $\mathcal{R}$ if for each $n \in \mathbb{N}$, $L_n(XY) = \sum\limits_{i+j=n} L_i(X)L_j(Y)$, for all $X, Y \in \mathcal{R}$.

(ii) a *Lie higher derivable* on $\mathcal{R}$ if for each $n \in \mathbb{N}$, $L_n([X, Y]) = \sum\limits_{i+j=n} [L_i(X), L_j(Y)]$, for all $X, Y \in \mathcal{R}$.

Note that if $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$ is the family of additive mappings $L_n : \mathcal{R} \to \mathcal{R}$ in the above definition then $\mathfrak{L}$ is said to be a higher derivation respectively a Lie higher derivation on $\mathcal{R}$.

Let $\mathcal{R}$ be a $*$-ring. An additive mapping $L : \mathcal{R} \to \mathcal{R}$ is said to be an additive $*$-derivation on $\mathcal{R}$ if $L(XY) = L(X)Y + XL(Y)$ and $L(X^*) = L(X)^*$ hold for all $X, Y \in \mathcal{R}$. More generally, a mapping

$L : \mathcal{R} \to \mathcal{R}$ is said to be a $*$-Lie derivable mapping if $L([X^*, Y]) = [L(X)^*, Y] + [X^*, L(Y)]$. Indeed, if $L(X^*) = L(X)^*$ for all $X \in \mathcal{R}$, then L is a Lie derivable mapping if and only if L is a $*$-Lie derivable mapping. An additive $*$-Lie derivable mapping is said to be a $*$-Lie derivation. It is not difficult to observe that any $*$-derivation is a $*$-Lie derivation but the converse is not true in general.

Let $\mathbb{N}$ be the set of all non-negative integers. A family $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$ of mappings $L_n : \mathcal{R} \to \mathcal{R}$ (not necessarily additive) such that $L_0 = I_{\mathcal{R}}$, the identity mapping on $\mathcal{R}$, is said to be

(i) a $*$-*higher derivable* on $\mathcal{R}$ if for each $n \in \mathbb{N}$, $L_n(X^*Y) = \sum\limits_{i+j=n} L_i(X)^* L_j(Y)$, for all $X, Y \in \mathcal{R}$.

(ii) a $*$-*Lie higher derivable* on $\mathcal{R}$ if for each $n \in \mathbb{N}$, $L_n([X^*, Y]) = \sum\limits_{i+j=n} [L_i(X)^*, L_j(Y)]$, for all $X, Y \in \mathcal{R}$.

It is to be noted that if $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$ is the family of additive mappings $L_n : \mathcal{R} \to \mathcal{R}$ in the above definition then $\mathfrak{L}$ is said to be a $*$-higher derivation respectively a $*$-Lie higher derivation on $\mathcal{R}$.

There has been a great interest in the study of characterizations of Lie derivations and $*$-Lie derivations for many years. The first quite surprising result is due to Martindale III who proved that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive (see [18]). Yu and Zhang [21] proved that every Lie derivable map of a triangular algebra is the sum of an additive derivation and a map from triangular algebra into its center sending commutators to zero. Mathieu and Villena [17] gave the characterizations of Lie derivations on C^*-algebras. W. Jing and F. Lu [13] showed that every Lie derivable map on a 2-torsion free prime ring $\mathcal{R}$ can be expressed as $L = d + \tau$, where d is a derivation of $\mathcal{R}$ into its central closure $\mathcal{T}$ and $\tau : \mathcal{R} \to \mathcal{C}$ (where $\mathcal{C}$ is extended centroid of $\mathcal{R}$) is nearly additive i.e. $\tau(X+Y) = \tau(X)+\tau(Y)+Z_{X,Y}$ where $Z_{X,Y} \in \mathcal{Z}(\mathcal{R})$ (depends on X and Y in $\mathcal{R}$) and vanishes on each commutator. In 2015, Ashraf and Parveen [3] proved that every Lie higher derivable map on a prime ring $\mathcal{R}$ is nearly additive. Yu and Zhang [22] proved that every $*$-Lie derivable mapping from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Chen and Wang [14] obtained the same result for $*$-Lie derivable mappings on a von Neumann algebras and proved that every $*$-Lie derivable mapping on a von Neumann algebra with no central abelian projections can be expressed as the sum of an additive $*$-derivation and a mapping with image in the centre vanishing at commutators. In addition, the characterization of Lie derivations and $*$-Lie derivations on various algebras are considered in [2], [4], [5], [8], [7], [9], [13], [15], [20], [23]. Very recently, Alkenani et al. [1] gave the characterization of $*$-Lie derivable mappings on $*$-rings, more precisely they proved the following result:

Theorem 1.1. *[1] Let $\mathcal{R}$ be a $*$-ring containing a nontrivial self adjoint idempotent P and satisfying the following conditions:*

(G_1) *If $X_{ii}Y_{ij} = Y_{ij}X_{jj}$ for all $Y_{ij} \in \mathcal{R}_{ij}$ and $1 \leq i \neq j \leq 2$, then $X_{ii} + X_{jj} \in \mathcal{Z}(\mathcal{R})$.*

(G_2) *If $X_{ij}Y_{jk} = 0$ for all $Y_{jk} \in \mathcal{R}_{jk}$ and $1 \leq i, j, k \leq 2$, then $Y_{ij} = 0$.*

If a mapping $L : \mathcal{R} \to \mathcal{R}$ satisfies $L([X^*, Y]) = [L(X)^*, Y] + [X^*, L(Y)]$, for all $X, Y \in \mathcal{R}$, then there exists $Z_{X,Y} \in \mathcal{Z}(\mathcal{R})$ such that $L(X + Y) = L(X) + L(Y) + Z_{X,Y}$.

Motivated by the above result, we investigate the additivity of $*$-Lie higher derivable mappings on $*$-rings and show that every $*$-Lie higher derivable mapping on $\mathcal{R}$ is almost additive in the sense that for any $X, Y \in \mathcal{R}$ there exists $Z_{X,Y} \in \mathcal{Z}(\mathcal{R})$ (depending on X and Y) such that $L_n(X+Y) = L_n(X) + L_n(Y) + Z_{X,Y}$. Finally, the above ring theoretic result has been applied to some special class of algebras such as nest algebras and von Neumann algebras.

2. $*$-Lie higher derivable mappings on $*$-rings

In this section, we study the additivity of a $*$-Lie higher derivable mappings on $*$-rings. In fact, we prove the following:

Theorem 2.1. *Let $\mathcal{R}$ be a $*$-ring containing a nontrivial self adjoint idempotent element P with center $\mathcal{Z}(\mathcal{R})$ and satisfying the following conditions:*

(G_1) *If $X_{ii}Y_{ij} = Y_{ij}X_{jj}$ for all $Y_{ij} \in \mathcal{R}_{ij}$ and $1 \leq i \neq j \leq 2$, then $X_{ii} + X_{jj} \in \mathcal{Z}(\mathcal{R})$.*

(G_2) *If $X_{ij}Y_{jk} = 0$ for all $Y_{jk} \in \mathcal{R}_{jk}$ and $1 \leq i, j, k \leq 2$, then $X_{ij} = 0$.*

Suppose that $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$ is the family of mappings $L_n : \mathcal{R} \to \mathcal{R}$ such that $L_0 = I_{\mathcal{R}}$, the identity mapping of $\mathcal{R}$, satisfying

$$L_n([X^*, Y]) = \sum_{i+j=n} [L_i(X)^*, L_j(Y)]$$

for all $X, Y \in \mathcal{R}$ and for each $n \in \mathbb{N}$. Then there exists $Z_{X,Y}$ (depending on X and Y) in $\mathcal{Z}(\mathcal{R})$ such that $L_n(X+Y) = L_n(X) + L_n(Y) + Z_{X,Y}$.

Let $\mathcal{R}$ be a $*$-ring with a nontrivial self adjoint idempotent P. We write $Q = 1 - P$. It is to be noted that $\mathcal{R}$ may be without identity element. It is obvious that $PQ = QP = 0$. By the Peirce decomposition of $\mathcal{R}$, we have $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}$, where $\mathcal{R}_{11} = P\mathcal{R}P$, $\mathcal{R}_{12} = P\mathcal{R}Q$, $\mathcal{R}_{21} = Q\mathcal{R}P$ and $\mathcal{R}_{22} = Q\mathcal{R}Q$. Throughout this paper, X_{ij} will denote an arbitrary element of $\mathcal{R}_{ij}$ and any element $X \in \mathcal{R}$ can be expressed as $X = X_{11} + X_{12} + X_{21} + X_{22}$. In view of Theorem 1.1, it is clear that $L_1(X+Y) = L_1(X) + L_1(Y) + Z_{X,Y}$, where $Z_{X,Y} \in \mathcal{Z}(\mathcal{R})$. We will use this result throughout the section whenever needed without specific mention.

Throughout assume that $\mathcal{R}$ satisfies the hypothesis of Theorem 2.1. The proof of the above theorem is given in a series of Lemmas.

Lemma 2.1. *For each $n \in \mathbb{N}$, $L_n(0) = 0$.*

Proof. For $n = 1$, we have $L_1(0) = 0$. Now by induction hypothesis, let the result hold for all $m < n$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} L_n(0) &= L_n([0^*, 0]) \\ &= [L_n(0)^*, 0] + [0^*, L_n(0)] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(0)^*, L_j(0)] = 0. \end{aligned}$$

□

Lemma 2.2. *For each $n \in \mathbb{N}$ and for any $X_{ii} \in \mathcal{R}_{ii}$, $Y_{ij} \in \mathcal{R}_{ij}$, $1 \leq i \neq j \leq 2$, there exists $Z_{X_{ii},Y_{ij}} \in \mathcal{Z}(\mathcal{R})$ such that*

(i) $L_n(X_{ii} + Y_{ij}) = L_n(X_{ii}) + L_n(Y_{ij}) + Z_{X_{ii},Y_{ij}}$,

(ii) $L_n(X_{ii} + Y_{ji}) = L_n(X_{ii}) + L_n(Y_{ji}) + Z_{X_{ii},Y_{ji}}$.

Proof. (i) For $n = 1$, we have $L_1(X_{ii} + Y_{ij}) = L_1(X_{ii}) + L_1(Y_{ij}) + Z_{X_{ii},Y_{ij}}$. Let $A = L_n(X_{ii} + Y_{ij}) - L_n(X_{ii}) - L_n(Y_{ij})$. Now by induction hypothesis, let the result hold for all $m < n$. For any $X_{ii} \in \mathcal{R}_{ii}$, $Y_{ij} \in \mathcal{R}_{ij}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} L_n(Y_{ij}) &= L_n([P^*, X_{ii} + Y_{ij}]) \\ &= [L_n(P)^*, X_{ii} + Y_{ij}] + [P^*, L_n(X_{ii} + Y_{ij})] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(P)^*, L_j(X_{ii} + Y_{ij})]. \end{aligned}$$

On the other hand by Lemma 2.1, we have

$$\begin{aligned} L_n(Y_{ij}) &= L_n([P^*, X_{ii}]) + L_n([P^*, Y_{ij}]) \\ &= [L_n(P)^*, X_{ii}] + [P^*, L_n(X_{ii})] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(P)^*, L_j(X_{ii})] \\ &\quad + [L_n(P)^*, Y_{ij}] + [P^*, L_n(Y_{ij})] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(P)^*, L_j(Y_{ij})] \\ &= [L_n(P)^*, X_{ii} + Y_{ij}] + [P^*, L_n(X_{ii}) + L_n(Y_{ij})] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(P)^*, L_j(X_{ii}) + L_j(Y_{ij})]. \end{aligned}$$

Comparing the above two identities, we get $[P, A] = 0$. Hence $A_{ij} = A_{ji} = 0$.

For any $W_{ji} \in \mathcal{R}_{ji}$, we compute

$$\begin{aligned} L_n(-X_{ii}W_{ji}^*) &= L_n([W_{ji}^*, X_{ii} + Y_{ij}]) \\ &= [L_n(W_{ji})^*, X_{ii} + Y_{ij}] + [W_{ji}^*, L_n(X_{ii} + Y_{ij})] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(W_{ji})^*, L_j(X_{ii} + Y_{ij})]. \end{aligned}$$

Using Lemma 2.1, $L_n(-X_{ii}W_{ji}^*)$ can also be expressed as

$$\begin{aligned} L_n(-X_{ii}W_{ji}^*) &= L_n([W_{ji}^*, X_{ii}]) + L_n([W_{ji}^*, Y_{ij}]) \\ &= [L_n(W_{ji})^*, X_{ii}] + [W_{ji}^*, L_n(X_{ii})] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(W_{ji})^*, L_j(X_{ii})] \\ &\quad + [L_n(W_{ji})^*, Y_{ij}] + [W_{ji}^*, L_n(Y_{ij})] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(W_{ji})^*, L_j(Y_{ij})] \\ &= [L_n(W_{ji})^*, X_{ii} + Y_{ij}] + [W_{ji}^*, L_n(X_{ii}) + L_n(Y_{ij})] \\ &\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(W_{ji})^*, L_j(X_{ii}) + L_j(Y_{ij})]. \end{aligned}$$

From the above two equations it follows that $[W_{ji}^*, A] = 0$. In other words $W_{ji}^* A = A W_{ji}^*$ for all $W_{ji} \in \mathcal{R}_{ji}$. By the condition (G_1), we see that $A_{ii} + A_{jj} \in \mathcal{Z}(\mathcal{R})$. Hence $L_n(X_{ii} + Y_{ij}) = L_n(X_{ii}) + L_n(Y_{ij}) + Z_{X_{ii},Y_{ij}}$ for some $Z_{X_{ii},Y_{ij}} \in \mathcal{Z}(\mathcal{R})$. Similarly, one can get (ii). □

Lemma 2.3. *For each $n \in \mathbb{N}$ and for any $X_{ij}, Y_{ij} \in \mathcal{R}_{ij}$, $1 \leq i \neq j \leq 2$, we have*

$$L_n(X_{ij} + Y_{ij}) = L_n(X_{ij}) + L_n(Y_{ij}).$$

Proof. For $n = 1$, we have $L_1(X_{ij} + Y_{ij}) = L_1(X_{ij}) + L_1(Y_{ij})$. Now by induction hypothesis, let the result hold for all $m < n$. By Lemma 2.2, we have

$$\begin{aligned}
L_n(X_{ij} + Y_{ij}) &= L_n([(X_{ij}^* + P)^*, Y_{ij} + Q]) \\
&= \sum_{i+j=n} [L_i(X_{ij}^* + P)^*, L_j(Y_{ij} + Q)] \\
&= \sum_{i+j=n} ([L_i(X_{ij}^*)^* + L_i(P)^*, L_j(Y_{ij}) + L_j(Q)]) \\
&= \sum_{i+j=n} ([L_i(X_{ij}^*)^*, L_j(Y_{ij})] + [L_i(X_{ij}^*)^*, L_j(Q)] + [L_i(P)^*, L_j(Y_{ij})] \\
&\quad + [L_i(P)^*, L_j(Q)]) \\
&= L_n([(X_{ij}^*)^*, Y_{ij}]) + L_n([(X_{ij}^*)^*, Q]) + L_n([P^*, Y_{ij}]) + L_n([P^*, Q]) \\
&= L_n(X_{ij}) + L_n(Y_{ij}).
\end{aligned}$$

□

Lemma 2.4. *For each $n \in \mathbb{N}$ and for any $X_{ii}, Y_{ii} \in \mathcal{R}_{ii}$, $i = 1, 2$, there exists $Z_{X_{ii},Y_{ii}} \in \mathcal{Z}(\mathcal{R})$ such that*

$$L_n(X_{ii} + Y_{ii}) = L_n(X_{ii}) + L_n(Y_{ii}) + Z_{X_{ii},Y_{ii}}.$$

Proof. We prove the result for $i = 1$. For $i = 2$ the proof follows similarly. For $n = 1$, we have $L_1(X_{11} + Y_{11}) = L_1(X_{11}) + L_1(Y_{11}) + Z_{X_{11},Y_{11}}$.
Let $A = L_n(X_{11} + Y_{11}) - L_n(X_{11}) - L_n(Y_{11})$. Now by induction hypothesis, let the result hold for all $m < n$. For any $X_{11}, Y_{11} \in \mathcal{R}_{11}$, we have

$$\begin{aligned}
0 &= L_n([X_{11} + Y_{11}, Q^*]) \\
&= \sum_{i+j=n} [L_i(X_{11} + Y_{11}), L_j(Q)^*] \\
&= [L_n(X_{11} + Y_{11}), Q^*] + [X_{11} + Y_{11}, L_n(Q)^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11} + Y_{11}), L_j(Q)^*].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
0 &= L_n([X_{11}, Q^*]) + L_n([Y_{11}, Q^*]) \\
&= [L_n(X_{11}), Q^*] + [X_{11}, L_n(Q)^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}), L_j(Q)^*]
\end{aligned}$$

$$
\begin{aligned}
&+[L_n(Y_{11}), Q^*] + [Y_{11}, L_n(Q)^*] + \sum_{\substack{i+j=n\\0<i,j\leq n-1}} [L_i(Y_{11}), L_j(Q)^*] \\
= \; & [L_n(X_{11}) + L_n(Y_{11}), Q^*] + [X_{11} + Y_{11}, L_n(Q)^*] \\
&+ \sum_{\substack{i+j=n\\0<i,j\leq n-1}} [L_i(X_{11}) + L_i(Y_{11}), L_j(Q)^*].
\end{aligned}
$$

Comparing the above two identities, we get $[A, Q] = 0$. Hence $A_{12} = A_{21} = 0$.

For any $W_{12} \in \mathcal{R}_{12}$, we compute

$$
\begin{aligned}
L_n(-W_{12}^*(X_{11} + Y_{11})) &= L_n([X_{11} + Y_{11}, W_{12}^*]) \\
&= \sum_{i+j=n} [L_i(X_{11} + Y_{11}), L_j(W_{12})^*] \\
&= [L_n(X_{11} + Y_{11}), W_{12}^*] + [X_{11} + Y_{11}, L_n(W_{12})^*] \\
&\quad + \sum_{\substack{i+j=n\\0<i,j\leq n-1}} [L_i(X_{11} + Y_{11}), L_j(W_{12})^*].
\end{aligned}
$$

On the other hand by using Lemma 2.3, we have

$$
\begin{aligned}
L_n(-W_{12}^*(X_{11} + Y_{11})) &= L_n(-W_{12}^* X_{11}) + L_n(-W_{12}^* Y_{11}) \\
&= L_n([X_{11}, W_{12}^*]) + L_n([Y_{11}, W_{12}^*]) \\
&= [L_n(X_{11}), W_{12}^*] + [X_{11}, L_n(W_{12})^*] + \sum_{\substack{i+j=n\\0<i,j\leq n-1}} [L_i(X_{11}), L_j(W_{12})^*] \\
&\quad +[L_n(Y_{11}), W_{12}^*] + [Y_{11}, L_n(W_{12})^*] + \sum_{\substack{i+j=n\\0<i,j\leq n-1}} [L_i(Y_{11}), L_j(W_{12})^*] \\
&= [L_n(X_{11}) + L_n(Y_{11}), W_{12}^*] + [X_{11} + Y_{11}, L_n(W_{12})^*] \\
&\quad + \sum_{\substack{i+j=n\\0<i,j\leq n-1}} [L_i(X_{11}) + L_i(Y_{11}), L_j(W_{12})^*].
\end{aligned}
$$

On comparing the above two equations, we have $[A, W_{12}^*] = 0$. Thus $A_{22}W_{12}^* = W_{12}^* A_{11}$ for all $W_{12} \in \mathcal{R}_{12}$. By using the condition (G_1), we see that $A_{11} + A_{22} \in \mathcal{Z}(\mathcal{R})$. Therefore $L_n(X_{11} + Y_{11}) = L_n(X_{11}) + L_n(Y_{11}) + Z_{X_{11}, Y_{11}}$ for all $X_{11}, Y_{11} \in \mathcal{R}_{11}$ and for some $Z_{X_{11}, Y_{11}} \in \mathcal{Z}(\mathcal{R})$. □

Lemma 2.5. *For each $n \in \mathbb{N}$ and for any $X_{12} \in \mathcal{R}_{12}$ and $Y_{21} \in \mathcal{R}_{21}$, we have*

$$
L_n(X_{12} + Y_{21}) = L_n(X_{12}) + L_n(Y_{21}).
$$

Proof. For $n = 1$, we have $L_1(X_{12} + Y_{21}) = L_1(X_{12}) + L_1(Y_{21})$. Let $A = L_n(X_{12} + Y_{21}) - L_n(X_{12}) - L_n(Y_{21})$. Now by induction hypothesis, let the result hold for all $m < n$. For any $X_{12} \in \mathcal{R}_{12}$, $Y_{21} \in \mathcal{R}_{21}$, we have

$$\begin{aligned}
L_n(X_{12}+Y_{21})) &= L_n([P^*, X_{12}-Y_{21}]) \\
&= [L_n(P)^*, X_{12}-Y_{21}] + [P^*, L_n(X_{12}-Y_{21})] \\
&\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(P)^*, L_j(X_{12}-Y_{21})] \\
&= [L_n(P)^*, X_{12}] + [L_n(P)^*, -Y_{21}] + [P^*, L_n(X_{12}-Y_{21})] \\
&\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(P)^*, L_j(X_{12})] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(P)^*, L_j(-Y_{21})] \\
&= L_n([P^*, X_{12}]) - [P^*, L_n(X_{12})] + L_n([P^*, -Y_{21}]) - [P^*, L_n(-Y_{21})] \\
&\quad +[P^*, L_n(X_{12}-Y_{21})] \\
&= L_n(X_{12}) + L_n(Y_{21}) + [P^*, L_n(X_{12}-Y_{21}) - L_n(X_{12}) - L_n(-Y_{21})].
\end{aligned}$$

Consequently $A = A_{11}+A_{12}+A_{21}+A_{22} = P(L_n(X_{12}-Y_{21})-L_n(X_{12})-L_n(-Y_{21}))Q - Q(L_n(X_{12}-Y_{21}) - L_n(X_{12}) - L_n(-Y_{21}))P = 0$. After solving this we get $A_{11}+A_{22} = 0$. Hence we see that $A_{11} = A_{22} = 0$.

For any $W_{12} \in \mathcal{R}_{12}$, by Lemma 2.1, we have

$$\begin{aligned}
L_n([X_{12}, W_{12}^*]) &= L_n([X_{12}+Y_{21}, W_{12}^*]) \\
&= [L_n(X_{12}+Y_{21}), W_{12}^*] + [X_{12}+Y_{21}, L_n(W_{12})^*] \\
&\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{12}+Y_{21}), L_j(W_{12})^*].
\end{aligned}$$

On the other hand, by using Lemma 2.1, we have

$$\begin{aligned}
L_n([X_{12}, W_{12}^*]) &= L_n([X_{12}, W_{12}^*]) + L_n([Y_{21}, W_{12}^*]) \\
&= [L_n(X_{12}), W_{12}^*] + [X_{12}, L_n(W_{12})^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{12}), L_j(W_{12})^*] \\
&\quad +[L_n(Y_{21}), W_{12}^*] + [Y_{21}, L_n(W_{12})^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(Y_{21}), L_j(W_{12})^*] \\
&= [L_n(X_{12}) + L_n(Y_{21}), W_{12}^*] + [X_{12}+Y_{21}, L_n(W_{12})^*] \\
&\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{12}) + L_i(Y_{21}), L_j(W_{12})^*].
\end{aligned}$$

Comparing the above two identities, we get $[A, W_{12}^*] = 0$. This gives that $A_{12}W_{12}^* = 0$ for all $W_{12} \in \mathcal{R}_{12}$. By the condition (G_2), we see that $A_{12} = 0$. Similarly, we obtain that $A_{21} = 0$. Thus we are done. □

Lemma 2.6. *For each $n \in \mathbb{N}$ and for any $X_{11} \in \mathcal{R}_{11}$, $Y_{12} \in \mathcal{R}_{12}$ and $W_{22} \in \mathcal{R}_{22}$, we have*

$$L_n(X_{11}+Y_{12}+W_{22}) = L_n(X_{11}) + L_n(Y_{12}) + L_n(W_{22}) + Z_{X_{11},Y_{12},W_{22}}.$$

Proof. For $n = 1$, we have $L_1(X_{11}+Y_{12}+W_{22}) = L_1(X_{11}) + L_1(Y_{12}) + L_1(W_{22}) + Z_{X_{11},Y_{12},W_{22}}$. Suppose $A = L_n(X_{11}+Y_{12}+W_{22}) - L_n(X_{11}) - L_n(Y_{12}) - L_n(W_{22})$. Now by induction hypothesis, let the result hold for all $m < n$. For any $X_{11} \in \mathcal{R}_{11}$, $Y_{12} \in \mathcal{R}_{12}$ and $W_{22} \in \mathcal{R}_{22}$, we compute

$$\begin{aligned} L_n(Y_{12}) &= L_n([X_{11}+Y_{12}+W_{22},Q^*]) \\ &= [L_n(X_{11}+Y_{12}+W_{22}),Q^*]+[X_{11}+Y_{12}+W_{22},L_n(Q)^*] \\ &\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}+Y_{12}+W_{22}),L_j(Q)^*]. \end{aligned}$$

On the other hand, by using Lemma 2.1, we have

$$\begin{aligned} L_n(Y_{12}) &= L_n([X_{11}+Y_{12}+W_{22},Q^*]) \\ &= L_n([X_{11},Q^*])+L_n([Y_{12},Q^*])+L_n([W_{22},Q^*]) \\ &= [L_n(X_{11}),Q^*]+[X_{11},L_n(Q)^*]+\sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}),L_j(Q)^*] \\ &\quad +[L_n(Y_{12}),Q^*]+[Y_{12},L_n(Q)^*]+\sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(Y_{12}),L_j(Q)^*] \\ &\quad +[L_n(W_{22}),Q^*]+[W_{22},L_n(Q)^*]+\sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(W_{22}),L_j(Q)^*] \end{aligned}$$

$$\begin{aligned} &= [L_n(X_{11})+L_n(Y_{12})+L_n(W_{22}),Q^*]+[X_{11}+Y_{12}+W_{22},L_n(Q)^*] \\ &\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11})+L_i(Y_{12})+L_i(W_{22}),L_j(Q)^*]. \end{aligned}$$

Comparing the above two identities, we get $[A,Q^*]=0$. This gives that $A_{12}=A_{21}=0$.

Now for any $S_{21}\in\mathcal{R}_{21}$, we see that

$$\begin{aligned} L_n([X_{11}+Y_{12}+W_{22},S_{21}^*]) &= [L_n(X_{11}+Y_{12}+W_{22}),S_{21}^*]+[X_{11}+Y_{12}+W_{22},L_n(S_{21})^*] \\ &\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}+Y_{12}+W_{22}),L_j(S_{21})^*]. \end{aligned}$$

On the other hand by Lemmas 2.1 & 2.3, we have

$$\begin{aligned} &L_n([X_{11}+Y_{12}+W_{22},S_{21}^*]) \\ &= L_n([X_{11}+W_{22},S_{21}^*])+L_n([Y_{12},S_{21}^*]) \\ &= L_n(X_{11}S_{21}^*-S_{21}^*W_{22})+L_n([Y_{12},S_{21}^*]) \\ &= L_n(X_{11}S_{21}^*)+L_n(-S_{21}^*W_{22})+L_n([Y_{12},S_{21}^*]) \\ &= L_n([X_{11},S_{21}^*])+L_n([W_{22},S_{21}^*])+L_n([Y_{12},S_{21}^*]) \\ &= [L_n(X_{11}),S_{21}^*]+[X_{11},L_n(S_{21})^*]+\sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}),L_j(S_{21})^*] \\ &\quad +[L_n(W_{22}),S_{21}^*]+[W_{22},L_n(S_{21})^*]+\sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(W_{22}),L_j(S_{21})^*] \end{aligned}$$

$$+[L_n(Y_{12}), S_{21}^*] + [Y_{12}, L_n(S_{21})^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(Y_{12}), L_j(S_{21})^*]$$
$$= [L_n(X_{11}) + L_n(Y_{12}) + L_n(W_{22}), S_{21}^*] + [X_{11} + Y_{12} + W_{22}, L_n(S_{21})^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}) + L_i(Y_{12}) + L_i(W_{22}), L_j(S_{21})^*].$$

Comparing the above two identities, we get $[A, S_{21}^*] = 0$. This gives that $A_{11}S_{21}^* = S_{21}^*A_{22}$ for all $S_{21} \in \mathcal{R}_{21}$. By the condition (G_1), we get $A_{11} + A_{22} \in \mathcal{Z}(\mathcal{R})$. Thus we have obtained that $L_n(X_{11} + Y_{12} + W_{22}) = L_n(X_{11}) + L_n(Y_{12}) + L_n(W_{22}) + Z_{X_{11},Y_{12},W_{22}}$ for some $Z_{X_{11},Y_{12},W_{22}} \in \mathcal{Z}(\mathcal{R})$. □

Lemma 2.7. *For each $n \in \mathbb{N}$ and for any $X_{11} \in \mathcal{R}_{11}$, $Y_{12} \in \mathcal{R}_{12}$, $W_{21} \in \mathcal{R}_{21}$ and $T_{22} \in \mathcal{R}_{22}$, we have*

$$L_n(X_{11} + Y_{12} + W_{21} + T_{22}) = L_n(X_{11}) + L_n(Y_{12}) + L_n(W_{21}) + L_n(T_{22}) + Z_{X_{11},Y_{12},W_{21},T_{22}}.$$

Proof. For $n = 1$, we have $L_1(X_{11} + Y_{12} + W_{21} + T_{22}) = L_1(X_{11}) + L_1(Y_{12}) + L_1(W_{21}) + L_1(T_{22})$. Suppose $A = L_n(X_{11}+Y_{12}+W_{21}+T_{22}) - L_n(X_{11}) - L_n(Y_{12}) - L_n(W_{21}) - L_n(T_{22})$. Now by induction hypothesis, let the result holds for all $m < n$. For any $X_{11} \in \mathcal{R}_{11}$, $Y_{12} \in \mathcal{R}_{12}$, $W_{21} \in \mathcal{R}_{21}$ and $T_{22} \in \mathcal{R}_{22}$, we see that

$$\begin{aligned} L_n(Y_{12} - W_{21}) &= L_n([X_{11} + Y_{12} + W_{21} + T_{22}, Q^*]) \\ &= [L_n(X_{11} + Y_{12} + W_{21} + T_{22}), Q^*] + [X_{11} + Y_{12} + W_{21} + T_{22}, L_n(Q)^*] \\ &\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11} + Y_{12} + W_{21} + T_{22}), L_j(Q)^*]. \end{aligned}$$

On the other hand, by using Lemmas 2.1 & 2.5, we have

$$\begin{aligned} L_n(Y_{12} - W_{21}) &= L_n([X_{11}, Q^*]) + L_n([Y_{12}, Q^*]) + L_n([W_{21}, Q^*]) + L_n([T_{22}, Q^*]) \\ &= [L_n(X_{11}), Q^*] + [X_{11}, L_n(Q)^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}), L_j(Q)^*] \\ &\quad +[L_n(Y_{12}), Q^*] + [Y_{12}, L_n(Q)^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(Y_{12}), L_j(Q)^*] \\ &\quad +[L_n(W_{21}), Q^*] + [W_{21}, L_n(Q)^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(W_{21}), L_j(Q)^*] \\ &\quad +[L_n(X_{22}), Q^*] + [X_{22}, L_n(Q)^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(T_{22}), L_j(Q)^*] \\ &= [L_n(X_{11}) + L_n(Y_{12}) + L_n(W_{21}) + L_n(T_{22}), Q^*] \\ &\quad +[X_{11} + Y_{12} + W_{21} + T_{22}, L_n(Q)^*] \\ &\quad + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}) + L_i(Y_{12}) + L_i(W_{21}) + L_i(T_{22}), L_j(Q)^*] \end{aligned}$$

Comparing the above two relations for $L_n(Y_{12} - W_{21})$, we have $[A, Q^*] = 0$. This gives that $A_{12} = A_{21} = 0$.

Now for any $S_{12} \in \mathcal{R}_{12}$, we compute

$$\begin{aligned} & L_n([X_{11} + Y_{12} + W_{21} + T_{22}, S_{12}^*]) \\ = \; & [L_n(X_{11} + Y_{12} + W_{21} + T_{22}), S_{12}^*] + [X_{11} + Y_{12} + W_{21} + T_{22}, L_n(S_{12})^*] \\ & + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11} + Y_{12} + W_{21} + T_{22}), L_j(S_{12})^*]. \end{aligned}$$

On the other hand, by using Lemmas 2.1 & 2.6, we have

$$\begin{aligned} & L_n([X_{11} + Y_{12} + W_{21} + T_{22}, S_{12}^*]) \\ = \; & L_n([X_{11} + Y_{12} + T_{22}, S_{12}^*]) + L_n([W_{21}, S_{12}^*]) \\ = \; & [L_n(X_{11} + Y_{12} + T_{22}), S_{12}^*] + [X_{11} + Y_{12} + T_{22}, L_n(S_{12})^*] \\ & + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11} + Y_{12} + T_{22}), L_j(S_{12})^*] \\ & + [L_n(W_{21}), S_{12}^*] + [W_{21}, L_n(S_{12})^*] + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(W_{21}), L_j(S_{12})^*] \\ = \; & [L_n(X_{11}) + L_n(Y_{12}) + L_n(W_{21}) + L_n(T_{22}), S_{12}^*] + [X_{11} + Y_{12} + W_{21} + T_{22}, L_n(S_{12})^*] \\ & + \sum_{\substack{i+j=n \\ 0<i,j\leq n-1}} [L_i(X_{11}) + L_i(Y_{12}) + L_i(W_{21}) + L_i(T_{22}), L_j(S_{12})^*]. \end{aligned}$$

Comparing the above two relations for $L_n([X_{11} + Y_{12} + W_{21} + T_{22}, S_{12}^*])$, we get $[A, S_{12}^*] = 0$. This gives that $S_{12}^* A_{11} = A_{22} S_{12}^*$ for all $S_{12} \in \mathcal{R}_{12}$. By using condition (G_1), we see that $A_{11} + A_{22} \in \mathcal{Z}(\mathcal{R})$. Thus we have obtained that $L_n(X_{11} + Y_{12} + W_{21} + T_{22}) = L_n(X_{11}) + L_n(Y_{12}) + L_n(W_{21}) + L_n(T_{22}) + Z_{X_{11},Y_{12},W_{21},T_{22}}$ for some $Z_{X_{11},Y_{12},W_{21},T_{22}} \in \mathcal{Z}(\mathcal{R})$. □

Proof of Theorem 2.1. Now take $X = X_{11} + X_{12} + X_{21} + X_{22}$ and $Y = Y_{11} + Y_{12} + Y_{21} + Y_{22}$. By using Lemmas 2.3, 2.4 & 2.7, we see that

$$\begin{aligned} L_n(X + Y) &= L_n(X_{11} + X_{12} + X_{21} + X_{22} + Y_{11} + Y_{12} + Y_{21} + Y_{22}) \\ &= L_n((X_{11} + Y_{11}) + (X_{12} + Y_{12}) + (X_{21} + Y_{21}) + (X_{22} + Y_{22})) \\ &= L_n(X_{11} + Y_{11}) + L_n(X_{12} + Y_{12}) + L_n(X_{21} + Y_{21}) + L_n(X_{22} + Y_{22}) + Z_1 \\ &= L_n(X_{11}) + L_n(Y_{11}) + Z_2 + L_n(X_{12}) + L_n(Y_{12}) + L_n(X_{21}) \\ &\quad + L_n(Y_{21}) + L_n(X_{22}) + L_n(Y_{22}) + Z_3 + Z_1 \\ &= (L_n(X_{11}) + L_n(X_{12}) + L_n(X_{21}) + L_n(X_{22})) + (L_n(Y_{11}) \\ &\quad + L_n(Y_{12}) + L_n(Y_{21}) + L_n(Y_{22}) + Z_1 + Z_2 + Z_3 \\ &= L_n(X_{11} + X_{12} + X_{21} + X_{22}) - Z_4 + L_n(Y_{11} + Y_{12} + Y_{21} + Y_{22}) \\ &\quad - Z_5 + Z_1 + Z_2 + Z_3 \\ &= L_n(X) + L_n(Y) + (Z_1 + Z_2 + Z_3 - Z_4 - Z_5). \end{aligned}$$

Take $Z_{X,Y} = Z_1 + Z_2 + Z_3 - Z_4 - Z_5$. Thus we see that $L_n(X+Y) = L_n(X) + L_n(Y) + Z_{X,Y}$ for some $Z_{X,Y} \in \mathcal{Z}(\mathcal{R})$. This completes the proof of our main theorem. □

Now we apply Theorem 2.1 to prime $*$-rings and nest algebras. We begin with the following important lemma.

Lemma 2.8. *Let $\mathcal{R}$ be a prime $*$-ring containing a nontrivial self-adjoint idempotent P with the centre $\mathcal{Z}(\mathcal{R})$.*

(i) *If $X_{ii}Y_{ij} = Y_{ij}X_{jj}$ for all $Y_{ij} \in \mathcal{R}_{ij}$ and $1 \leq i \neq j \leq 2$ then $X_{ii} = 0$.*

(ii) *If $X_{11}Y_{12} = Y_{12}X_{22}$ for all $Y_{12} \in \mathcal{R}_{12}$, then $X_{11} + X_{22} \in \mathcal{Z}(\mathcal{R})$.*

Proof. (i) is the direct consequence of the primeness of $\mathcal{R}$.

(ii) For any $Y_{11} \in \mathcal{R}_{11}$ and $Y_{12} \in \mathcal{R}_{12}$, we get $X_{11}Y_{11}Y_{12} = Y_{11}Y_{12}X_{22} = Y_{11}X_{11}Y_{12}$ for all $Y_{12} \in \mathcal{R}_{12}$. As $\mathcal{R}$ is prime, we have $X_{11}Y_{11} = Y_{11}X_{11}$.

For any $Y_{12} \in \mathcal{R}_{12}$ and $Y_{22} \in \mathcal{R}_{22}$, we get $Y_{12}Y_{22}X_{22} = X_{11}Y_{12}Y_{22} = Y_{12}X_{22}Y_{22}$ for all $Y_{12} \in \mathcal{R}_{12}$. It follows by the primeness of $\mathcal{R}$ that $Y_{22}X_{22} = X_{22}Y_{22}$.

For any $Y_{12} \in \mathcal{R}_{12}$ and $Y_{21} \in \mathcal{R}_{21}$, we get $X_{22}Y_{21}Y_{12} = Y_{21}Y_{12}X_{22} = Y_{21}X_{11}Y_{12}$ for all $Y_{12} \in \mathcal{R}_{12}$. It follows that $X_{22}Y_{21} = Y_{21}X_{22}$.

For any $Y \in \mathcal{R}$, we have

$$\begin{aligned}
(X_{11} + X_{22})Y &= (X_{11} + X_{22})(Y_{11} + Y_{12} + Y_{21} + Y_{22}) \\
&= X_{11}Y_{11} + X_{11}Y_{12} + X_{22}Y_{21} + X_{22}Y_{22} \\
&= Y_{11}X_{11} + Y_{12}X_{11} + Y_{21}X_{22} + Y_{22}X_{22} \\
&= (Y_{11} + Y_{12} + Y_{21} + Y_{22})(X_{11} + X_{22}) \\
&= Y(X_{11} + X_{22}).
\end{aligned}$$

Hence it follows that $X_{11} + X_{22} \in \mathcal{Z}(\mathcal{R})$. □

It follows from Lemma 2.8 that every prime $*$-ring with nontrivial self-adjoint idempotent satisfies the conditions (G_1) and (G_2) of Theorem 2.1. So we have the following immediate corollary.

Corollary 2.1. *Let $\mathcal{R}$ be a prime $*$-ring containing a nontrivial self adjoint idempotent element P with center $\mathcal{Z}(\mathcal{R})$. Suppose that $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$ is the family of mappings $L_n : \mathcal{R} \to \mathcal{R}$ such that $L_0 = I_{\mathcal{R}}$, the identity mapping of $\mathcal{R}$, satisfying*

$$L_n([X^*, Y]) = \sum_{i+j=n} [L_i(X)^*, L_j(Y)]$$

for all $X, Y \in \mathcal{R}$ and for each $n \in \mathbb{N}$. Then there exists $Z_{X,Y}$ (depending on X and Y) in $\mathcal{Z}(\mathcal{R})$ such that $L_n(X+Y) = L_n(X) + L_n(Y) + Z_{X,Y}$.

Let $\mathcal{H}$ be a complex Hilbert space. Recall that a nest $\mathcal{N}$ of projections on $\mathcal{H}$ is a chain of orthogonal projections on $\mathcal{H}$ containing zero operator 0 and the identity operator I and is closed in the strong operator topology. By $\mathcal{B}(\mathcal{H})$, we mean the algebra of all bounded linear operators on $\mathcal{H}$. The nest algebra $\mathcal{T}(\mathcal{N})$ corresponding to the nest $\mathcal{N}$ is the set of all operators X in $\mathcal{B}(\mathcal{H})$ such that $XP = PXP$ for all $P \in \mathcal{N}$. It is to be noted that $\mathcal{T}(\mathcal{N})$ is a weak $*$- closed operator algebra. A nest is said to be nontrivial if it contains at least one nontrivial projection. The centre of the nest algebra $\mathcal{T}(\mathcal{N})$ is $\mathbb{C}I$, where $\mathbb{C}$ is a complex field. It is to be noted that every nest algebra $\mathcal{T}(\mathcal{N})$ with non trivial projection P satisfies the conditions (G_1) and (G_2) of Theorem 2.1. Thus we have the following immediate corollary.

Corollary 2.2. *Let $\mathcal{N}$ be a nontrivial nest on a complex Hilbert space $\mathcal{H}$ and $\mathcal{T}(\mathcal{N})$ be the associated nest algebra. Suppose that $\mathfrak{L} = \{L_n\}_{n\in\mathbb{N}}$ is the family of mappings $L_n : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ such that $L_0 = I_{\mathcal{T}(\mathcal{N})}$, the identity mapping of $\mathcal{T}(\mathcal{N})$, satisfying*

$$L_n([X^*, Y]) = \sum_{i+j=n} [L_i(X)^*, L_j(b)]$$

for all $X, Y \in \mathcal{T}(\mathcal{N})$ and for each $n \in \mathbb{N}$. Then there exists $Z_{X,Y}$ (depending on X and Y) in $\mathcal{Z}(\mathcal{T}(\mathcal{N}))$ such that $L_n(X + Y) = L_n(X) + L_n(Y) + Z_{X,Y}I$.

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