Advances in Pure and Applied Mathematics
2024, vol. 15, n° 4, 21-83 pages, DOI: 10.21494/ISTE.OP.2024.1199 ISTE OpenScience

Kan extendable subcategories and fibrewise topology

Sous-catégories extensibles au sens de Kan et la topologie par fibre

Moncef Ghazel

Faculty of Sciences of Tunis, University of Tunis El manar
moncef.ghazel@fst.utm.tn

ABSTRACT. We use pointwise Kan extensions to generate new subcategories out of old ones. We investigate the properties of these newly produced categories and give sufficient conditions for their cartesian closedness to hold. Our methods are of general use. Here we apply them particularly to the study of the properties of certain categories of fibrewise topological spaces. In particular, we prove that the categories of fibrewise compactly generated spaces, fibrewise sequential spaces and fibrewise Alexandroff spaces are cartesian closed provided that the base space satisfies the right separation axiom.
Mathematics Subject Classification 2010: 18A40, 18D15, 54B30, 55R70
KEYWORDS. Reflective subcategory, Reflective hull, Dense functor, Kan extension, Codensity monad, Cartesian closed category, Fibrewise topological space

Introduction

In this paper, a subcategory of any category is always assumed to be full.

A subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is said to be reflective if the inclusion functor $\mathcal{B} \longrightarrow \mathcal{C}$ has a left adjoint. Examples of such are the subcategories of Hausdorff spaces, Tychonoff spaces, compact spaces and realcompact spaces of the category Top of topological spaces. The reflective hull of a subcategory $\mathcal{W}$ of $\mathcal{C}$ is the smallest replete, reflective subcategory of $\mathcal{C}$ containing $\mathcal{W}$. Such a subcategory does not always exist, for the intersection of all replete reflective subcategories of $\mathcal{C}$ containing $\mathcal{W}$ may not be reflective, as is shown by Adámek and Rosický [2]. The existence of reflective hulls and their properties have been extensively studied by several authors [2, 3, 20, 21, 28, 38].

We, in Theorem 1.8, show that a replete reflective subcategory of $\mathcal{C}$ containing $\mathcal{W}$ as a codense subcategory is necessarily the reflective hull of $\mathcal{W}$, and is therefore unique when it exists. We call such a subcategory the strong reflective hull of $\mathcal{W}$. Coreflective subcategories, coreflective hulls and strong coreflective hulls are dually defined. The notion of the so-called strong reflective hull is strictly stronger than that of the reflective hull, in the sense that there are examples of reflective hulls which are not strong (see Remark 4.4 and Example 1.11).

When it exists, a (pointwise) right Kan extension R of a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ along itself has a monad structure. This monad R is called the codensity monad of F; for R reduces to the identity functor $1_{\mathcal{B}}$ iff the functor F is codense. One has a dual notion of density comonad of the functor F.

A monad (T, η, μ) is said to be idempotent when its multiplication $\mu : T^2 \longrightarrow T$ is an isomorphism. Similarly, a comonad is said to be idempotent if its comultiplication is an isomorphism.

We define a subcategory $\mathcal{W}$ of $\mathcal{C}$ to be left Kan extendable if the inclusion functor $\mathcal{W} \longrightarrow \mathcal{C}$ has an idempotent density comonad (L, ϵ, δ). When this is the case, then the category of L-coalgebras is denoted by $\mathcal{W}_l[\mathcal{C}]$ and the forgetful functor $U : \mathcal{W}_l[\mathcal{C}] \longrightarrow \mathcal{C}$ is fully faithful and injective on objects.

Consequently, $\mathcal{W}_l[\mathcal{C}]$ is viewed as a subcategory of $\mathcal{C}$. Dually, the subcategory $\mathcal{W}$ of $\mathcal{C}$ is said to be right Kan extendable provided that the inclusion functor $\mathcal{W} \longrightarrow \mathcal{C}$ has an idempotent codensity monad (R, η, μ). In this case, the category of R-algebras is denoted by $\mathcal{W}_r[\mathcal{C}]$.

The two notions of strong reflective hull and right Kan extendability are closely related: a subcategory $\mathcal{W}$ of $\mathcal{C}$ has a strong reflective hull iff $\mathcal{W}$ is right Kan extendable in $\mathcal{C}$. When this is the case, then the strong reflective hull of $\mathcal{W}$ is precisely the subcategory $\mathcal{W}_r[\mathcal{C}]$ of $\mathcal{C}$ (dual of Theorem 3.6).

As applications, we prove that the subcategory of Top whose only object is the square of the unit interval has a strong reflective hull which is the subcategory of compact Hausdorff spaces. Similarly, we prove that the subcategory of Top whose only object is the square of the real line is the subcategory of realcompact spaces. Consequently, one recovers the Stone–Čech compactification and the Hewitt realcompactification procedures.

Fibrewise topology is a branch of topology which studies the slice categories of Top. It plays an important role in homotopy theory as shown by Crabb and James in their book [10], and is now considered as a subject in its own right. One of the main objectives of this paper is to extend some of the categorical properties of certain subcategories of Top to their fibrewise counterparts.

It is a well known fact that the subcategories of Top of Fréchet spaces, Hausdorff spaces, Urysohn spaces, completely Hausdorff spaces, weak Hausdorff spaces and k-Hausdorff spaces are reflective. Let B be a topological space and let Top_B be the category of fibrewise spaces over B. We use the theory of Kan extendable subcategories to present a general theorem allowing one to recognize reflective subcategories of Top_B. We then use it to prove, in a harmonized and systematic manner, that the fibrewise versions of the above subcategories of Top are again reflective subcategories of Top_B.

It is a classical result of Herrlich and Strecker that any subcategory $\mathcal{W}$ of Top containing a nonempty space is, in our terminology, left Kan extendable ([18, Proposition 2.17], [20, Theorem 12] and [19, page 283]). Moreover, if the objects of $\mathcal{W}$ are exponentiable in Top and if $\mathcal{W}$ satisfies an additional condition, then a celebrated theorem of Day asserts $\mathcal{W}_l[\mathsf{Top}]$ is cartesian closed [11, Theorem 3.1]. In the most famous application, one takes $\mathcal{W}$ to be the subcategory Comp of compact Hausdorff spaces to deduce that the category of compactly generated spaces, which is the strong reflective hull of Comp, is cartesian closed [11, Corollary 3.3]. Similarly, by taking $\mathcal{W}$ to be the subcategory of Top whose only object is the one-point compactification of a discrete countable space, we deduce that the subcategory of Top of sequential spaces is cartesian closed and, by taking $\mathcal{W}$ to be the subcategory of Top whose only object is the Sierpinski space, one gets the fact that the subcategory of Alexandroff spaces is cartesian closed.

The category Top_B is not cartesian closed. Lots of work with varying success has been done to provide a convenient substitute for it. In [5, Theorem 2.2], Booth proves that the category of fibrewise quasi-topological spaces, in which the category of fibrewise topological spaces embeds, is cartesian closed. In a later work, Booth and Brown defined a partial map version of the compact-open topology and use it to describe a fibrewise mapping-space satisfying certain exponential laws [4, Section 3]. Variants of the Booth-Brown topology on the mapping space were used by James to show that an exponential law holds in certain situations ([24, Proposition 5.6] and [25, Proposition 10.15]).

Let $\mathcal{W}$ be a left Kan extendable subcategory of $\mathcal{C}$ whose objects are exponentiable in $\mathcal{C}$. We show that under mild conditions, the subcategory $\mathcal{W}_l[\mathcal{C}]$ of $\mathcal{C}$ is cartesian closed (Theorem 9.6).

We here prove that a subcategory $\mathcal{W}$ of Top_B, which is suitable in a specified sense, is left Kan extendable (Theorem 8.2). We then use Theorem 9.6 to derive a fibrewise version of Day's theorem. As application, we prove that the category of fibrewise compactly generated spaces is cartesian closed provided that the base B is T_1 (Theorem 11.12); a result which is not proved neither in [5, 6, 4] nor in [24, 25] and is new to author's knowledge. Further applications include the cartesian closedness of the category of fibrewise sequential spaces (Proposition 13.4) and that of fibrewise Alexandroff spaces (Proposition 14.9), provided that the base B satisfies the right separation axiom.

The paper is structured as follows: Section 1 contains a brief discussion of reflective subcategories and their properties that are being used throughout. In particular, the concept of strong reflective hull is introduced and its connection with the ordinary reflective hull is investigated. In Section 2, we recall the basic definitions and facts about codensity monads and their idempotency. These are used in Section 3 to define the notion of Kan extendable subcategories and study their properties. In Section 4, we use the theory of Kan extendable subcategories to derive the Stone–Čech compactification and the Hewitt realcompactification procedures. In Section 5, we prove that subcategories of fibrewise topological spaces over B which satisfy certain separation axioms are reflective subcategories of Top_B. In Section 6, we investigate the concept of fibrewise compact spaces. We in particular prove that a fibrewise compact fibrewise Hausdorff space over a T_1 base B is an exponential object of Top_B, a fact that is needed to give one of the main applications of the paper. In Section 7, we introduce the subcategories of Top_B of fibrewise weak Hausdorff spaces and fibrewise k-Hausdorff spaces and prove that they are reflective in Top_B. In Section 8, we present a sufficient condition for a subcategory of Top_B to be left Kan extendable. In Section 9, we state conditions that ensure the cartesian closedness of the strong coreflective hull of a subcategory (Theorem 9.6). The fibrewise Day's theorem is presented and proved in Section 10 (Theorem 10.2). It is then used in Sections 11 to prove that the category kTop_B of fibrewise compactly generated topological spaces over a T_1 base B is cartesian closed. Properties of certain subcategories of kTop_B are inspected in Section 12. Sections 13 and 14 are devoted to the study of fibrewise sequential spaces and fibrewise Alexandroff spaces respectively. In Appendix A, limits in a slice category are investigated. Specializations of the results of Appendix A to either a slice category of sets or a category of fibrewise topological spaces are given in appendices B and C.

Conventions and notations

Throughout this paper, the product of two categories $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \times \mathcal{B}$. A subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is always assumed to be full. Given two objects $X, Y \in \mathcal{C}$, the set of morphisms from X to Y is denoted by $\mathcal{C}(X, Y)$. When it exists, the cartesian product of X and Y in $\mathcal{C}$ is denoted by $X \times_{\mathcal{C}} Y$. If X and Y are in the subcategory $\mathcal{B}$ of $\mathcal{C}$, then their cartesian product $X \times_{\mathcal{B}} Y$ in $\mathcal{B}$, when it exists may be different from their product $X \times_{\mathcal{C}} Y$ in the larger category $\mathcal{C}$ and should not be confused with it. Given a family of objects $(X_i)_{i \in I}$ of $\mathcal{C}$, when they exist, the product and coproduct over I of the X_i's are denoted by $\prod_{\mathcal{C}}^{i\in I} X_i$ and $\coprod_{\mathcal{C}}^{i\in I} X_i$ respectively.

Throughout this paper, B denotes a fixed topological space. The slice category Top/B is called the category of fibrewise topological spaces over B and denoted simply by Top_B. (see Appendix C).

Acknowledgments

I would like to greatly thank Inès Saihi for the careful revision of an early draft and the several corrections she made. I am also deeply grateful to the referees whose comments considerably helped streamline the paper and improve its readability.

1 Reflective subcategories

In this section, we briefly recall the notion of reflective subcategories and discuss some of their relevant properties.

Definition 1.1. *Let $\mathcal{C}$ be a category.*

1. *A subcategory $\mathcal{C}_0$ of $\mathcal{C}$ is said to be reflective if the inclusion functor $\mathcal{C}_0 \overset{U}{\hookrightarrow} \mathcal{C}$ is a right adjoint functor. In this case, a left adjoint functor F of U is called a reflector and the adjoint pair $(F \dashv U)$ is called a reflection of $\mathcal{C}$ on $\mathcal{C}_0$. The unit $1_{\mathcal{C}} \overset{\eta}{\Longrightarrow} UF$ and counit $FU \overset{\epsilon}{\Longrightarrow} 1_{\mathcal{C}_0}$ of the adjunction $(F \dashv U)$ are also called the unit and counit of the reflection $(F \dashv U)$ of $\mathcal{C}$ on $\mathcal{C}_0$.*

2. *Dually, a subcategory $\mathcal{C}^0$ of $\mathcal{C}$ is said to be coreflective if the inclusion functor $\mathcal{C}^0 \overset{U}{\hookrightarrow} \mathcal{C}$ is a left adjoint functor. In this case, a right adjoint G of U is called a coreflector and the adjoint pair $(U \dashv G)$ is called a coreflection of $\mathcal{C}$ on $\mathcal{C}^0$. The unit $1_{\mathcal{C}^0} \overset{\eta}{\Longrightarrow} GU$ and counit $UG \overset{\epsilon}{\Longrightarrow} 1_{\mathcal{C}}$ of the adjunction $(U \dashv G)$ are also called the unit and counit of the coreflection $(U \dashv G)$ of $\mathcal{C}$ on $\mathcal{C}^0$.*

Under the conditions of Definition 1.1.1, the objects of $\mathcal{C}_0$ are often identified with their images in $\mathcal{C}$ by the inclusion functor U. In particular, the components of the unit η of the reflection $(F \dashv U)$ are viewed as maps $\eta_C : C \longrightarrow F(C)$ in $\mathcal{C}$.

Lemma 1.2.

1. *Let $(F \dashv U)$ be a reflection of $\mathcal{C}$ on $\mathcal{C}_0$ with unit $1_{\mathcal{C}} \overset{\eta}{\Longrightarrow} UF$ and counit $FU \overset{\epsilon}{\Longrightarrow} 1_{\mathcal{C}_0}$. Then*
   (a) *The natural transformation ϵ is an isomorphism.*
   (b) *An object $C \in \mathcal{C}$ is isomorphic to an object in $\mathcal{C}_0$ iff the map $\eta_C : C \longrightarrow F(C)$ is an isomorphism.*

2. *Dually, let $(U \dashv G)$ be a coreflection of $\mathcal{C}$ on $\mathcal{C}^0$ with unit $1_{\mathcal{C}} \overset{\eta}{\Longrightarrow} GU$ and counit $UG \overset{\epsilon}{\Longrightarrow} 1_{\mathcal{C}_0}$. Then*
   (a) *The natural transformation η is an isomorphism.*
   (b) *An object $C \in \mathcal{C}$ is isomorphic to an object in $\mathcal{C}^0$ iff the map $\epsilon_C : G(C) \longrightarrow C$ is an isomorphism.*

Proof.

1. (a) The functor $U : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is fully faithful, therefore by [32, Theorem 1 page 90], the counit $FU \overset{\epsilon}{\Longrightarrow} 1_{\mathcal{C}_0}$ is an isomorphism.

(b) If $C \in \mathcal{C}$ is such that $\eta_C : C \longrightarrow F(C)$ is an isomorphism, then obviously, C is isomorphic to the object $F(C)$ of $\mathcal{C}_0$. Conversely, assume that there is an isomorphism $C_0 \xrightarrow{f} C$, where $C_0 \in \mathcal{C}_0$. The counit $FU \overset{\epsilon}{\Longrightarrow} 1_{\mathcal{C}_0}$ is an isomorphism, therefore $UFU \overset{U\epsilon}{\Longrightarrow} U$ is an isomorphism. By [32, Theorem 1 page 82] the composite

$$U \overset{\eta U}{\Longrightarrow} UFU \overset{U\epsilon}{\Longrightarrow} U$$

is the identity natural transformation. Therefore $U \overset{\eta U}{\Longrightarrow} UFU$ is an isomorphism. It follows that $\eta_{C_0} : C_0 \longrightarrow F(C_0)$ is an isomorphism. In the following commutative diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{\eta_{C_0}} & F(C_0) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ C & \xrightarrow[\eta_C]{} & F(C) \end{array}$$

f, $F(f)$ and η_{C_0} are isomorphisms. Therefore η_C is an isomorphism.

2. The second property is dual to the first.

□

Definition 1.3. *Let $\mathcal{B}$ be a subcategory of a category $\mathcal{C}$ and $C \xrightarrow{f} C'$ a morphism in $\mathcal{C}$. Then*

1. *f is said to be $\mathcal{B}$-monic if given two maps α, β from an object B in $\mathcal{B}$ to C, then $f\alpha = f\beta \Longrightarrow \alpha = \beta$.*

2. *Dually, f is said to be $\mathcal{B}$-epic if given two maps α, β in $\mathcal{C}$ from C' to an object B in $\mathcal{B}$, then $\alpha f = \beta f \Longrightarrow \alpha = \beta$.*

Lemma 1.4.

1. *Assume that $(F \dashv U)$ is a reflection of a category $\mathcal{C}$ on a subcategory $\mathcal{C}_0$ with unit $1_{\mathcal{C}} \overset{\eta}{\Longrightarrow} UF$. Then for every $C \in \mathcal{C}$, the morphism $C \xrightarrow{\eta_C} F(C)$ is $\mathcal{C}_0$-epic.*

2. *Dually, assume that $(U \dashv G)$ is a coreflection of a category $\mathcal{C}$ on a subcategory $\mathcal{C}^0$ with counit $UG \overset{\epsilon}{\Longrightarrow} 1_{\mathcal{C}^0}$. Then for every $C \in \mathcal{C}$, the morphism $G(C) \xrightarrow{\epsilon_C} C$ is $\mathcal{C}^0$-monic.*

Proof.

1. Let $C \in \mathcal{C}$ and $C_0 \in \mathcal{C}_0$. The map $\mathcal{C}_0(F(C), C_0) \xrightarrow{\mathcal{C}_0(\eta_C, C_0)} \mathcal{C}(C, C_0)$ is bijective, therefore injective. Thus η_C is $\mathcal{C}_0$-epic.

2. The second property is dual to the first.

□

Proposition 1.5. *(Riehl, [36, Proposition 4.5.15])*

1. *Let $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ be a reflective subcategory, then*
   (a) *The inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ creates all limits that $\mathcal{C}$ admits.*
   (b) *The subcategory $\mathcal{C}_0$ has all colimits that $\mathcal{C}$ admits, formed by applying the reflector to the colimit in $\mathcal{C}$.*

In particular if $\mathcal{C}$ is either complete or cocomplete, then so is $\mathcal{C}_0$.

2. *Let $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ be a coreflective subcategory, then*
   - *(a) The inclusion functor $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ creates all colimits that $\mathcal{C}$ admits.*
   - *(b) The subcategory $\mathcal{C}^0$ has all limits that $\mathcal{C}$ admits, formed by applying the coreflector to the limit in $\mathcal{C}$.*

   In particular if $\mathcal{C}$ is either complete or cocomplete, then so is $\mathcal{C}^0$.

The following result is a generalization of [21, Proposition 3].

Lemma 1.6. *Let $\mathcal{C}_0$ be a subcategory of a category $\mathcal{C}$ which is either reflective or coreflective. Then the retract in $\mathcal{C}$ of an object in $\mathcal{C}_0$ is isomorphic to an object of $\mathcal{C}_0$.*

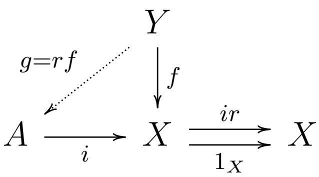
Proof. We only need to prove the property in the reflective case. Let

$$A \xrightarrow{i} X \xrightarrow{r} A$$

be a retraction in $\mathcal{C}$ of an object $X \in \mathcal{C}_0$. The diagram

$$A \xrightarrow{\ i\ } X \underset{1_X}{\overset{ir}{\rightrightarrows}} X \tag{1}$$

is an equalizer in $\mathcal{C}$. For $iri = i = 1_X i$. Let $f : Y \longrightarrow X$ be such that $irf = f$. Assume that $g : Y \longrightarrow A$ is such that $ig = f$.



Then $ig = f = irf$. The morphism i is monic, thus $g = rf$ and g is unique. Now define $g = rf$, $ig = irf = f$. It follows that (1) is an equalizer. By Proposition 1.5.1.(a), A is isomorphic to an object of $\mathcal{C}_0$. □

Recall that a subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is said to be replete if any object of $\mathcal{C}$ which is isomorphic to an object of $\mathcal{A}$ is itself in $\mathcal{A}$.

Definition 1.7. *Let $\mathcal{W}$ be a subcategory of a category $\mathcal{C}$.*

1. *A subcategory $\mathcal{C}_0$ of $\mathcal{C}$ is called the reflective hull of $\mathcal{W}$ in $\mathcal{C}$ if it is the smallest replete, reflective subcategory of $\mathcal{C}$ containing $\mathcal{W}$.*

2. *Dually, a subcategory $\mathcal{C}^0$ of $\mathcal{C}$ is called the coreflective hull of $\mathcal{W}$ in $\mathcal{C}$ if it is the smallest replete, coreflective subcategory of $\mathcal{C}$ containing $\mathcal{W}$.*

A reflective (resp. coreflective) hull of a subcategory may not always exist, as is shown in [2], but if it does, then it is unique. A subcategory $\mathcal{W}$ of a category $\mathcal{C}$ has a reflective (resp. coreflective) hull iff the intersection of all reflective (resp. coreflective), replete subcategories of $\mathcal{C}$ containing $\mathcal{W}$ is again a reflective (resp. coreflective) subcategory of $\mathcal{C}$. In which case, this intersection is precisely the reflective (resp. coreflective) hull of $\mathcal{W}$.

Let $F : \mathcal{A} \longrightarrow \mathcal{C}$ be a functor. For $C \in \mathcal{C}$, let F/C be standard comma category, $D_C : F/C \longrightarrow \mathcal{A}$ be the functor which takes an arrow-object $F(A) \xrightarrow{\sigma} C$ to A and F_C be the composite functor

$$F/C \xrightarrow{D_C} \mathcal{A} \xrightarrow{F} \mathcal{C} \tag{2}$$

Recall that the functor F is said to be dense if for each $C \in \mathcal{C}$, F_C has a colimit and the natural map $\mathsf{colim} F_C \longrightarrow C$ is an isomorphism. If $\mathcal{A}$ is a subcategory of $\mathcal{C}$ and $J : \mathcal{A} \longrightarrow \mathcal{C}$ is the inclusion functor, then for $C \in \mathcal{C}$, the comma category J/C is also denoted by $\mathcal{A}/C$. The functor J_C is the composite

$$\mathcal{A}/C \xrightarrow{D_C} \mathcal{A} \xrightarrow{J} \mathcal{C}. \tag{3}$$

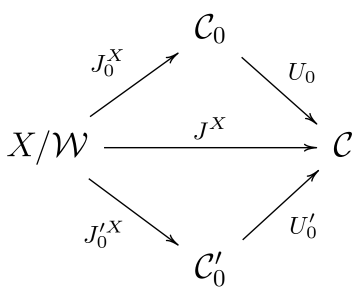
The subcategory $\mathcal{A}$ of $\mathcal{C}$ is said to be dense in $\mathcal{C}$ if the functor J is dense. One has dual notions of codense functor and codense subcategory.

Theorem 1.8. *Let $\mathcal{W}$ be a subcategory of a category $\mathcal{C}$.*

1. *Assume that $\mathcal{C}_0$ is a replete reflective subcategory of $\mathcal{C}$ in which $\mathcal{W}$ is codense. Then $\mathcal{C}_0$ is the reflective hull of $\mathcal{W}$ in $\mathcal{C}$.*

2. *Dually, assume that $\mathcal{C}^0$ is a replete coreflective subcategory of $\mathcal{C}$ in which $\mathcal{W}$ is dense. Then $\mathcal{C}^0$ is the coreflective hull of $\mathcal{W}$ in $\mathcal{C}$.*

Proof. We prove the first property, the second one is the dual of the first. Let $\mathcal{C}'_0$ be a replete reflective subcategory of $\mathcal{C}$ containing $\mathcal{W}$. Define $\mathcal{C}_0 \xrightarrow{U_0} \mathcal{C}$ and $\mathcal{C}'_0 \xrightarrow{U'_0} \mathcal{C}$ to be the inclusion functors and let $X \in \mathcal{C}_0$. Define $X/\mathcal{W}$ to be the subcategory of the under category $X/\mathcal{C}$ whose objects are arrows $X \to V$ with $V \in \mathcal{W}$. Let $J^X : X/\mathcal{W} \longrightarrow \mathcal{C}$ be the functor which takes an arrow-object $X \to V$ to its codomain V. The functor J^X takes values in $\mathcal{W}$ which is contained in $\mathcal{C}_0$ and $\mathcal{C}'_0$, therefore J^X factors through $\mathcal{C}_0$ and $\mathcal{C}'_0$ as shown in the following commutative diagram



The subcategory $\mathcal{W}$ is codense in $\mathcal{C}_0$, thus $\mathsf{lim} J_0^X = X$. Being a right adjoint, U_0 preserves limits. Thus $J_X = U_0 J_0^X$ has a limit and $\mathsf{lim} J^X = X$. We have $J_X = U'_0 J_0'^X$ and $\mathcal{C}'_0$ be a replete. By Proposition 1.5.1.(a), $J_0'^X$ has a limit, $X \in \mathcal{C}'_0$ and $\mathsf{lim} J_0'^X = X$. It follows that $\mathcal{C}_0$ is a subcategory of $\mathcal{C}'_0$. Therefore $\mathcal{C}_0$ is the reflective hull of $\mathcal{W}$ in $\mathcal{C}$. □

Recall that subcategories are always assumed to be full.

Remark 1.9. *Given a subcategory $\mathcal{W}$ of $\mathcal{C}$. Theorem 1.8 shows that:*

1. *There is at most one replete reflective subcategory of $\mathcal{C}$ in which $\mathcal{W}$ is codense. When it exists, it is certainly the reflective hull of $\mathcal{W}$, and is called the* ***strong reflective hull*** *of $\mathcal{W}$ in $\mathcal{C}$.*
2. *Dually, there is at most one replete coreflective subcategory of $\mathcal{C}$ in which $\mathcal{W}$ is dense. When it exists, it is certainly the coreflective hull of $\mathcal{W}$, and is called the* ***strong coreflective hull*** *of $\mathcal{W}$ in $\mathcal{C}$.*

Corollary 1.10. *Given a subcategory $\mathcal{W}$ of $\mathcal{C}$.*

1. *The subcategory $\mathcal{W}$ has a strong reflective hull iff it has a reflective hull in which it is codense.*
2. *Dually, $\mathcal{W}$ has a strong coreflective hull iff it has a coreflective hull in which it is dense.*

Proof. This is a consequence of Theorem 1.8. □

We next give an example of a subcategory which has a coreflective hull but has no strong coreflective hull.

Example 1.11. *Let* Vect *be the category of $\mathbb{Z}/2\mathbb{Z}$-vector spaces and $\mathcal{Z}$ the subcategory of* Vect *containing $\mathbb{Z}/2\mathbb{Z}$ as its unique object. Let $\mathcal{C}$ be a replete coreflective subcategory of* Vect *containing $\mathcal{Z}$. By Proposition 1.5.2.(a), $\mathcal{C}$ contains $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. By [32, page 247 Exercise 1], $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ is dense in* Vect. *Thus by Proposition 1.5.2.(a), $\mathcal{C}=$* Vect. *It follows that* Vect *is the unique coreflective subcategory of* Vect *containing $\mathcal{Z}$ and it is consequently its coreflective hull. Let $A=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\in$* Vect *and let $J_A:\mathcal{Z}/A\longrightarrow$* Vect *be the functor which takes an arrow-object $\mathbb{Z}/2\mathbb{Z}\longrightarrow A$ in $\mathcal{Z}/A$ to its domain $\mathbb{Z}/2\mathbb{Z}$. Clearly, $\mathrm{colim} J_A\cong\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. It follows that the subcategory $\mathcal{Z}$ of* Vect *is not dense in* Vect. *By Corollary 1.10.2, $\mathcal{Z}$ has no strong coreflective hull.*

We close this section with the following observation.

Remark 1.12. *Let $\mathcal{C}$ be a cartesian closed category with internal* hom *functor*

$$\begin{array}{rrcl}(.)^{(.)}: & \mathcal{C}^{op}\times\mathcal{C} & \longrightarrow & \mathcal{C}\\ & (Y,Z) & \longmapsto & Z^Y\end{array}$$

Let $\mathcal{C}_0$ be a reflective subcategory of $\mathcal{C}$ and assume that for every $Y,Z\in\mathcal{C}_0$, the power object $Z^Y\in\mathcal{C}_0$. Then:

1. *By Proposition 1.5.1.(a), for every $X,Y\in\mathcal{C}_0$, $X\times_{\mathcal{C}_0}Y$ exists and is isomorphic to the product $X\times_{\mathcal{C}}Y$ of X and Y in $\mathcal{C}$.*
2. *The category $\mathcal{C}_0$ is cartesian closed with internal* hom *functor induced by that of $\mathcal{C}$.*

2 Idempotent codensity monads

Here, we recall the concepts of monads, codensity monads and their algebras. Details may be found in [32, Ch. VI], [36, Ch. 5], [7, Ch. 4], [13, page 67] and [9, Section 2]. These notions are needed to define the main concept of this paper, which that of Kan extendable subcategories.

Let $\mathcal{C}$ be a category.

- The category $\mathcal{C}^{\mathcal{C}}$ of endofunctors of $\mathcal{C}$ is a monoidal category with composition of functors as its monoidal product.
- A monad on $\mathcal{C}$ is an unital associative monoid in $\mathcal{C}^{\mathcal{C}}$. It consists then of a triple (T, η, μ), where $T : \mathcal{C} \longrightarrow \mathcal{C}$ is a functor, $\mu : T^2 \longrightarrow T$ is an associative multiplication with unit $\eta : 1_{\mathcal{C}} \longrightarrow T$.

Let (T, η, μ) be a monad on the category $\mathcal{C}$.

- The monad (T, η, μ) is said to be idempotent if the multiplication $\mu : T^2 \longrightarrow T$ is an isomorphism.
- An algebra over T is a pair (A, u) consisting of an object $A \in \mathcal{C}$ and a morphism $u : TA \longrightarrow A$ rendering commutative the diagrams:

$$\begin{array}{ccc} T^2A & \xrightarrow{Tu} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle u} \\ TA & \xrightarrow[u]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle u} \\ & & A \end{array}$$

- Given two T-algebras (A, u) and (B, v). A morphism of T-algebras from A to B is an arrow $f : A \longrightarrow B$ rendering commutative the diagram

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[f]{} & B \end{array}$$

- Algebras over T and their morphisms form a category denoted by $\mathcal{C}^T$. It admits a forgetful functor $U : \mathcal{C}^T \longrightarrow \mathcal{C}$ which is right adjoint to the free T-algebras functor $F : \mathcal{C} \longrightarrow \mathcal{C}^T$.

Proposition 2.1. *(Borceaux, [7, Proposition 4.1.4])*

Let (T, η, μ) be a monad on a category $\mathcal{C}$. Let $U : \mathcal{C}^T \longrightarrow \mathcal{C}$ be the forgetful functor and $F : \mathcal{C} \longrightarrow \mathcal{C}^T$ the free T-algebras functor. Then $(F \dashv U)$ is an adjunction with unit the unit $\eta : 1_{\mathcal{C}} \longrightarrow T = UF$ of the monad T.

In order to recall the notion of idempotent monad, we state the following result which is part of [29, Proposition 7.2] of Kelly and Lack.

Proposition 2.2. *Let (T, η, μ) be a monad on a category $\mathcal{C}$. Then the following properties are equivalent:*

1. *The monad T is idempotent.*
2. *The natural transformation $\eta T : T \longrightarrow T^2$ is an isomorphism.*
3. *The natural transformation $T\eta : T \longrightarrow T^2$ is an isomorphism.*

4. *The functors μ and ηT are mutually inverse.*

5. *The functors μ and $T\eta$ are mutually inverse.*

6. *The functors ηT and $T\eta$ are equal.*

7. *For each object A of $\mathcal{C}$, a map $u : TA \longrightarrow A$ defines an algebra structure on A iff it is inverse to η_A.*

8. *The forgetful functor $U : \mathcal{C}^T \longrightarrow \mathcal{C}$ is full.*

9. *The forgetful functor $U : \mathcal{C}^T \longrightarrow \mathcal{C}$ is full and faithful.*

Proposition 2.3. *Let (T, η, μ) be an idempotent monad on a category $\mathcal{C}$ and let A be an object of $\mathcal{C}$. Then the following three conditions are equivalent:*

1. *The object A of $\mathcal{C}$ has a T-algebra structure.*

2. *The unit map $\eta_A : A \longrightarrow TA$ is an isomorphism.*

3. *The object A of $\mathcal{C}$ is isomorphic to a certain T-algebra.*

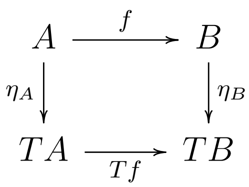
In particular, $\mathcal{C}^T$ is a replete subcategory of $\mathcal{C}$.

Proof.

- $1 \iff 2$: The monad T is idempotent. Therefore by Proposition 2.2.7, an object A of $\mathcal{C}$ has a T-algebra structure iff $\eta_A : A \longrightarrow TA$ is an isomorphism.

- $2 \implies 3$: If $\eta_A : A \longrightarrow TA$ is an isomorphism, then the object A of $\mathcal{C}$ is isomorphic to the free algebra TA.

- $3 \implies 2$: Assume that $f : A \longrightarrow B$ is an isomorphism, where B is a T-algebra. In the following commutative diagram

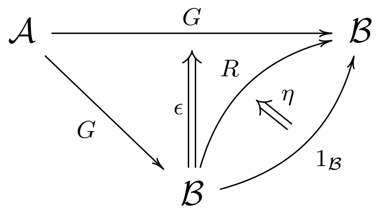


  The maps f, Tf and η_B are isomorphisms, therefore η_A is an isomorphism. It follows that A is a T-algebra.

- The monad T is idempotent. By Proposition 2.2.9, the forgetful functor $U : \mathcal{C}^T \longrightarrow \mathcal{C}$ is fully faithful. By Proposition 2.2.7, any object of $\mathcal{C}$ admits at most one algebra structure. Therefore U is injective on objects. The category $\mathcal{C}^T$ may then be identified to a subcategory of $\mathcal{C}$. The fact that $\mathcal{C}^T$ is a replete follows from the fact that properties 1. and 3. are equivalent.

□

Let $G : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and assume that G has a pointwise right Kan extension R along itself with counit $\epsilon : RG \longrightarrow G$. Then one has a diagram



where $\eta : 1_{\mathcal{B}} \longrightarrow R$ is the unique natural transformation rendering commutative the diagram

$$\begin{array}{ccc} G & \overset{1_G}{\Longrightarrow} & G \\ & {\scriptstyle \eta G} \searrow \quad \nearrow {\scriptstyle \epsilon} & \\ & RG & \end{array} \tag{4}$$

Let $B \in \mathcal{B}$, $D^B : B/G \longrightarrow \mathcal{A}$ be the functor which takes an arrow-object $B \xrightarrow{\sigma} G(A)$ to A and G^B be the composite functor

$$B/G \xrightarrow{D^B} \mathcal{A} \xrightarrow{G} \mathcal{B}. \tag{5}$$

Then by [32, Theorem 1 page 237],

$$R(B) = \lim G^B. \tag{6}$$

By [32, (6), page 238], the unit

$$\eta_B : B \longrightarrow R(B) \tag{7}$$

is the map induced by the cone

$$B \overset{\lambda^B}{\Longrightarrow} G^B \tag{8}$$

whose component λ^B_σ along an arrow-object $B \xrightarrow{\sigma} G(A)$ is the map $\sigma : B \longrightarrow G(A)$. By the universal property of (R, ϵ), there exists a unique natural transformation $\mu : R^2 \longrightarrow R$ rendering commutative the diagram

$$\begin{array}{ccc} R^2G & \overset{R\epsilon}{\Longrightarrow} & RG \\ {\scriptstyle \mu G}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \epsilon} \\ RG & \underset{\epsilon}{\Longrightarrow} & G \end{array}$$

Then the triple (R, η, μ) is a monad called the codensity monad of the functor G.

Assume that a functor $G : \mathcal{A} \longrightarrow \mathcal{B}$ has an idempotent codensity monad (R, η, μ). Then as explained in the proof of the final statement of Proposition 2.3, the forgetful functor $U : \mathcal{B}^R \longrightarrow \mathcal{B}$ is fully faithful and injective on objects. The category $\mathcal{B}^R$ is then identified to its image by U, which is, by Proposition 2.3, a replete subcategory of $\mathcal{C}$.

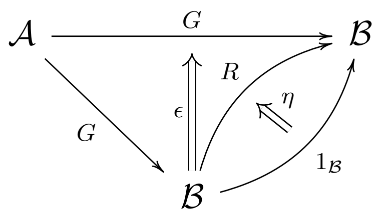
Examples 2.4.

1. *The unit of a monoidal category is a unital associative monoid.*
2. *Let $\mathcal{B}$ be a category. The trivial monod $I_{\mathcal{B}}$ on $\mathcal{B}$ is the unit of the monoidal category of endofunctors $\mathcal{B}^{\mathcal{B}}$. It is the identity functor $1_{\mathcal{B}}$ with the identity natural transformation of $1_{\mathcal{B}}$ as its unit and its multiplication, and it is idempotent. The trivial comonad on $\mathcal{B}$ is dually defined.*
3. *Clearly, a functor $G : \mathcal{A} \longrightarrow \mathcal{B}$ is codense iff the trivial monad $I_{\mathcal{B}}$ is a codensity monad of G [32, Proposition 1 page 246].*

Theorem 2.5. *Let $G : \mathcal{A} \longrightarrow \mathcal{B}$ be a fully faithful functor which has an idempotent codensity monad (R, η, μ). Then*

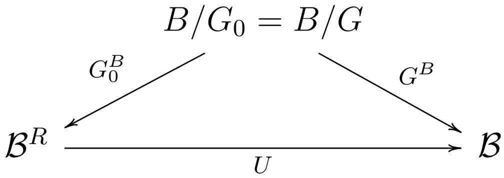
1. *The functor G takes values in the subcategory of R-algebras. That is, $G(\mathcal{A}) \subset \mathcal{B}^R$.*
2. *The functor $G_0 : \mathcal{A} \longrightarrow \mathcal{B}^R$ induced by G is a codense functor.*

Proof. Let $\epsilon : RG \longrightarrow G$ be the counit of the pointwise right Kan extension R of G along itself.



1. The functor G is fully faithful. By [32, Corollary 3, page 239], ϵ is an isomorphism. Let $A \in \mathcal{A}$, by (4), the composite
$$G(A) \xrightarrow{\eta_{G(A)}} RG(A) \xrightarrow{\epsilon_A} G(A)$$
is $1_{G(A)}$. It follows that $\eta_{G(A)}$ is an isomorphism. By Proposition 2.3, $G(A) \in \mathcal{B}^R$.
2. Let B be an R-algebra and $G_0 : \mathcal{A} \longrightarrow \mathcal{B}^R$ be the functor induced by G. Let
$$G^B : B/G \longrightarrow \mathcal{B} \quad \text{and} \quad G_0^B : B/G_0 \longrightarrow \mathcal{B}^R$$
be as defined by (5). Moreover, let
$$B \overset{\lambda^B}{\Longrightarrow} G^B \quad \text{and} \quad B \overset{\lambda_0^B}{\Longrightarrow} G_0^B$$
be as defined by (8). As explained above, $\lim G^B = R(B)$ and the map $B \longrightarrow \lim G^B$ induced by the cone λ^B is just the unit $\eta_B : B \longrightarrow R(B)$, which is an isomorphism by Proposition 2.3. It follows that λ^B is a limiting cone.

   The category B/G_0 is isomorphic to B/G and may be identified with it. The functor G^B factors through G_0^B as follows



By Proposition 1.5.1.(a), the functor U creates limits. In particular, any cone setting above a limit cone is itself a limit cone (see [36, page 90]). We have $U(\lambda_0^B) = \lambda^B$. Thus the cone λ_0^B is a limiting cone. It follows that G_0 is a codense functor.

□

The notions of comonads, idempotent comonads, coalgebras over them and density comonads are dually defined and satisfy the appropriate dual properties.

3 Left and right Kan extendable subcategories and their properties

In this section, we introduce the key notion of left Kan extendable subcategories and investigate some of its properties. We conclude the section by briefly introducing the dual notion of right Kan extendable subcategories.

Definition 3.1. [1] *A subcategory $\mathcal{W}$ of a category $\mathcal{C}$ is said to be left Kan extendable provided that:*

1. *The inclusion functor $J : \mathcal{W} \longrightarrow \mathcal{C}$ has a density comonad (L, ϵ, δ).*
2. *The comonad (L, ϵ, δ) is idempotent.*

Let $\mathcal{W}$ be a left Kan extendable subcategory of $\mathcal{C}$ and let (L, ϵ, δ) be the idempotent density comonad of the inclusion functor $J : \mathcal{W} \longrightarrow \mathcal{C}$. The category of L-coalgebras is denoted by $\mathcal{W}_l[\mathcal{C}]$. It is, by the dual of Proposition 2.3, a replete subcategory of $\mathcal{C}$ and is called the subcategory of $\mathcal{W}$-generated objects of $\mathcal{C}$.

Examples 3.2. *We here give examples of left Kan extendable subcategories.*

1. *Let* Ab *be the category of abelian groups. The subcategory* Fin *of* Ab *of finite abelian groups is left Kan extendable,* $\mathsf{Fin}_l[\mathsf{Ab}]$ *is the subcategory* Tor *of torsion abelian groups [30, page 42]. The functor* $\mathsf{Ab} \longrightarrow \mathsf{Tor}$ *which takes an abelian group to its torsion subgroup is a coreflector.*
2. *Let* P *the subcategory of the category* Top *consisting of just one object which is the one point topological space. The subcategory* P *is left Kan extendable in* Top *and* $\mathsf{P}_l[\mathsf{Top}]$ *is the category* Dis *of discrete topological spaces [8, page 18]. Furthermore, the discretization functor* $\mathsf{Top} \longrightarrow \mathsf{Dis}$ *is a coreflector.*
3. *The simplicial category Δ has objects $[n] = \{0, 1, ..., n\}$, $n \geq 0$. A map in Δ is an order preserving function $\alpha : [n] \longrightarrow [m]$. Let S be the category of simplicial sets and let $\Delta^n \in S$ be the standard n-simplex. Fix a non-negative integer n and let $\mathcal{W}_n$ be the (full) subcategory of S whose objects are Δ^k, $k \leq n$. Then $\mathcal{W}_n$ is left Kan extendable in S and $\mathcal{W}_n[S]$ is the subcategory S^n of S of simplicial sets of dimension $\leq n$. Furthermore, the left Kan extension of the inclusion functor $\mathcal{W}_n \longrightarrow S$ along itself is just the functor n-skeleton functor $Sk^n : S \longrightarrow S$ as defined in [26, page 11].*

1 The author is greatly grateful to Richard Garner for helping him introduce this final form of the definition of Kan extendability.

Proposition 3.3. *Let $\mathcal{W}$ be a left Kan extendable subcategory of $\mathcal{C}$, (L, ϵ, δ) the density comonad of the inclusion functor $J : \mathcal{W} \longrightarrow \mathcal{C}$ and $U : \mathcal{W}_l[\mathcal{C}] \longrightarrow \mathcal{C}$ the forgetful functor. Then:*

1. *The subcategory $\mathcal{W}_l[\mathcal{C}]$ is the strong coreflective hull of $\mathcal{W}$ in $\mathcal{C}$.*
2. *The free L-coalgebra functor $F_L : \mathcal{C} \longrightarrow \mathcal{W}_l[\mathcal{C}]$ is a coreflector.*
3. *The coreflection $(U \dashv F_L)$ has ϵ as its counit.*

Proof. This follows from the duals of Propositions 2.1, 2.2, 2.3 and the dual of Theorem 2.5. □

Proposition 3.4. *Let $\mathcal{W}$ be a left Kan extendable subcategory of $\mathcal{C}$.*

1. *The inclusion functor $\mathcal{W}_l[\mathcal{C}] \xrightarrow{U} \mathcal{C}$ creates all colimits that $\mathcal{C}$ admits.*
2. *The subcategory $\mathcal{W}_l[\mathcal{C}]$ has all limits that $\mathcal{C}$ admits formed by applying the coreflector F_L to the limit in $\mathcal{C}$.*

In particular, if $\mathcal{C}$ is either complete, cocomplete or bicomplete, then so is $\mathcal{W}_l[\mathcal{C}]$.

Proof. This follows from Proposition 3.3 and Proposition 1.5.2. □

Corollary 3.5. *Let $\mathcal{W}$ be a left Kan extendable subcategory of $\mathcal{C}$ and $C \in \mathcal{C}$. Then the following two properties are equivalent:*

1. *The object C is $\mathcal{W}$-generated.*
2. *There exists a functor $F : \mathcal{K} \longrightarrow \mathcal{W}$ such that $C \cong \mathrm{colim} JF$, where $J : \mathcal{W} \hookrightarrow \mathcal{C}$ is the inclusion functor.*

Proof.

- $2 \Longrightarrow 1$: This follows from Proposition 3.4.1.
- $1 \Longrightarrow 2$: Let (L, ϵ, δ) be the density comonad of the inclusion functor $J : \mathcal{W} \hookrightarrow \mathcal{C}$. Define $D_C : \mathcal{W}/C \longrightarrow \mathcal{W}$ to be the functor which associates to an arrow-object $V \longrightarrow C$ its domain V and let J_C be the composite functor

$$J_C : \mathcal{W}/C \xrightarrow{D_C} \mathcal{W} \overset{J}{\hookrightarrow} \mathcal{C}$$

Then, $L(C) \cong \mathrm{colim} J_C$. By the dual of Proposition 2.3, $\epsilon_C : L(C) \longrightarrow C$ is an isomorphism. Therefore $C \cong \mathrm{colim} J_C$. □

The next result presents a criteria for the existence of a strong reflective hull of a subcategory.

Theorem 3.6. *Let $\mathcal{W}$ be a subcategory of a category $\mathcal{C}$ and $J : \mathcal{W} \longrightarrow \mathcal{C}$ the inclusion functor. The following two properties are equivalent:*

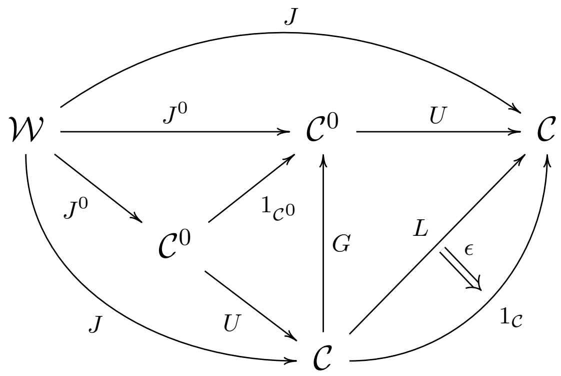
1. *The subcategory $\mathcal{W}$ of $\mathcal{C}$ is left Kan extendable.*
2. *The subcategory $\mathcal{W}$ of $\mathcal{C}$ has a strong coreflective hull.*
3. *The functor J has a density comonad (L, ϵ, δ) and the morphism $\epsilon_C : L(C) \longrightarrow C$ is $\mathcal{W}$-monic for all $C \in \mathcal{C}$.*

When these conditions are satisfied, then $\mathcal{W}_l[\mathcal{C}]$ is the strong coreflective hull of $\mathcal{W}$.

Proof.

- $1 \Longrightarrow 2$: By Proposition 3.3.1, $\mathcal{W}_l[\mathcal{C}]$ is the strong coreflective hull of $\mathcal{W}$ in $\mathcal{C}$.
- $2 \Longrightarrow 3$: Let $\mathcal{C}^0$ be the strong coreflective hull of $\mathcal{W}$ in $\mathcal{C}$, $\mathcal{W} \xrightarrow{J^0} \mathcal{C}^0$ the inclusion functor and $(U \dashv G)$ a coreflection of $\mathcal{C}$ on $\mathcal{C}^0$. The subcategory $\mathcal{W}$ is dense in $\mathcal{C}^0$, thus J^0 has a trivial density comonad (dual of Example 2.4.3).

  The functor $1_{\mathcal{C}^0} : \mathcal{C}^0 \longrightarrow \mathcal{C}^0$ is a left pointwise Kan extension of J^0 along itself. The functor G is a right adjoint of U, thus by [36, Proposition 6.5.2], G is a left pointwise Kan extension of $1_{\mathcal{C}^0}$ along U. Therefore G is a left pointwise Kan extension of J^0 along $J = UJ^0$. The functor U is a left adjoint functor, therefore it preserves left pointwise Kan extensions. It follows that $L = UG$ is a density comonad of J.

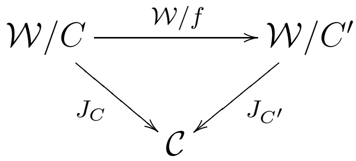
  Let ϵ be the counit of the comonad L. By Proposition 3.3.3, $\epsilon : L = UG \Longrightarrow 1_{\mathcal{C}}$ is the counit of the coreflection (U, G). By Lemma 1.4.2, $\epsilon_C : L(C) \longrightarrow C$ is $\mathcal{W}$-monic for all $C \in \mathcal{C}$.
- $3 \Longrightarrow 1$: For $C \in \mathcal{C}$, let $\mathcal{W}/C$ be the subcategory of the over category $\mathcal{C}/C$ whose objects are arrows $V \longrightarrow C$ with domain $V \in \mathcal{W}$. Define $D_C : \mathcal{W}/C \longrightarrow \mathcal{W}$ to be the functor which associates to an arrow-object $V \longrightarrow C$ its domain V and let J_C be the composite functor

  $$J_C : \mathcal{W}/C \xrightarrow{D_C} \mathcal{W} \xrightarrow{J} \mathcal{C}$$

The functor J has a density comonad (L, ϵ, δ). Therefore by the dual of [32, Theorem 3 page 244],

$$\forall C \in \mathcal{C}, \quad \mathsf{colim} J_C \text{ exists and } L(C) = \mathsf{colim} J_C.$$

A morphism $C \xrightarrow{f} C'$ in $\mathcal{C}$ induces a functor $\mathcal{W}/C \xrightarrow{\mathcal{W}/f} \mathcal{W}/C'$ rendering commutative the diagram



This last diagram induces a map

$$\mathsf{colim} J_C \longrightarrow \mathsf{colim} J_{C'}$$

which is just

$$L(f) : L(C) \longrightarrow L(C').$$

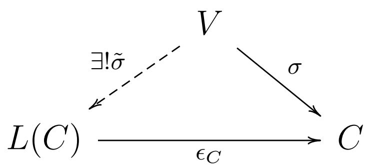
Let $\eta : J \Longrightarrow LJ$ be the unit of the left Kan extension L of J along itself. The functor J is fully faithful. By the dual of [32, Corollary 3 page 239], η is an isomorphism. We may therefore assume that

$$L(V) = V \text{ and } \eta_V = 1_V : V \longrightarrow V, \quad \text{for all } V \in \mathcal{W}.$$

In which case, by the commutativity of the diagram which is dual to (4),

$$\epsilon_V = 1_V : V \longrightarrow V, \quad \text{for all } V \in \mathcal{W}.$$

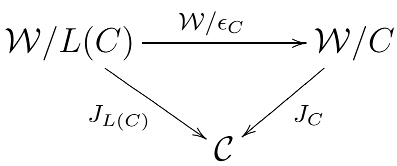
Therefore by the naturality of ϵ, for each $C \in \mathcal{C}$ and each arrow-object $\sigma : V \longrightarrow C$ in $\mathcal{W}/C$, there exists a map $\tilde{\sigma} : V \longrightarrow L(C)$ rendering commutative the diagram



Moreover, $\tilde{\sigma}$ is unique since ϵ_C is $\mathcal{W}$-monic. It follows that the functor

$$\mathcal{W}/L(C) \xrightarrow{\mathcal{W}/\epsilon_C} \mathcal{W}/C$$

is an isomorphism. The following diagram commutes



Therefore the map $L(\epsilon_C) : L^2(C) \longrightarrow L(C)$ is an isomorphism. By the dual of Proposition 2.2, L is an idempotent comonad and $\mathcal{W}$ is left Kan extendable in $\mathcal{C}$.

□

Example 3.7. *The subcategory $\mathcal{Z}$ of* Vect *of Example 1.11 has no strong coreflective hull. By Theorem 3.6, $\mathcal{Z}$ is not left Kan extendable.*

Corollary 3.8. *Let $\mathcal{W}$ be a left Kan extendable subcategory of a category $\mathcal{C}$ and assume that $\mathcal{C}^0$ is a replete coreflective subcategory of $\mathcal{C}$ containing $\mathcal{W}$. Then $\mathcal{W}$ is left Kan extendable as a subcategory $\mathcal{C}^0$. Furthermore, $\mathcal{W}_l[\mathcal{C}^0] = \mathcal{W}_l[\mathcal{C}]$.*

Proof. By Theorem 3.6, $\mathcal{W}_l[\mathcal{C}]$ is the coreflective hull of $\mathcal{W}$. The subcategory $\mathcal{C}^0$ is a replete, coreflective subcategory of $\mathcal{C}$ containing $\mathcal{W}$, thus $\mathcal{W}_l[\mathcal{C}]$ is a subcategory of $\mathcal{C}^0$, which is replete coreflective as a subcategory of $\mathcal{C}^0$, in which $\mathcal{W}$ is dense. By Theorem 1.8.2, $\mathcal{W}_l[\mathcal{C}]$ is the strong coreflective hull of $\mathcal{W}$ in $\mathcal{C}^0$. By Theorem 3.6, $\mathcal{W}$ is left Kan extendable in $\mathcal{C}^0$ and $\mathcal{W}_l[\mathcal{C}^0] = \mathcal{W}_l[\mathcal{C}]$. $\square$

Corollary 3.9. *Let $\mathcal{W}$, $\mathcal{W}'$ be left Kan extendable subcategories of $\mathcal{C}$.*

1. *If $\mathcal{W}' \subset \mathcal{W}_l[\mathcal{C}]$, then $\mathcal{W}'_l[\mathcal{C}]$ is a coreflective subcategory of $\mathcal{W}_l[\mathcal{C}]$.*
2. *If $\mathcal{W}' \subset \mathcal{W}_l[\mathcal{C}]$ and $\mathcal{W} \subset \mathcal{W}'_l[\mathcal{C}]$, then $\mathcal{W}_l[\mathcal{C}] = \mathcal{W}'_l[\mathcal{C}]$.*

Proof.

1. By Corollary 3.8, $\mathcal{W}'$ is left Kan extendable as a subcategory of $\mathcal{W}_l[\mathcal{C}]$ and $\mathcal{W}'_l[\mathcal{W}_l[\mathcal{C}]] = \mathcal{W}'_l[\mathcal{C}]$. By Proposition 3.3, $\mathcal{W}'_l[\mathcal{C}]$ is a coreflective subcategory of $\mathcal{W}_l[\mathcal{C}]$.
2. This follows from 1.

$\square$

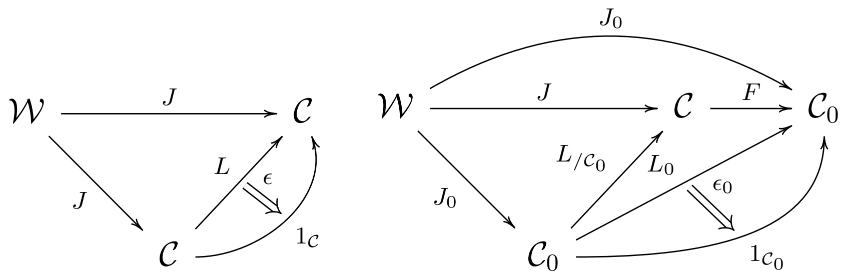
Theorem 3.10. *Let $\mathcal{C}_0$ be a reflective subcategory of $\mathcal{C}$, $\mathcal{W}$ a left Kan extendable subcategory of $\mathcal{C}$ contained in $\mathcal{C}_0$ and $(U \dashv F_L)$ the coreflection of $\mathcal{C}$ on $\mathcal{W}_l[\mathcal{C}]$ given by Proposition 3.3. Assume further that $F_L(\mathcal{C}_0) \subset \mathcal{C}_0$. Then:*

1. *The subcategory $\mathcal{W}$ is left Kan extendable as a subcategory of $\mathcal{C}_0$.*
2. *A reflection of $\mathcal{C}$ on $\mathcal{C}_0$ induces a reflection of $\mathcal{W}_l[\mathcal{C}]$ on $\mathcal{W}_l[\mathcal{C}_0]$.*
3. *We have $\mathcal{W}_l[\mathcal{C}_0] = \mathcal{C}_0 \cap \mathcal{W}_l[\mathcal{C}]$.*
4. *The coreflection $(U \dashv F_L)$ of $\mathcal{C}$ on $\mathcal{W}_l[\mathcal{C}]$ induces a coreflection of $\mathcal{C}_0$ on $\mathcal{W}_l[\mathcal{C}_0]$.*

Proof. Let $J : \mathcal{W} \longrightarrow \mathcal{C}$, $J_0 : \mathcal{W} \longrightarrow \mathcal{C}_0$ be the inclusion functors and $(F \dashv V)$ a reflection of $\mathcal{C}$ on $\mathcal{C}_0$. We may, by Lemma 1.2.1, assume that the composite $\mathcal{C}_0 \xrightarrow{V} \mathcal{C} \xrightarrow{F} \mathcal{C}_0$ is the identity $1_{\mathcal{C}_0}$ functor. We then have $FJ = J_0$.

1. Let (L, ϵ, δ) be a density comonad of J. The subcategory $\mathcal{C}_0$ of $\mathcal{C}$ contains $\mathcal{W}$, thus J has a pointwise left Kan extension along $J_0 : \mathcal{W} \longrightarrow \mathcal{C}_0$ which is $L_{/\mathcal{C}_0}$. The functor F is left adjoint, it is then cocontinuous and therefore preserves left pointwise Kan extensions. Thus $J_0 = FJ$ has a pointwise left Kan extension along itself which is $L_0 = FL_{/\mathcal{C}_0}$. We have $L_{/\mathcal{C}_0}(\mathcal{C}_0) = L(\mathcal{C}_0) = F_L(\mathcal{C}_0) \subset \mathcal{C}_0$ and $F_{/\mathcal{C}_0} = 1_{\mathcal{C}_0}$, therefore L induces an endofunctor of $\mathcal{C}_0$ which is simply the functor L_0. Let $\epsilon_0 : L_0 \longrightarrow 1_{\mathcal{C}_0}$ be the natural transformation induced by ϵ. Clearly, ϵ_0 is the counit of the density

comonad L_0. By Theorem 3.6, ϵ is $\mathcal{W}$-monic, hence ϵ_0 is $\mathcal{W}$-monic. Again, by Theorem 3.6, $\mathcal{W}$ is left Kan extendable as a subcategory of $\mathcal{C}_0$.



2. We just need to prove that $\mathcal{W}_l[\mathcal{C}_0] \subset \mathcal{W}_l[\mathcal{C}]$ and $F(\mathcal{W}_l[\mathcal{C}]) \subset \mathcal{W}_l[\mathcal{C}_0]$. Let $X_0 \in \mathcal{W}_l[\mathcal{C}_0]$. By the dual of Proposition 2.3, $\epsilon_{X_0} = (\epsilon_0)_{X_0}$ is an isomorphism, therefore $X_0 \in \mathcal{W}_l[\mathcal{C}]$ and $\mathcal{W}_l[\mathcal{C}_0] \subset \mathcal{W}_l[\mathcal{C}]$. Let $X \in \mathcal{W}_l[\mathcal{C}]$ and $J_X : \mathcal{W}/X \longrightarrow \mathcal{C}$ be the functor which takes an arrow-object $V \to X$ in $\mathcal{W}/X$ to its domain V. The functor F preserves colimits, thus $\mathrm{colim} FJ_X \cong F(X)$. The functor FJ_X takes values in $\mathcal{W}$, by Corollary 3.5, $\mathrm{colim} FJ_X \in \mathcal{W}_l[\mathcal{C}_0]$. It follows that $F(X) \in \mathcal{W}_l[\mathcal{C}_0]$ and $F(\mathcal{W}_l[\mathcal{C}]) \subset \mathcal{W}_l[\mathcal{C}_0]$.

3. We have $\mathcal{W}_l[\mathcal{C}_0] \subset \mathcal{W}_l[\mathcal{C}]$, thus $\mathcal{W}_l[\mathcal{C}_0] \subset \mathcal{C}_0 \cap \mathcal{W}_l[\mathcal{C}]$. The induced functor $F_{/\mathcal{C}_0} = 1_{\mathcal{C}_0}$, thus $\mathcal{C}_0 \cap \mathcal{W}_l[\mathcal{C}] = F(\mathcal{C}_0 \cap \mathcal{W}_l[\mathcal{C}]) \subset F(\mathcal{W}_l[\mathcal{C}]) \subset \mathcal{W}_l[\mathcal{C}_0]$. Therefore $\mathcal{W}_l[\mathcal{C}_0] = \mathcal{C}_0 \cap \mathcal{W}_l[\mathcal{C}]$.

4. One has $F_L(\mathcal{C}_0) \subset \mathcal{C}_0 \cap \mathcal{W}_l[\mathcal{C}] = \mathcal{W}_l[\mathcal{C}_0]$. Thus $F_L(\mathcal{C}_0) \subset \mathcal{W}_l[\mathcal{C}_0]$ and the result follows.

□

We next introduce the dual notion of right Kan extendable subcategories.

Definition 3.11. *A subcategory $\mathcal{W}$ of a category $\mathcal{C}$ is said to be right Kan extendable provided that:*

1. *The inclusion functor $J : \mathcal{W} \longrightarrow \mathcal{C}$ has a codensity monad (R, η, μ).*

2. *The monad (R, η, μ) is idempotent.*

Examples of right Kan extendable subcategories are given in the next section.

Let $\mathcal{W}$ be a right Kan extendable subcategory of a category $\mathcal{C}$ and (R, η, μ) the codensity monad of the inclusion functor $J : \mathcal{W} \longrightarrow \mathcal{C}$. Define $\mathcal{W}_r[\mathcal{C}]$ to be the category of R-algebras. Then by proposition 2.3, $\mathcal{W}_r[\mathcal{C}]$ may be viewed as a replete subcategory of $\mathcal{C}$. It is called the subcategory of $\mathcal{W}$-cogenerated objects of $\mathcal{C}$.

Corollary 3.12. *Let $\mathcal{W}$ be a right Kan extendable subcategory of $\mathcal{C}$, (R, η, μ) the codensity monad of the inclusion functor $J : \mathcal{W} \longrightarrow \mathcal{C}$ and $U : \mathcal{W}_r[\mathcal{C}] \longrightarrow \mathcal{C}$ the forgetful functor. Then*

1. *The subcategory $\mathcal{W}_r[\mathcal{C}]$ is the strong reflective hull of $\mathcal{W}$ in $\mathcal{C}$.*

2. *The free R-algebra functor $F^R : \mathcal{C} \longrightarrow \mathcal{W}_r[\mathcal{C}]$ is a reflector.*

3. *The reflection $(F^R \dashv U)$ has η as its unit.*

Proof. This is the dual of Proposition 3.3. □

4 Compactifications

Stone–Čech compactification and Hewitt realcompactification are procedures exhibiting the subcategories of compact Hausdorff and realcompact spaces as reflective subcategories of Top. Our objective in this section is to show how can these two facts be established using the notion of Kan extendable subcategories. We begin with the following technical result.

Lemma 4.1. *Let $\mathcal{C}$ be a complete category, $\mathcal{C}_0$ is a subcategory of $\mathcal{C}$ and $J : \mathcal{C}_0 \longrightarrow \mathcal{C}$ the inclusion functor. Assume that for any small category $\mathcal{I}$ and any functor $F : \mathcal{I} \longrightarrow \mathcal{C}_0$, the limit of the composite functor JF is in $\mathcal{C}_0$. Then:*

1. *The subcategory $\mathcal{C}_0$ is complete.*
2. *The functor $J : \mathcal{C}_0 \longrightarrow \mathcal{C}$ preserves and creates small limits.*

Observe that such a subcategory $\mathcal{C}_0$ is necessarily replete.

Proof. Clear. □

Theorem 4.2. *Let $\mathcal{C}$ be a complete category, $\mathcal{W}$ a small subcategory of $\mathcal{C}$, $\mathcal{C}_0$ a subcategory of $\mathcal{C}$ containing $\mathcal{W}$ as a codense subcategory and $J_0 : \mathcal{C}_0 \longrightarrow \mathcal{C}$ the inclusion functor. Assume further that for any small category $\mathcal{I}$ and any functor $F : \mathcal{I} \longrightarrow \mathcal{C}_0$, the limit of the composite functor J_0F is in $\mathcal{C}_0$. Then $\mathcal{C}_0$ is the strong reflective hull of $\mathcal{W}$.*

Proof. The category $\mathcal{C}$ is complete and $\mathcal{W}$ is small, therefore the inclusion functor $J : \mathcal{W} \longrightarrow \mathcal{C}$ has a codensity monad (R, η, μ). By hypothesis, $R(\mathcal{C}) \subseteq \mathcal{C}_0$. Moreover, the subcategory $\mathcal{W}$ is codense in $\mathcal{C}_0$. Therefore by lemma 4.1, for each $X \in \mathcal{C}$,

$$\text{the morphism } \eta_X : X \longrightarrow R(X) \text{ is an isomorphism iff } X \in \mathcal{C}_0. \tag{9}$$

As observed above, the functor R takes $\mathcal{C}$ into $\mathcal{C}_0$. Therefore by (9), $\eta R : R \longrightarrow R^2$ is an isomorphism. By Proposition 2.2, R is an idempotent monad. It follows that $\mathcal{W}$ is right Kan extendable.

By Proposition 2.3, and object $X \in \mathcal{C}$ has an R-algebra structure iff $\eta_X : X \longrightarrow R(X)$ is an isomorphism. Therefore by (9), $\mathcal{W}_r[\mathcal{C}] = \mathcal{C}_0$. By the dual of Theorem 3.6, $\mathcal{C}_0$ is the strong reflective hull of $\mathcal{W}$. □

As before, let Comp be the subcategory of Top of compact Hausdorff spaces and let I the unit interval, $I^2 = I \times_{\mathsf{Top}} I$ and Square the subcategory of Top having I^2 as its unique object. The following result strengthens the standard Stone–Čech compactification

Corollary 4.3. *The subcategory* Comp *of* Top *is the strong reflective hull of* Square.

Proof. Let J : Comp $\longrightarrow$ Top be the inclusion functor, $\mathcal{I}$ be a small category and $F : \mathcal{I} \longrightarrow$ Comp a functor. The limit of JF is clearly in Comp. By (Isbell, [22, Theorem 2.6]), Square is a codense subcategory of Comp. By Theorem 4.2, Comp is the strong reflective hull of Square. □

Remark 4.4. *An algebraic example of a coreflective hull which is not strong is given in Example 1.11. We next provide another example which is topological.*

Let U *be the subcategory of* Top *having the unit interval I as its unique object. The subcategory* Comp *of* Top *is reflective and contains* U*. Let* top *be a reflective subcategory of* Top *containing* U*. Clearly,* top *contains* Square*, therefore it contains the reflective hull of* Square *which is* Comp*. It follows that* Comp *is precisely the reflective hull of* U*. By [22, Theorem 2.6],* U *is not dense in* Comp*. Therefore* U *has no strong reflective hull.*

Let Rng be the category of commutative rings and let

$$C : \mathsf{Top}^{op} \longrightarrow \mathsf{Rng}$$

be the functor which takes a space X to the ring of real-valued continuous maps defined on X. Recall that a topological space is said to be realcompact if it is homeomorphic to a closed subspace of a product of real lines [16, 11.12]. Let Rcomp be the subcategory of Top of realcompact spaces.

Theorem 4.5. *([16, Theorem 10.6])*

The restriction functor $C_/$: $\mathsf{Rcomp}^{op} \longrightarrow \mathsf{Rng}$ of C is fully faithful.

Let P be the subcategory of Rcomp having precisely one object which is $\mathbb{R}^2 = \mathbb{R} \times_{\mathsf{Top}} \mathbb{R}$.

Theorem 4.6. *The subcategory* P *of* Rcomp *is codense.*

Proof. The proof is based on Theorem 4.5, and is strictly similar to Isbell's proof of the fact that Square is codense in Comp [22, Theorem 2.6]. □

The following result strengthens the standard Hewitt Realcompactification.

Corollary 4.7. *The subcategory* Rcomp *of* Top *is the strong reflective hull of* P*.*

Proof. Top is complete and Rcomp is a subcategory Top containing P as a codense subcategory. The product in Top of a small set of realcompact spaces is realcompact. Similarly, the equalizer in Top of two parallel maps in Rcomp is again in Rcomp. Therefore by Theorem 4.2, Rcomp is the strong reflective hull of P. □

5 Reflective subcategories of Top_B

In this section, we apply the theory developed previously to prove that subcategories of fibrewise topological spaces over B satisfying certain separation axioms are reflective subcategories of Top_B.

Recall that a subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is said to be closed under subobjects if whenever we have a monomorphism $X \longrightarrow Y$ in $\mathcal{C}$ with codomain $Y \in \mathcal{A}$, then X is isomorphic to an object of $\mathcal{A}$. Observe that the next theorem may also be derived from [1, Theorem 16.8].

Theorem 5.1. *Let top_B be a subcategory of Top_B such that:*

1. *top_B is replete and contains the fibrewise topological space B (over itself).*
2. *top_B is closed under subobjects as a subcategory of Top_B.*
3. *For every family $(V_i)_{i\in I}$ of objects of top_B (indexed by a small set I), the product $\prod_{\mathsf{Top}_B}^{i\in I} V_i$ is an object of top_B.*

Then top_B is a reflective subcategory of Top_B. In particular, top_B is bicomplete. Furthermore, the unit η of this reflection is such that the maps $\eta_X : X \to R(X)$ are quotient maps, where $R : \mathsf{Top}_B \longrightarrow \mathsf{top}_B$ is a reflector.

Proof. Let $X \in \mathsf{Top}_B$ and let $J^X : X/\mathsf{top}_B \longrightarrow \mathsf{Top}_B$ be the functor which takes an arrow-object $X \longrightarrow V$ to its codomain V. Define $\mathcal{R}_X$ to be the equivalence relation on X given by $x_1\mathcal{R}_X x_2$ iff $f(x_1) = f(x_2)$ for every continuous fibrewise map f from X to any fibrewise topological space in top_B. The projection $p_X : X \longrightarrow B$ defines a continuous fibrewise map from X to B as follows:

$$\begin{array}{ccc} X & \xrightarrow{\quad p_X \quad} & B \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle 1_B} & \\ & B & \end{array}$$

Therefore if $x_1\mathcal{R}_X x_2$, then $p_X(x_1) = p_X(x_2)$. It follows that the projection p_X factors through $X/\mathcal{R}_X$ as follows:

$$\begin{array}{ccc} X & \xrightarrow{\quad \eta_X \quad} & X/\mathcal{R}_X \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle \widetilde{p}_X} & \\ & B & \end{array}$$

In other words, $\mathcal{R}_X$ is a fibrewise equivalence relation on X, thus $X/\mathcal{R}_X$ is a fibrewise topological space over B and the quotient map $\eta_X : X \longrightarrow X/\mathcal{R}_X$ is a fibrewise map. Define $A_X = \{\{x_1, x_2\} \subset X \mid p_X(x_1) = p_X(x_2) \text{ and } x_1 \not\mathcal{R}_X x_2\}$ and let $(f_i)_{i\in I}$ be a family of maps in Top_B such that:

- $f_i : X \longrightarrow V_i$, where $V_i \in \mathsf{top}_B$ for all $i \in I$.
- I is small and nonempty.
- For each $\{x_1, x_2\} \in A_X$, there exists $i \in I$ such that $f_i(x_1) \neq f_i(x_2)$.

Define $f : X \longrightarrow \prod_{\mathsf{Top}_B}^{i\in I} V_i$ to be the map whose i-component is f_i. Observe that $f(x_1) = f(x_2) \Leftrightarrow x_1\mathcal{R}_X x_2$. Thus there exists a unique continuous fibrewise map $\tilde{f} : X/\mathcal{R}_X \longrightarrow \prod_{\mathsf{Top}_B}^{i\in I} V_i$ rendering commutative the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & \prod\limits_{\mathsf{Top}_B}^{i\in I} V_i \\ {\scriptstyle \eta_X}\big\downarrow & \nearrow_{\tilde{f}} & \\ X/\mathcal{R}_X & & \end{array}$$

$\prod\limits_{\mathsf{Top}_B}^{i\in I} V_i \in \mathsf{top}_B$, $\tilde{f}$ is monic and top_B is closed under subobjects. Therefore $X/\mathcal{R}_X \in \mathsf{top}_B$ and the arrow $X \xrightarrow{\eta_X} X/\mathcal{R}_X$ is an object of X/top_B which is initial. Then clearly, $\lim J^X \cong X/\mathcal{R}_X \in \mathsf{top}_B$ exists. It follows that the inclusion functor $\mathsf{top} \xrightarrow{J} \mathsf{Top}_B$ has a codensity monad R given by $R(X) \cong X/\mathcal{R}_X \in \mathsf{top}_B$, with unit the natural transformation $\eta : 1_{\mathsf{Top}} \Longrightarrow R$ whose component along X is the quotient map $X \xrightarrow{\eta_X} X/\mathcal{R}_X$ which is epic. By the dual of Theorem 3.6, top_B is right Kan extendable in Top_B. The codensity monad R of the inclusion functor $J : \mathsf{top}_B \longrightarrow \mathsf{Top}_B$ takes values in top_B which is replete. By Proposition 2.3, $\mathsf{top}_B[\mathsf{Top}_B] \subset \mathsf{top}_B$. If $X \in \mathsf{top}_B$, then $\mathcal{R}_X$ is the trivial equivalence relation and $\eta_X = 1_X$. Therefore by Proposition 2.3.3, $X \in \mathsf{top}_B[\mathsf{Top}_B]$ and $\mathsf{top}_B \subset \mathsf{top}_B[\mathsf{Top}_B]$. It follows that $\mathsf{top}_B[\mathsf{Top}_B] = \mathsf{top}_B$. By Corollary 3.12, top_B is a reflective subcategory of Top_B with reflector $F^R : \mathsf{Top}_B \longrightarrow \mathsf{top}_B$ the functor induced by R. Furthermore, the reflection of Top_B on top_B has unit η which is an objectwise quotient map. □

Examples 5.2. *According to James [25, Chapter I, section 2], a fibrewise topological space X is said to be fibrewise*

- *Fréchet (or T_1) if each fibre X_b of X is an ordinary T_1-topological space. The category of fibrewise Fréchet spaces is denoted by fTop_B.*
- *Hausdorff (or T_2) if any two distinct points of X laying in the same fibre can be separated by neighborhoods in X. The category of fibrewise Hausdorff spaces is denoted by hTop_B.*

Observe that if X is a fibrewise T_i-space over B, $i = 1, 2$ and B is an ordinary T_i-space, then X is a T_i-space in the ordinary sense.

Similarly, define a fibrewise topological space X to be fibrewise

- *Urysohn space (or $\mathsf{T}_{2\frac{1}{2}}$) if any two distinct points of X laying in the same fibre can be separated by closed neighborhoods in X. The category of fibrewise Urysohn spaces is denoted by uTop_B.*
- *completely Hausdorff space (or functionally Hausdorff space) if any two distinct points of X laying in the same fibre can be separated by a continuous function (or equivalently, by a continuous fibrewise map into $B \times_{\mathsf{Top}} \mathbb{R}$). The category of fibrewise completely Hausdorff spaces is denoted by $\mathsf{h_cTop}_B$.*

By Theorem 5.1, the categories fTop_B, hTop_B, uTop_B and $\mathsf{h_cTop}_B$ are reflective subcategories of Top_B.

A one point space pt is a terminal object of Top. Therefore one has the standard isomorphism

$$P : \mathsf{Top}_{\mathsf{pt}} \longrightarrow \mathsf{Top}. \tag{10}$$

By substituting pt for B, Theorem 5.1 reduces to the following.

Corollary 5.3. *Let* top *be a subcategory of* Top *such that:*

1. top *is replete and contains a nonempty space.*
2. top *is closed under subobjects as a subcategory of* Top.
3. *For every family* $(V_i)_{i\in I}$ *of objects of* top *(indexed by a small set* I*), the product* $\prod_{\mathsf{Top}}^{i\in I} V_i$ *is an object of* top.

Then top *is a reflective subcategory of* Top. *In particular,* top *is bicomplete. Furthermore, the unit* η *of this reflection is such that the map* $\eta_X : X \longrightarrow R(X)$*,* $X \in \mathsf{Top}$*, is a quotient map, where* $R : \mathsf{Top} \longrightarrow \mathsf{top}$ *is the reflector.*

Examples 5.4. *A space* $X \in \mathsf{Top}$ *is Fréchet (resp. Hausdorff, Urysohn, completely Hausdorff) if it corresponds, under the isomorphism* P *of (10), to a fibrewise Fréchet (resp. Hausdorff, Urysohn, completely Hausdorff) space over* pt. *The subcategory of* Top *of such spaces is reflective and is denoted by* fTop *(resp.* hTop, uTop, $\mathsf{h_c}$Top*).*

6 Fibrewise compact spaces

Let $\mathcal{W}$ be a left Kan extendable subcategory of a category $\mathcal{C}$. One of the main objectives of this paper is to present sufficient conditions, under which, the category $\mathcal{W}_l[\mathcal{C}]$ is cartesian closed. Among other conditions, one requires that the objects of $\mathcal{W}$ be exponentiable as objects of the category $\mathcal{C}$. To be able to apply this result to prove that the category of fibrewise compactly generated spaces over a T_1-base B is cartesian closed, one then needs to prove that a fibrewise compact fibrewise Hausdorff space over B is an exponentiable object of Top_B. This last result is precisely what this section is after.

We begin by introducing the notion of fibrewise compact spaces and recalling their relevant properties. The main references of what is discussed here are the books of Bourbaki [8, Chapter I, Section 10] and James [25, Chapter I].

Recall that a continuous map $f : X \longrightarrow Y$ between two topological spaces X and Y is said to be proper if the product map $f \times_{\mathsf{Top}} 1_Z : X \times_{\mathsf{Top}} Z \longrightarrow Y \times_{\mathsf{Top}} Z$ is closed for all $Z \in \mathsf{Top}$ [8, Section 10.1]. A fibrewise space X over the fixed topological space B is said to be fibrewise compact if its projection $p : X \longrightarrow B$ is a proper map.

The next proposition is an immediate consequence of [8, Proposition 5.b, Section 10.1].

Proposition 6.1. *A continuous map Let* $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$ *be continuous maps. If* $g \circ f$ *is proper, then the map* $f(X) \longrightarrow Z$ *induced by* g *is proper.*

The next theorem presents a criteria for a continuous map to be proper.

Theorem 6.2. *[8, Theorem 1, Section 10.2]*

A continuous map $f : X \longrightarrow Y$ *is proper iff* f *is closed and* $f^{-1}(y)$ *is compact for all* $y \in Y$.

Proposition 6.3. *Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be two continuous fibrewise maps over B. Assume that f and f' are proper. Then the map*

$$f \times_{\mathsf{Top}_B} f' : X \times_{\mathsf{Top}_B} X' \longrightarrow Y \times_{\mathsf{Top}_B} Y'$$

is proper.

Proof. The maps f and f' are proper. By [8, Proposition 4, Section 10.1], the product map $f \times_{\mathsf{Top}} f' : X \times_{\mathsf{Top}} X' \longrightarrow Y \times_{\mathsf{Top}} Y'$ is proper. The commutative diagram

$$\begin{array}{ccc} X \times_{\mathsf{Top}_B} X' & \hookrightarrow & X \times_{\mathsf{Top}} X' \\ {\scriptstyle f \times_{\mathsf{Top}_B} f'} \downarrow & & \downarrow {\scriptstyle f \times_{\mathsf{Top}} f'} \\ Y \times_{\mathsf{Top}_B} Y' & \hookrightarrow & Y \times_{\mathsf{Top}} Y' \end{array}$$

is a pullback diagram in Top. By [8, Proposition 3, Section 10.1], the map

$$f \times_{\mathsf{Top}_B} f' : X \times_{\mathsf{Top}_B} X' \longrightarrow Y \times_{\mathsf{Top}_B} Y'$$

is proper. □

Corollary 6.4. *Let X and Y be two fibrewise compact spaces over B. Then $X \times_{\mathsf{Top}_B} Y$ fibrewise compact.*

Proposition 6.5. *[25, Proposition 2.7]*

A fibrewise space X is fibrewise Hausdorff iff its diagonal Δ_X is closed in $X \times_{\mathsf{Top}_B} X$.

Definition 6.6. *[25, Definition 2.15]*

A fibrewise topological space $p : X \longrightarrow B$ is fibrewise regular if for each point $x_0 \in X$, and for each open neighborhood V of x_0 in X, there exist an open neighborhood Ω of $b_0 = p(x_0)$ in B and an open neighborhood U of x_0 in X such that $\overline{U} \cap X_\Omega \subset V$.

Proposition 6.7. *[25, Proposition 3.19] Let $\phi : K \longrightarrow X$ be a continuous fibrewise map, where K is fibrewise compact and X is fibrewise Hausdorff over B. Then ϕ is a proper map. In particular,*

1. *$\phi(K)$ is closed in X.*
2. *$\phi(K)$ is fibrewise compact fibrewise Hausdorff over B.*

Corollary 6.8. *[25, Corollary 3.20] A fibrewise compact subspace of a fibrewise Hausdorff space is closed.*

Corollary 6.9. *A subspace of a fibrewise compact fibrewise Hausdorff space is fibrewise compact iff it is closed.*

Proof. The result follows from Corollary 6.8 and Theorem 6.2. □

Proposition 6.10. *([8, Proposition 6 page 104]) Let $p : X \longrightarrow B$ be a proper map and let K be a compact subspace of B, then $p^{-1}(K)$ is a compact subspace of X.*

Proposition 6.11. *[25, Proposition 3.22] Every fibrewise compact, fibrewise Hausdorff space over B is fibrewise regular.*

The next result reduces to the standard tube lemma [33, Lemma 26.8] in the case where B is a one point space.

Lemma 6.12. *(A fibrewise tube lemma)*

Let X and K be two fibrewise spaces over B with K fibrewise compact. Let $x_0 \in X$, O an open subset of $X \times_{\mathsf{Top}_B} K$ and assume that $\{x_0\} \times_{\mathsf{Top}_B} K \subset O$. Then there exists an open neighborhood V of x_0 in X such that $V \times_{\mathsf{Top}_B} K \subset O$.

Proof.

- Case 1: $X = B$ and $x_0 = b_0 \in B$.
  Observe that $B \times_{\mathsf{Top}_B} K = K$, $\{b_0\} \times_{\mathsf{Top}_B} K = K_{b_0}$ and O is an open subset of K containing K_{b_0}. Define C be the closed subset of K given by $C = K \setminus O$. The projection $p_K : K \longrightarrow B$ is a proper map, it is therefore closed. It follows that $p_K(C)$ is closed and does not contain b_0. Define $V = B \setminus p_K(C)$. Then clearly, V is an open neighborhood of b_0 and $V \times_{\mathsf{Top}_B} K = p_K^{-1}(V) = K_V \subset O$ as desired.

- Case 2: The general case.
  Let $p_X : X \longrightarrow B$ be the projection of the fibrewise space X and let $b_0 = p_X(x_0)$. We have $\{x_0\} \times_{\mathsf{Top}_B} K = \{x_0\} \times_{\mathsf{Top}_B} K_{b_0} \subset O$. For every $y \in K_{b_0}$, there exist open neighborhoods U_y of x_0 in X and W_y of y in K such that $U_y \times_{\mathsf{Top}_B} W_y \subset O$. The family $(W_y)_{y \in K_{b_0}}$ is an open cover of K_{b_0} which is compact. There exist $y_1, y_2, \ldots, y_n \in K_{b_0}$ such that $K_{b_0} \subset \bigcup_{i=1}^n W_{y_i}$. Define $U = \bigcap_{i=1}^n U_{y_i}$ and $W = \bigcup_{i=1}^n W_{y_i}$. Then U is an open neighborhood of x_0, W is an open subset of K containing K_{b_0} and $U \times_{\mathsf{Top}_B} W \subset O$. By Case 1, there exists an open subset Ω of B such that $K_\Omega \subset W$. Define $V = X_\Omega \cap U$, then

$$V \times_{\mathsf{Top}_B} K = U \times_{\mathsf{Top}_B} K_\Omega \subset U \times_{\mathsf{Top}_B} W \subset O.$$

□

We next present a special case of the fibrewise compact-open topology defined in [25, page 64], (see also [34, page 152]).

Let $K, Y \in \mathsf{Top}_B$ with K fibrewise compact, fibrewise Hausdorff space. A subspace of K (or Y) may be viewed as a fibrewise space over B. For Ω open in B, C closed in K and O open in Y, let

$$(C, O, \Omega) = \coprod_{\mathsf{Set}}^{b \in \Omega} \{\gamma \in \mathsf{Top}(K_b, Y_b) \mid \gamma(C_b) \subset O_b\}. \tag{11}$$

Define $\mathrm{map}_B(K, Y)$ to be the topological space whose underlying set is $\coprod_{\mathsf{Set}}^{b \in B} \mathsf{Top}(K_b, Y_b)$ and whose topology is generated [2] by the subsets (C, O, Ω), where Ω is open in B, C is closed in K and O is open in Y.

2 The topology of $\mathrm{map}_B(K, Y)$ is then the coarsest topology on the set $\coprod_{\mathsf{Set}}^{b \in B} \mathsf{Top}(K_b, Y_b)$ containing (C, O, Ω)'s as open subsets.

Our definition agrees with that of James mentioned above with the difference that in our case, $\mathrm{map}_B(K,Y)$ is only defined when K is fibrewise compact, while in [25], $\mathrm{map}_B(X,Y)$ is defined for any fibrewise space X in precisely the same way.

Open subsets given by (11) are called elementary open subsets of $\mathrm{map}_B(K,Y)$. For $b \in B$, let $\mathrm{map}(K_b, Y_b)$ be the subspace of $\mathrm{map}_B(K,Y)$ whose underlying set is $\mathsf{Top}(K_b, Y_b)$[3]. Define

$$p_{\mathrm{map}_{B(K,Y)}} : \mathrm{map}_B(K,Y) \longrightarrow B \tag{12}$$

to be the map whose fibre over b is $\mathrm{map}(K_b, Y_b)$. Let Ω be open in B, then

$$p^{-1}_{\mathrm{map}_{B(K,Y)}}(\Omega) = \coprod_{\mathsf{Set}}^{b\in B} \mathsf{Top}(K_b, Y_b) = (K, Y, \Omega)$$

is open in $\mathrm{map}_B(K,Y)$. It follows that $p_{\mathrm{map}_{B(K,Y)}}$ is continuous. The space $\mathrm{map}_B(K,Y)$ is therefore viewed as a fibrewise space over B.

Example 6.13. *For $b \in B$, let B^b be the fibrewise subspace of B having b as its unique point. Then B^b is fibrewise Hausdorff. Assume that B is T_1. Then by Theorem 6.2, B^b is fibrewise compact. If $Z \in \mathsf{Top}_B$, then $map_B(B^b, Z)$ is a fibrewise space over B. It is such that*

$$map_B(B^b, Z)_{b'} \cong \begin{cases} Z_b & \text{if } b' = b \\ \text{One point space} & \text{if } b' \neq b \end{cases} \tag{13}$$

The next proposition is a special case of that of James [25, Corollary 9.13].

Proposition 6.14. *Let K, Y be fibrewise topological spaces over B with K fibrewise compact fibrewise Hausdorff. Then the evaluation map*

$$ev : map_B(K,Y) \times_{\mathsf{Top}_B} K \longrightarrow Y$$

is continuous.

Proof. Let $b_0 \in B$, $\gamma_0 \in \mathrm{map}(K_{b_0}, Y_{b_0})$, $x_0 \in K_{b_0}$, O open in Y and suppose that $\gamma_0(x_0) \in O$. The map $\gamma_0 : K_{b_0} \longrightarrow Y_{b_0}$ is continuous, therefore there exists an open neighborhood V of x_0 in K such that $\gamma_0(V \cap K_{b_0}) \subset O$. The fibrewise space K fibrewise compact, fibrewise Hausdorff, by Definition 6.6, K is regular. There exists an open neighborhood Ω of $b_0 \in B$ and an open neighborhood U of x_0 in K such that $\overline{U} \cap K_\Omega \subset V$. Define $W = U \cap K_\Omega$. Then $(\overline{U}, O, \Omega) \times_{\mathsf{Top}_B} W$ is a neighborhood of $(\gamma_0, x_0) \in \mathrm{map}_{\mathsf{Top}_B}(K,Y) \times_{\mathsf{Top}_B} K$ and $ev((\overline{U}, O, \Omega) \times_{\mathsf{Top}_B} W) \subset O$. It follows that ev is continuous. □

Recall that an object Y in a category $\mathcal{C}$ is said to be exponentiable if for each $X \in \mathcal{C}$, the binary product $X \times_{\mathcal{C}} Y$ exists and the functor $. \times_{\mathcal{C}} Y : \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint.

The following fact is a consequence of [25, Proposition 9.7 and Corollary 9.13] of James.

Theorem 6.15. *Let K be a fibrewise compact fibrewise Hausdorff space over B. Then the functor*

$$. \times_{\mathsf{Top}_B} K : \mathsf{Top}_B \longrightarrow \mathsf{Top}_B$$

3 Observe that if K_b is empty, then $\mathsf{Top}(K_b, Y_b)$ contains precisely one element.

has a right adjoint which is the functor

$$map_B(K, .) : \mathsf{Top}_B \longrightarrow \mathsf{Top}_B.$$

In particular, K is an exponentiable object of Top_B.

Proof. Let $X, Y \in \mathsf{Top}_B$ and $f : X \times_{\mathsf{Top}_B} K \longrightarrow Y$ a fibrewise function with adjoint (as a fibrewise map between sets) the fibrewise function $\widehat{f} : X \longrightarrow \mathrm{map}_B(K, Y)$. We need to prove that f is continuous iff $\widehat{f}$ is.

Assume that $f : X \times_{\mathsf{Top}_B} K \longrightarrow Y$ is continuous and let $x_0 \in X$, (C, O, Ω) be an elementary open subset of $\mathrm{map}_B(K, Y)$ and assume that $\widehat{f}(x_0) \in (C, O, \Omega)$. Then $f(\{x_0\} \times_{\mathsf{Top}_B} C) \subset O$. By the fibrewise tube Lemma 6.12, there exists an open neighborhood U of x_0 such that $f(U \times_{\mathsf{Top}_B} C) \subset O$. Define $V = U \cap X_\Omega$. Then $\widehat{f}(V) \in (C, O, \Omega)$. It follows that $\widehat{f}$ is continuous.

Conversely, assume that $\widehat{f} : X \longrightarrow \mathrm{map}_B(K, Y)$ is continuous. By Proposition 6.14, the evaluation map

$$ev : \mathrm{map}_B(K, Y) \times_{\mathsf{Top}_B} K \longrightarrow Y$$

is continuous. Therefore f which is the composite

$$X \times_{\mathsf{Top}_B} K \xrightarrow{\widehat{f} \times_{\mathsf{Top}_B} 1_K} \mathrm{map}_B(K, Y) \times_{\mathsf{Top}_B} K \xrightarrow{ev} Y \tag{14}$$

is continuous. □

Proposition 6.16. *Assume that B is T_1. Let $K, Z \in \mathsf{Top}_B$ with K fibrewise compact fibrewise Hausdorff and let $map_B(K, Z)$ be the exponential object defined by (12). If Z is fibrewise T_1, then so is $map_B(K, Z)$.*

Proof.

- Step 1: K is the fibrewise space B^b defined by Example 6.13, $b \in B$.

  B is T_1 and the fibre Z_b is closed T_1-subspace of Z. Then by Example 6.13, $\mathrm{map}_B(B^b, Z)$ is a fibrewise T_1-subspace.

- Step 2: The general case.

  Let $\gamma \in \mathrm{map}_B(K, Z)$. we need to show that $\{\gamma\}$ is closed in $\mathrm{map}_B(K, Z)$. Let $b = p(\gamma)$, where p is the projection of the fibrewise space $\mathrm{map}_B(K, Z)$. Then $\gamma \in \mathsf{Top}(K_b, Z_b)$. For each $x \in K_b$, define

  $$f_x : B^b \longrightarrow K$$

  to be the fibrewise map given by $f_x(b) = x$ and let

  $$\mathrm{map}_B(f_x, Z) : \mathrm{map}_B(K, Z) \longrightarrow \mathrm{map}_B(B^b, Z)$$

be the fibrewise map induced by f_x. Furthermore, let $\gamma_x \in \mathsf{Top}(\{b\}, Z_b)$ to be the map given by $\gamma_x(b) = \gamma(x)$. Then $\gamma_x \in \mathrm{map}_B(B^b, Z)$. We have

$$\{\gamma\} = \bigcap_{x \in K_b} \mathrm{map}_B(f_x, Z)^{-1}(\{\gamma_x\}).$$

By Step 1, $\{\gamma\}$ is closed in $\mathrm{map}_B(K, Z)$.

□

Remark 6.17. *Let $K, Z \in \mathsf{Top}_B$ with K fibrewise compact, fibrewise Hausdorff and Z fibrewise Hausdorff. Then the exponential space $\mathrm{map}_B(K, Z)$ is not in general fibrewise Hausdorff even if Z is Hausdorff (not just fibrewise Hausdorff) and B is T_1.*

7 Fibrewise weak and k-Hausdorfifications

Our objective in this section is to prove that if B is T_1, then the subcategories of fibrewise weak Hausdorff spaces and fibrewise k-Hausdorff spaces are reflective subcategories of Top_B. We adopt a definition of fibrewise weak Hausdorff spaces that is seemingly weaker than that of James [24, Definition 1.1]. Our definition has the advantage that it agrees with the ordinary definition of weak Hausdorff spaces when B is reduced to a point (Strickland, [37, Definition 1.2]).

Definition 7.1.

1. *A fibrewise space X over B is said to be fibrewise weak Hausdorff if for each open set Ω of B, each fibrewise compact, fibrewise Hausdorff space K over Ω and each fibrewise map $\alpha : K \longrightarrow X_\Omega$, the image $\alpha(K)$ is closed in X_Ω.*

2. *The subcategory of Top_B whose objects are the weak Hausdorff spaces is denoted by $\mathsf{h_w Top}_B$.*

Proposition 7.2. *A fibrewise Hausdorff space is fibrewise weak Hausdorff.*

Proof. Let X be a fibrewise Hausdorff space, Ω open in B, K a fibrewise compact, fibrewise Hausdorff space over Ω and $u : K \longrightarrow X_\Omega$ a continuous fibrewise map. By Proposition 6.7, $u(K)$ is closed in X_Ω. Hence X is weak Hausdorff. □

Proposition 7.3. *Let $f : X \longrightarrow Y$ be an injective, continuous fibrewise map with Y fibrewise weak Hausdorff. Then X is fibrewise weak Hausdorff. In particular, a subspace of a fibrewise weak Hausdorff space is fibrewise weak Hausdorff.*

Proof. Clear. □

Proposition 7.4. *Assume that the base space B is a T_1-space. Then every fibrewise weak Hausdorff space over B is fibrewise T_1.*

Proof. Let X be a fibrewise weak Hausdorff space over B and let $x \in X$. B is T_1, thus the fibrewise subspace $\{x\}$ of X is fibrewise compact, fibrewise Hausdorff space. X is weak Hausdorff, thus $\{x\}$ is closed in X and X is T_1. □

Proposition 7.5. *Assume that the base space B is a T_1-space. Let u be a fibrewise continuous map from a fibrewise compact, fibrewise Hausdorff space K to a fibrewise weak Hausdorff space X. Then:*

1. *The map $u : K \longrightarrow X$ is proper.*
2. *The subspace $u(K)$ is a closed, fibrewise Hausdorff subspace of X.*

Proof.

1. We use the characterization of proper maps given by Theorem 6.2: Let C be a closed subset of K. C is fibrewise compact, fibrewise Hausdorff space over B, X is weak Hausdorff thus $u(C)$ is closed. u is then a closed map. B is T_1, by Proposition 7.4, X is T_1. Let $x \in X$ and $b = p(x)$ where p is the projection of X on B. The subset $\{x\}$ is closed in X, thus $u^{-1}(x)$ is closed in the compact space X_b. It follows that $u^{-1}(x)$ is compact. Therefore u is proper.
2. The map u is proper, thus $u(K)$ is closed. By Proposition 7.3, the subspace of a fibrewise weak Hausdorff space is fibrewise weak Hausdorff. We therefore may assume without loss of generalities that u is onto. By the first point, u is proper, thus by Proposition 6.3, the map

$$u \times_{\mathsf{Top}_B} u : K \times_{\mathsf{Top}_B} K \longrightarrow X \times_{\mathsf{Top}_B} X$$

is proper. K is fibrewise Hausdorff, therefore by Proposition 6.5, the diagonal $\Delta(K)$ of K is closed in $K \times_{\mathsf{Top}_B} K$. It follows that $\Delta(X) = u \times_{\mathsf{Top}_B} u(K \times_{\mathsf{Top}_B} K)$ is closed in $X \times_{\mathsf{Top}_B} X$. By Proposition 6.5, X is fibrewise Hausdorff.

□

Proposition 7.6. *Assume that the base space B is T_1 and let $(X_i)_{i\in I}$ be a family of fibrewise weak Hausdorff spaces indexed by a (small) set I. Then $\prod_{\mathsf{Top}_B}^{i\in I} X_i$ is fibrewise weak Hausdorff.*

Proof. Let $X = \prod_{\mathsf{Top}_B}^{i\in I} X_i$ and $p : X \longrightarrow B$ the projection of X on B.

- Step 1: Let K be a fibrewise compact, fibrewise Hausdorff space over B, $u : K \longrightarrow X$ a continuous fibrewise map, $u_i : K \longrightarrow X_i$ the i-component of u and $K_i = u_i(K)$, $i \in I$. Each K_i is closed and by Proposition 7.5.2, each K_i is a fibrewise Hausdorff subspace of X_i. It follows that $\prod_{\mathsf{Top}_B}^{i\in I} K_i$ is closed, fibrewise Hausdorff subspace of X. By Proposition 6.7, $u(K)$ is closed in $\prod_{\mathsf{Top}_B}^{i\in I} K_i$. Thus $u(K)$ is closed in X.
- Step 2: Let Ω be an open subset of B, K a fibrewise compact, fibrewise Hausdorff space over Ω, $Y = X_\Omega$ and $u : K \longrightarrow Y$ a continuous, fibrewise map. Define $Y_i = p_i^{-1}(\Omega)$. Then $Y = \prod_{\mathsf{Top}_\Omega}^{i\in I} Y_i$. By Proposition 7.3, each Y_i is weak Hausdorff, thus by Step 1, $u(K)$ is closed in Y. It follows that $\prod_{\mathsf{Top}_B}^{i\in I} X_i$ is weak Hausdorff.

□

Theorem 7.7. *Assume that the base space B is T_1. Then the category $\mathsf{h_wTop}_B$ is a reflective subcategory of Top_B. In particular, $\mathsf{h_wTop}$ is bicomplete.*

Proof. This follows from Theorem 5.1, Proposition 7.3 and Proposition 7.6. □

k-Hausdorff spaces are defined by Rezk in [35, Section 4]. We here introduce the notion of fibrewise k-Hausdorff spaces.

Definition 7.8.

1. *A fibrewise space X over B is said to be fibrewise k-Hausdorff if for each open set Ω of B, each fibrewise compact, fibrewise Hausdorff space K over Ω and each continuous fibrewise map $u : K \longrightarrow X_\Omega \times_{\mathsf{Top}_\Omega} X_\Omega$, the inverse image by u of the diagonal of X_Ω is closed in K.*

2. *The subcategory of Top_B whose objects are the fibrewise k-Hausdorff spaces is denoted by $\mathsf{h_kTop}_B$.*

By Proposition 6.5, a fibrewise Hausdorff space is fibrewise k-Hausdorff. The product in Top_B of fibrewise k-Hausdorff spaces is fibrewise k-Hausdorff. Similarly, a subobject of a fibrewise k-Hausdorff spaces is fibrewise k-Hausdorff space. We can apply Theorem 5.1 to get the following result.

Proposition 7.9. *Assume that the base space B is a T_1-space. The subcategory $\mathsf{h_kTop}_B$ of Top_B is reflective. In particular, $\mathsf{h_kTop}_B$ is bicomplete.*

The next result generalizes that of Rezk [35, Proposition 11.2].

Proposition 7.10. *Assume that the base space B is a T_1-space. Then $\mathsf{h_wTop}_B$ is a reflective subcategory of $\mathsf{h_kTop}_B$.*

Proof. In the light of Theorem 7.7, we just need to prove that $\mathsf{h_wTop}_B$ is a subcategory of $\mathsf{h_kTop}_B$. Let X be a fibrewise weak Hausdorff space.

- Step 1: Let $f : K \longrightarrow X \times_{\mathsf{Top}_B} X$ be a continuous, fibrewise map, where K is fibrewise compact, fibrewise Hausdorff space. Let f_1 and f_2 be the components of the map f. Define $K_1 = f_1(K)$, $K_2 = f_2(K)$ and $L = K_1 \cup K_2$. The subspace L of X is the image of the continuous, fibrewise map $f_1 \coprod_{\mathsf{Top}_B} f_2 : K \coprod_{\mathsf{Top}_B} K \longrightarrow X$. The space $K \coprod_{\mathsf{Top}_B} K$ is fibrewise compact fibrewise Hausdorff, thus L is closed and by Proposition 7.5.2, L is fibrewise Hausdorff. f factors through $L \times_{\mathsf{Top}_B} L$ as follows

$$\begin{array}{ccc} K & \xrightarrow{\quad f \quad} & X \times_{\mathsf{Top}_B} X \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle j} & \\ & L \times_{\mathsf{Top}_B} L & \end{array}$$

where $g : K \longrightarrow L \times_{\mathsf{Top}_B} L$ is continuous, fibrewise map and j is the inclusion map. Let Δ_X and Δ_L be the diagonals of X and L respectively. By Proposition 6.5, Δ_L is closed in $L \times_{\mathsf{Top}_B} L$, thus

$$f^{-1}(\Delta_X) = g^{-1}(j^{-1}(\Delta_X)) = g^{-1}(\Delta_L)$$

is closed in K.

- Step 2: Let Ω be open in B, K a fibrewise compact, fibrewise Hausdorff space over Ω and $u : K \longrightarrow X_\Omega \times_{\mathsf{Top}_\Omega} X_\Omega$ a continuous, fibrewise map. X is fibrewise weak Hausdorff, by Proposition 7.3, X_Ω is fibrewise weak Hausdorff. Therefore by Step 1, $u(K)$ is closed in $X_\Omega \times_{\mathsf{Top}_\Omega} X_\Omega$. It follows that X is k-Hausdorff.

□

Remark 7.11. *A space $X \in \mathsf{Top}$ is weak Hausdorff (resp. k-Hausdorff) if it corresponds, under the isomorphism P of (10) to a fibrewise weak Hausdorff (resp. k-Hausdorff) space over Pt. The subcategory of Top of such spaces is reflective and is denoted by $\mathsf{h_w Top}$ (resp. $\mathsf{h_k Top}$)*

8 Left Kan extendable subcategories of Top_B

It is well known that any subcategory of Top containing a nonempty space has a coreflective hull ([20, Theorem 12], [18, Proposition 2.17] and [19, page 283]). In this section, we prove that any subcategory of Top_B, which is suitable in the sense of the definition below, has a strong coreflective hull.

Definition 8.1. *A subcategory $\mathcal{W}$ of Top_B is said to be suitable if for every $b \in B$, there exists a fibrewise topological space $E(b)$ in $\mathcal{W}$ such that*

$$\begin{cases} E(b)_b \neq \emptyset \\ E(b)_c = \emptyset \quad \textit{for all } c \neq b \end{cases} \tag{15}$$

where $E(b)_c$ is the fibre of $E(b)$ over $c \in B$.

Let $\mathcal{W}$ be a suitable subcategory of Top_B (See Definition 8.1). For $X \in \mathsf{Top}_B$, let

$$J_X : \mathcal{W}/X \longrightarrow \mathsf{Top}_B \tag{16}$$

be the functor which takes an arrow $V \to X$ to its domain V,

$$|J_X| : \mathcal{W}/X \longrightarrow \mathsf{Set}_{|B|} \tag{17}$$

its underlying functor as defined by (71) and

$$P_{|B|} : \mathsf{Set}_{|B|} \longrightarrow \mathsf{Set}$$

the functor defined by (61). For $(V \xrightarrow{\sigma} X) \in \mathcal{W}/X$, define a map

$$\begin{array}{rccc} \lambda_\sigma : & |V| & \longrightarrow & |X| \\ & v & \mapsto & |\sigma|(v) \end{array}$$

The maps λ_σ define a cone

$$P_{|B|}\,|J_X| \overset{\lambda}{\Longrightarrow} |X| \tag{18}$$

- Let $V \xrightarrow{\sigma} X$, $V' \xrightarrow{\sigma'} X$ be in $\mathcal{W}/X$, p_σ and $p_{\sigma'}$ the projections of the fibrewise spaces V and V' and $v \in V$, $v' \in V'$. The fact that $\mathcal{W}$ is suitable implies that $\lambda_\sigma(v) = \lambda_{\sigma'}(v')$ iff the objects (σ, v) and (σ', v') of $\int P_{|B|}\,|J_X|$ are in the same connected component.

- Let $x_0 \in |X|$ and let $b_0 = |p_X|(x_0)$, where $p_X : X \longrightarrow B$ is the projection of the fibrewise space X over B. Let $E(b_0)$ be as in (15) and define $\sigma_0 : E(b_0) \longrightarrow X$ to be the fibrewise map given by $\lambda_{\sigma_0}(e) = x_0$ for all $e \in E(b_0)$. Then $\sigma_0 \in \mathcal{W}/X$ and $x_0 \in \lambda_{\sigma_0}(E(b_0))$.

Therefore by Remark B.4.1.(c), the cone $P_{|B|}|J_X| \stackrel{\lambda}{\Longrightarrow} |X|$ given by (18) is a colimiting cone. By Remark B.4.2, $|J_X|$ has a colimit. Therefore by Lemma C.3, J_X has a colimit whose underlying set is $|X|$ and whose topology is the final topology defined by the functions $P_{|B|}\lambda_\sigma = |P_B(\sigma)| : |V| \longrightarrow |X|$, $\sigma \in \mathcal{W}/X$.

This proves that the inclusion functor $\mathcal{W} \stackrel{J}{\hookrightarrow} \mathsf{Top}_B$ has a density comonad (L, ϵ, δ) satisfying $|L(X)| = |X|$, $\forall X \in \mathsf{Top}_B$. Furthermore, the underlying map $|\epsilon_X|$ of the counit $\epsilon_X : L(X) \longrightarrow X$ of the subcategory $\mathcal{W}$ of Top_B is just the identity map $1_{|X|}$. In particular, ϵ_X is monic and by Theorem 3.6, we have the following result.

Theorem 8.2. *Let $\mathcal{W}$ be a suitable subcategory of Top_B. Then:*

1. *The subcategory $\mathcal{W}$ is left Kan extendable in Top_B.*
2. *The coreflector $\mathsf{Top}_B \stackrel{\omega}{\to} \mathcal{W}_l[\mathsf{Top}_B]$ takes a fibrewise topological space X to the fibrewise topological space $\omega(X)$ having the same underlying set as X and whose topology is the final topology induced by the functions $|V| \xrightarrow{|P_B(\sigma)|} |X|$, $\sigma \in \mathcal{W}/X$.*
3. *A fibrewise topological space X over B is $\mathcal{W}$-generated iff X has the final topology defined by all continuous fibrewise maps $V \to X$, where V is a fibrewise space in $\mathcal{W}$.*

Example 8.3. *For $b \in B$, let B^b be the fibrewise subspace of B defined by Example 6.13. Let $\mathcal{D}$ be the subcategory of Top_B whose objects are the fibrewise spaces B^b, $b \in B$. Then $\mathcal{D}$ is a suitable subcategory of Top_B. It is then left Kan extendable and $\mathcal{D}_l[\mathsf{Top}_B]$ is precisely the subcategory Dis_B of Top_B of discrete fibrewise spaces over B.*

A subcategory $\mathcal{W}$ of Top is said to be suitable if it corresponds, under the isomorphism P of (10) to a suitable subcategory of Top_B. That is, if $\mathcal{W}$ contains a nonempty space. By substituting pt for B, one partially recovers a result of Herrlich and Strecker [18, Proposition 2.17].

Corollary 8.4. *Let $\mathcal{W}$ be a suitable subcategory of* Top. *Then:*

1. *$\mathcal{W}$ is left Kan extendable in* Top.
2. *The coreflector $\mathsf{Top} \stackrel{\omega}{\to} \mathcal{W}_l[\mathsf{Top}]$ takes a topological space X to the topological space $\omega(X)$ having the same underlying set as X and whose topology is the final topology induced by the functions $|V| \xrightarrow{|\sigma|} |X|$, $\sigma \in \mathcal{W}/X$.*
3. *A topological space X is $\mathcal{W}$-generated iff X has the final topology defined by all continuous maps $V \longrightarrow X, V \in \mathcal{W}$.*

Let $\mathcal{W}$ be a suitable subcategory of Top_B. For $b \in B$, let $E(b)$ in $\mathcal{W}$ be as in (15) and let B^b to be as defined in Example 8.3. B^b is a retract of $E(b)$. By Lemma 1.6, B^b is $\mathcal{W}$-generated. Therefore by Example 8.3 and Corollary 3.9.1, every discrete fibrewise space is $\mathcal{W}$-generated and we have the following.

Lemma 8.5. *Let $\mathcal{W}$ be a suitable subcategory of Top_B. Then every discrete fibrewise space over B is $\mathcal{W}$-generated.*

Proposition 8.6. *Let $\mathcal{W}$ be a suitable subcategory of Top_B. Then:*

1. *The fibrewise quotient of a $\mathcal{W}$-generated fibrewise space is $\mathcal{W}$-generated.*
2. *A fibrewise space is $\mathcal{W}$-generated iff it is the fibrewise quotient of a coproduct of spaces in $\mathcal{W}$.*

Proof.

1. Let X be a $\mathcal{W}$-generated fibrewise space and $\sim$ a fibrewise equivalence relation on $|X|$. Let R the discrete topological space whose underlying set is the graph of the equivalence relation $\sim$. The space R is a fibrewise space over B. The fibrewise quotient quotient space $X/\sim$ is the coequalizer in Top_B
$$R \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} X \xrightarrow{\ q\ } X/\sim$$
where pr_1 and pr_2 are induced by the projections on the first and second factors. The fibrewise space X is $\mathcal{W}$-generated and by Lemma 8.5, R is $\mathcal{W}$-generated. Therefore by Proposition 3.4.1, $X/\sim$ is $\mathcal{W}$-generated.
2. Straight forward generalization of the Escardó-Lawson proof of the same result when B is a one point space [15, Lemma 3.2.(iv)]. □

Corollary 8.7. *Let $\mathcal{W}$ be a suitable subcategory of Top_B. Assume that a fibrewise space X is such that every point of X has a neighborhood which is in $\mathcal{W}$. Then X is $\mathcal{W}$-generated.*

Proof. For each $x \in X$, choose a neighborhood V_x of x which is in $\mathcal{W}$ and let $i_x : V_x \longrightarrow X$ be the inclusion map. Then the map

$$\coprod_{\mathsf{Top}_B}^{x\in X} V_x \longrightarrow X \tag{19}$$

whose restriction to V_x is i_x, is a fibrewise quotient map. By Proposition 8.6.2, X is $\mathcal{W}$-generated. □

Proposition 8.8. *Let top_B be a reflective subcategory of Top_B that is closed under subobjects and let $\mathcal{W}$ be a suitable subcategory of top_B. Then*

1. *The subcategory $\mathcal{W}$ of top_B is left Kan extendable.*
2. $\mathcal{W}_l[\mathsf{top}_B] = \mathsf{top}_B \cap \mathcal{W}_l[\mathsf{Top}_B]$.
3. *A reflection of Top_B on top_B induces a reflection of $\mathcal{W}_l[\mathsf{Top}_B]$ on $\mathcal{W}_l[\mathsf{top}_B]$.*
4. *A coreflection of Top_B on $\mathcal{W}_l[\mathsf{Top}_B]$ induces a coreflection of top on $\mathcal{W}_l[\mathsf{top}]$.*

Proof. By Theorem 8.2.1, $\mathcal{W}$ is left Kan extendable subcategory of Top_B. Let (L, ϵ, δ) be the density comonad of the inclusion functor $J : \mathcal{W} \longrightarrow \mathsf{Top}_B$. Let $X_0 \in \mathsf{top}_B$, by Theorem 3.6, the map $\epsilon_{X_0} : L(X_0) \longrightarrow X_0$ is $\mathcal{W}$-monic. The subcategory $\mathcal{W}$ of Top_B is suitable, therefore ϵ_{X_0} is monic. The subcategory top_B is closed under subobject, thus $L(X_0) \in \mathsf{top}_B$. Therefore $L(\mathsf{top}_B) \subset \mathsf{top}_B$ and then the points 1-4 follow from Theorem 3.10. □

9 Cartesian closed category of $\mathcal{W}$-generated objects

Given a left Kan extendable subcategory $\mathcal{W}$ of a category $\mathcal{C}$, In this section, we present sufficient conditions for the category $\mathcal{W}_l[\mathcal{C}]$ to be cartesian closed.

Assume that Y is an exponentiable object in a category $\mathcal{C}$ and let $G : \mathcal{C} \longrightarrow \mathcal{C}$ be a right adjoint of the functor $. \times_{\mathcal{C}} Y : \mathcal{C} \longrightarrow \mathcal{C}$. For $Z \in \mathcal{C}$, the object $G(Z)$ is called an exponential object and denoted by Z^Y.

Examples 9.1.

1. *In* Top, *the exponentiable objects are precisely the core compact spaces [12, 14, 23]. In particular, locally compact Hausdorff spaces are exponentiable.*
2. *By Theorem 6.15, every fibrewise compact fibrewise Hausdorff space over B is exponentiable in the category* Top_B.
3. *By [34, Corollary 2.9], every local homeomorphism $X \longrightarrow B$ is an exponentiable object of* Top_B.

Lemma 9.2. *Let $\mathcal{W}$ be a left Kan extendable subcategory of a bicomplete category $\mathcal{C}$. Assume that*

1. *Every object in $\mathcal{W}$ is exponentiable in $\mathcal{C}$.*
2. *For every $V, W \in \mathcal{W}$, the object $V \times_{\mathcal{C}} W \in \mathcal{W}_l[\mathcal{C}]$.*

Then for every $V \in \mathcal{W}$ and every $Y \in \mathcal{W}_l[\mathcal{C}]$, $V \times_{\mathcal{C}} Y$ is a $\mathcal{W}$-generated object. That is $V \times_{\mathcal{W}_l[\mathcal{C}]} Y \cong V \times_{\mathcal{C}} Y$.

Proof. Let $V \in \mathcal{W}$ and $Y \in \mathcal{W}_l[\mathcal{C}]$. By Corollary 3.5, there exists a functor $F : \mathcal{K} \longrightarrow \mathcal{C}$ taking values in $\mathcal{W}$ such that $Y \cong \mathsf{colim} F$. Define $V \times_{\mathcal{C}} F$ to be the composite functor $\mathcal{K} \xrightarrow{F} \mathcal{C} \xrightarrow{V \times_{\mathcal{C}}} \mathcal{C}$. Then

$$\begin{aligned} V \times_{\mathcal{C}} Y &\cong V \times_{\mathcal{C}} \mathsf{colim} F \\ &\cong \mathsf{colim} V \times_{\mathcal{C}} F \quad (\text{because } V \text{ is exponentiable in } \mathcal{C}) \end{aligned}$$

By 2., $V \times_{\mathcal{C}} F$ takes values in $\mathcal{W}_l[\mathcal{C}]$. Therefore, by Proposition 3.4.1, $V \times_{\mathcal{C}} Y \cong \mathsf{colim} V \times_{\mathcal{C}} F$ is in $\mathcal{W}_l[\mathcal{C}]$. Thus by Proposition 3.4.2, $V \times_{\mathcal{W}_l[\mathcal{C}]} Y$ exists and

$$V \times_{\mathcal{W}_l[\mathcal{C}]} Y \cong F_L(V \times_{\mathcal{C}} Y) \cong V \times_{\mathcal{C}} Y.$$

□

Assume next that $\mathcal{W}$ and $\mathcal{C}$ are as in Lemma 9.2.

- For $X, Y \in \mathcal{W}_l[\mathcal{C}]$, let $J_X : \mathcal{W}/X \longrightarrow \mathcal{C}$ be as defined by (3) and let $J_X \times_{\mathcal{C}} Y$ be the composite functor

$$J_X \times_{\mathcal{C}} Y : \mathcal{W}/X \xrightarrow{J_X} \mathcal{C} \xrightarrow{- \times_{\mathcal{C}} Y} \mathcal{C} \tag{20}$$

By Proposition 3.4, $\mathcal{W}_l[\mathcal{C}]$ is complete. For $(V \xrightarrow{\sigma} X) \in \mathcal{W}/X$, define

$$\theta_\sigma = \sigma \times_{\mathcal{W}_l[\mathcal{C}]} 1_Y : V \times_{\mathcal{C}} Y = V \times_{\mathcal{W}_l[\mathcal{C}]} Y \longrightarrow X \times_{\mathcal{W}_l[\mathcal{C}]} Y.$$

The maps θ_σ define a cone

$$J_X \times_{\mathcal{C}} Y \overset{\theta}{\Longrightarrow} X \times_{\mathcal{W}_l[\mathcal{C}]} Y \tag{21}$$

- Let

$$\mathrm{Hom}(V, .) : \mathcal{C} \longrightarrow \mathcal{C}$$

be a right adjoint of the functor $. \times_{\mathcal{C}} V : \mathcal{C} \longrightarrow \mathcal{C}$. For $Y, Z \in \mathcal{W}_l[\mathcal{C}]$, define

$$S_Z^Y = \mathrm{Hom}(J_Y(.), Z) : \begin{array}{ccc} (\mathcal{W}/Y)^{op} & \longrightarrow & \mathcal{C} \\ (V \xrightarrow{\sigma} Y) & \longmapsto & \mathrm{Hom}(V, Z) \end{array} \tag{22}$$

Definition 9.3. *A left Kan extendable subcategory $\mathcal{W}$ of a bicomplete category $\mathcal{C}$ is said to be **closeable** if*

1. *Every object in $\mathcal{W}$ is exponentiable in $\mathcal{C}$.*
2. *For every $V, W \in \mathcal{W}$, the object $V \times_{\mathcal{C}} W \in \mathcal{W}_l[\mathcal{C}]$.*
3. *For all $X, Y \in \mathcal{W}_l[\mathcal{C}]$, the cone $J_X \times_{\mathcal{C}} Y \overset{\theta}{\Longrightarrow} X \times_{\mathcal{W}_l[\mathcal{C}]} Y$ given by (21) is a colimiting cone.*
4. *For all $Y, Z \in \mathcal{W}_l[\mathcal{C}]$, the functor $S_Z^Y : (\mathcal{W}/Y)^{op} \longrightarrow \mathcal{C}$ given by (22) has a limit.*

For the remainder of this section, we assume that $\mathcal{W}$ is a closeable left Kan extendable subcategory of a bicomplete category $\mathcal{C}$. Define

$$\mathrm{hom}(., .) : \mathcal{W}_l[\mathcal{C}]^{op} \times \mathcal{W}_l[\mathcal{C}] \longrightarrow \mathcal{C} \tag{23}$$

by

$$\mathrm{hom}(Y, Z) = \mathsf{lim} S_Z^Y = \underset{(V \xrightarrow{\sigma} Y) \in \mathcal{W}|Y}{\mathsf{lim}} \mathrm{Hom}(V, Z)$$

Then for $V \in \mathcal{W}$, the arrow-object 1_V of $\mathcal{W}/V$ is terminal, it is therefore an initial object in the opposite category $(\mathcal{W}/V)^{op}$. It follows that the limit of the functor

$$S_Z^V : (\mathcal{W}/V)^{op} \longrightarrow \mathcal{C}$$

is just $S_Z^V(1_V)$ which is $\mathrm{Hom}(V, Z)$. That is $\mathrm{hom}(V, Z) \cong \mathrm{Hom}(V, Z)$.

Lemma 9.4. *Let $V \in \mathcal{W}$ and $Y, Z \in \mathcal{W}_l[\mathcal{C}]$. There exists a natural bijection*

$$\mathcal{C}(V, \mathrm{hom}(Y, Z)) \cong \mathcal{C}(V \times_{\mathcal{C}} Y, Z).$$

Proof.

$$\begin{array}{rcll} \mathcal{C}(V, \mathrm{hom}(Y, Z)) & \cong & \mathcal{C}(V, \underset{(W \xrightarrow{\sigma} Y) \in \mathcal{W}|Y}{\mathsf{lim}} \mathrm{Hom}(W, Z)) & \\ & \cong & \underset{(W \xrightarrow{\sigma} Y) \in \mathcal{W}|Y}{\mathsf{lim}} \mathcal{C}(V, \mathrm{Hom}(W, Z)) & \\ & \cong & \underset{(W \xrightarrow{\sigma} Y) \in \mathcal{W}|Y}{\mathsf{lim}} \mathcal{C}(V \times_{\mathcal{C}} W, Z) & \\ & \cong & \mathcal{C}(\underset{(W \xrightarrow{\sigma} Y) \in \mathcal{W}|Y}{\mathsf{colim}} V \times_{\mathcal{C}} W, Z) & \\ & \cong & \mathcal{C}(V \times_{\mathcal{C}} Y, Z) & \text{(because V is exponentiable in } \mathcal{C}) \end{array}$$

□

Let $F_L : \mathcal{C} \longrightarrow \mathcal{W}_l[\mathcal{C}]$ be the coreflector. Define

$$\begin{array}{rrcl} (-)^{(-)}: & \mathcal{W}_l[\mathcal{C}]^{op} \times \mathcal{W}_l[\mathcal{C}] & \longrightarrow & \mathcal{W}_l[\mathcal{C}] \\ & (Y, Z) & \mapsto & Z^Y \end{array} \tag{24}$$

to be the composite functor

$$\mathcal{W}_l[\mathcal{C}]^{op} \times \mathcal{W}_l[\mathcal{C}] \xrightarrow{\text{hom}} \mathcal{C} \xrightarrow{F_L} \mathcal{W}_l[\mathcal{C}] \tag{25}$$

Then for $Y, Z \in \mathcal{W}_l[\mathcal{C}]$, $Z^Y = F_L(\text{hom}(Y, Z))$.

Lemma 9.5. *Let $V \in \mathcal{W}$ and $Y, Z \in \mathcal{W}_l[\mathcal{C}]$. There exists a natural bijection*

$$\mathcal{W}_l[\mathcal{C}](V, Z^Y) \cong \mathcal{W}_l[\mathcal{C}](V \times_{\mathcal{W}_l[\mathcal{C}]} Y, Z).$$

Proof.

$$\begin{array}{rcll} \mathcal{W}_l[\mathcal{C}](V, Z^Y) & \cong & \mathcal{W}_l[\mathcal{C}](V, F_L(\text{hom}(Y, Z))) & \\ & \cong & \mathcal{C}(V, \text{hom}(Y, Z)) & \text{(by Proposition 3.3)} \\ & \cong & \mathcal{C}(V \times_{\mathcal{C}} Y, Z) & \text{(by Lemma 9.4)} \\ & \cong & \mathcal{C}(V \times_{\mathcal{W}_l[\mathcal{C}]} Y, Z) & \text{(by Lemma 9.2)} \\ & \cong & \mathcal{W}_l[\mathcal{C}](V \times_{\mathcal{W}_l[\mathcal{C}]} Y, Z) & \end{array}$$

□

Theorem 9.6. *$\mathcal{W}_l[\mathcal{C}]$ is cartesian closed with internal hom functor the functor*

$$(-)^{(-)} : \mathcal{W}_l[\mathcal{C}]^{op} \times \mathcal{W}_l[\mathcal{C}] \longrightarrow \mathcal{W}_l[\mathcal{C}]$$

defined by (25).

Proof. Let $X, Y, Z \in \mathcal{W}_l[\mathcal{C}]$.

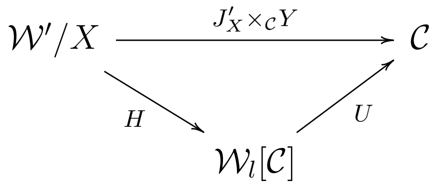
$$\begin{array}{rcll} \mathcal{W}_l[\mathcal{C}](X, Z^Y) & \cong & \mathcal{W}_l[\mathcal{C}](\operatorname{colim}_{(V \xrightarrow{\sigma} X) \in \mathcal{W}|X} V, Z^Y) & \\ & \cong & \lim_{(V \xrightarrow{\sigma} X) \in \mathcal{W}|X} \mathcal{W}_l[\mathcal{C}](V, Z^Y) & \\ & \cong & \lim_{(V \xrightarrow{\sigma} X) \in \mathcal{W}|X} \mathcal{W}_l[\mathcal{C}](V \times_{\mathcal{W}_l[\mathcal{C}]} Y, Z) & \text{(by Lemma 9.5)} \\ & \cong & \lim_{(V \xrightarrow{\sigma} X) \in \mathcal{W}|X} \mathcal{W}_l[\mathcal{C}](J_X \times_{\mathcal{C}} Y(\sigma), Z) & \\ & \cong & \mathcal{W}_l[\mathcal{C}](\operatorname{colim} J_X \times_{\mathcal{C}} Y, Z) & \\ & \cong & \mathcal{W}_l[\mathcal{C}](X \times_{\mathcal{W}_l[\mathcal{C}]} Y, Z) & \text{(by Definition 9.3.3)} \end{array}$$

□

The next result is a generalization of that of Escardó-Lawson [15, Corollary 5.5].

Corollary 9.7. *Let $\mathcal{W}'$ be another closeable, left Kan extendable subcategory of $\mathcal{C}$ which is contained in $\mathcal{W}$. Then the inclusion functor $\mathcal{W}'_l[\mathcal{C}] \longrightarrow \mathcal{W}_l[\mathcal{C}]$ preserves finite products.*

Proof. Let $J' : \mathcal{W}' \longrightarrow \mathcal{C}$ be the inclusion functor, $X, Y \in \mathcal{W}'_l[\mathcal{C}]$ and $J'_X \times_{\mathcal{C}} Y : \mathcal{W}'/X \longrightarrow \mathcal{C}$ be as in (20). The functor $J'_X \times_{\mathcal{C}} Y$ factors through $\mathcal{W}_l[\mathcal{C}]$ as follows:



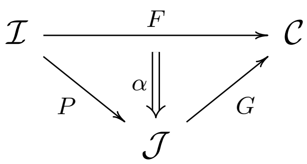
$$
\begin{aligned}
X \times_{\mathcal{W}'_l[\mathcal{C}]} Y &\cong \mathsf{colim} J'_X \times_{\mathcal{C}} Y && \text{(by Definition 9.3.3)}\\
&\cong \mathsf{colim} UH \\
&\cong \mathsf{colim} H && \text{(by Proposition 3.4.1)}\\
&\cong X \times_{\mathcal{W}_l[\mathcal{C}]} Y && \text{(by Lemma 9.2 and Theorem 9.6)}
\end{aligned}
$$

□

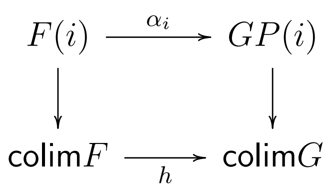
10 A fibrewise Day's theorem

The aim of this section is to use the notion of Kan extendable subcategories to provide a fibrewise version of Day's theorem ([11, Theorem 3.1]). We begin with the following simple observation.

Remark 10.1. *Let $\mathcal{I} \xrightarrow{F} \mathcal{C}$, $\mathcal{J} \xrightarrow{G} \mathcal{C}$ and $\mathcal{I} \xrightarrow{P} \mathcal{J}$ be functors. Assume that F and G have colimits and let $F \overset{\alpha}{\Longrightarrow} GP$ be a natural transformation.*



Then there exists a unique map $h : \mathsf{colim} F \longrightarrow \mathsf{colim} G$ rendering commutative the diagram



for all $i \in I$.

Theorem 10.2. *Assume that*

1. *The space B is a T_1-space.*
2. *The subcategory $\mathcal{W}$ of Top_B is suitable (See Definition 8.1).*
3. *Every fibrewise space in $\mathcal{W}$ is exponentiable as an object of Top_B.*
4. *For every $V, W \in \mathcal{W}$, the fibrewise space $V \times_{\mathsf{Top}_B} W$ is $\mathcal{W}$-generated.*

Then $\mathcal{W}$ is left Kan extendable. Moreover, $\mathcal{W}_l[\mathsf{Top}_B]$ is a cartesian closed subcategory of Top_B.

Proof. By Theorem 8.2, $\mathcal{W}$ is left Kan extendable. In the light of Theorem 9.6, we just need to prove that conditions 3 and 4 of Definition 9.3 are satisfied.

Let $X, Y \in \mathcal{W}_l[\mathsf{Top}_B]$, $J_X : \mathcal{W}/X \longrightarrow \mathsf{Top}_B$ be the functor defined by (16) and $J_X \times_{\mathsf{Top}_B} Y : \mathcal{W}/X \longrightarrow \mathsf{Top}_B$ be the composite functor

$$J_X \times_{\mathsf{Top}_B} Y : \mathcal{W}/X \xrightarrow{J_X} \mathsf{Top}_B \xrightarrow{-\times_{\mathsf{Top}_B} Y} \mathsf{Top}_B \tag{26}$$

By Theorem 8.2, the functor J_X has a colimit. Therefore by Lemma C.3, the functor $|J_X| : \mathcal{W}/X \longrightarrow \mathsf{Set}_{|B|}$ has a colimit. $\mathsf{Set}_{|B|}$ is cartesian closed, thus the functor $- \times_{\mathsf{Set}_B} |Y| : \mathsf{Set}_{|B|} \longrightarrow \mathsf{Set}_{|B|}$ is left adjoint and preserves colimits. It follows that the composite of these last two functors, which is $\left|J_X \times_{\mathsf{Top}_B} Y\right|$, has a colimit. Again by Lemma C.3, the functor $J_X \times_{\mathsf{Top}_B} Y$ has a colimit. Let $J_{X \times_{\mathsf{Top}_B} Y} \overset{\lambda}{\Longrightarrow} X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y$ and $J_X \times_{\mathsf{Top}_B} Y \overset{\mu}{\Longrightarrow} \mathsf{colim}(J_X \times_{\mathsf{Top}_B} Y)$ be colimiting cones. Observe that for $(f : V \longrightarrow X \times_{\mathsf{Top}_B} Y) \in \mathcal{W}/X \times_{\mathsf{Top}_B} Y$, the component λ_f of the cone λ along f is the map

$$\lambda_f = F_L(f) : V \longrightarrow X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y \tag{27}$$

where $F_L : \mathsf{Top} \longrightarrow \mathcal{W}_l[\mathsf{Top}_B]$ is the coreflector.

The cone $\theta : J_X \times_{\mathsf{Top}_B} Y \Longrightarrow X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y$ defined by (21) induces a map

$$\mathsf{colim}(J_X \times_{\mathsf{Top}_B} Y) \xrightarrow{\tilde{\theta}} X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y \tag{28}$$

It is such that for every $(V \xrightarrow{\sigma} X) \in \mathcal{W}/X$, the diagram commutes

$$\begin{array}{ccc} V \times_{\mathsf{Top}_B} Y = V \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y & & \\ \downarrow{\scriptstyle \mu_\sigma} & \searrow{\scriptstyle \sigma \times_{\mathcal{W}_l[\mathsf{Top}_B]} 1_Y} & \\ \mathsf{colim}(J_X \times_{\mathsf{Top}_B} Y) & \xrightarrow[\tilde{\theta}]{} & X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y \end{array} \tag{29}$$

We need to prove that $\tilde{\theta}$ is an isomorphism. Let $P : \mathcal{W}/X \times_{\mathsf{Top}_B} Y \longrightarrow \mathcal{W}/X$ be the functor which takes an object in $\mathcal{W}/X \times_{\mathsf{Top}_B} Y$, which is an arrow $f = (\sigma, \tau) : V \longrightarrow X \times_{\mathsf{Top}_B} Y$, to its first component $\sigma : V \longrightarrow X$, which is an object in $\mathcal{W}/X$. Define a natural transformation

$$J_{X \times_{\mathsf{Top}_B} Y} \overset{\alpha}{\Longrightarrow} (J_X \times_{\mathsf{Top}_B} Y)P \tag{30}$$

as follows:

For $f = (\sigma, \tau) : V \longrightarrow X \times_{\mathsf{Top}_B} Y$, $\alpha_f = (1_V, \tau) : V \longrightarrow V \times_{\mathsf{Top}_B} Y$

$$\begin{array}{ccc} \mathcal{W}/X \times_{\mathsf{Top}_B} Y & \xrightarrow{J_{X \times_{\mathsf{Top}_B} Y}} & \mathsf{Top}_B \\ & \Downarrow{\scriptstyle \alpha} & \\ {\scriptstyle P}\searrow & \mathcal{W}/X & \nearrow{\scriptstyle J_X \times_{\mathsf{Top}_B} Y} \end{array} \tag{31}$$

The natural transformation α is such that the following diagram commutes

$$\begin{array}{ccc} & V & \\ {\scriptstyle \alpha_f}\swarrow & & \searrow{\scriptstyle f} \\ V \times_{\mathsf{Top}_B} Y & \xrightarrow[\sigma \times_{\mathsf{Top}_B} 1_Y]{} & X \times_{\mathsf{Top}_B} Y \end{array} \tag{32}$$

Applying the coreflector $F_L : \mathsf{Top} \longrightarrow \mathcal{W}_l[\mathsf{Top}]$ to (32), we get a new commutative diagram

$$\begin{array}{ccc} & V & \\ {}^{\alpha_f}\swarrow & & \searrow^{\lambda_f} \\ V \times_{\mathsf{Top}_B} Y & \xrightarrow[\sigma \times_{\mathcal{W}_l[\mathsf{Top}]} 1_Y]{} & X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y \end{array} \tag{33}$$

By Remark 10.1, the natural transformation $J_{X \times_{\mathsf{Top}_B} Y} \xRightarrow{\alpha} (J_X \times_{\mathsf{Top}_B} Y)P$ induces a map $X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y \xrightarrow{h} \mathsf{colim} J_X \times_{\mathsf{Top}_B} Y$. It is such that for every $(f = (\sigma, \tau) : V \longrightarrow X \times_{\mathsf{Top}_B} Y) \in \mathcal{W}/X \times_{\mathsf{Top}_B} Y$, the diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\alpha_f} & V \times_{\mathsf{Top}_B} Y \\ {\scriptstyle \lambda_f}\downarrow & & \downarrow{\scriptstyle \mu_\sigma} \\ X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y & \xrightarrow[h]{} & \mathsf{colim} J_X \times_{\mathsf{Top}_B} Y \end{array} \tag{34}$$

Gluing together diagrams (34) and (29) along their common edge, we get the following commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\alpha_f} & V \times_{\mathsf{Top}_B} Y & & \\ {\scriptstyle \lambda_f}\downarrow & & \downarrow{\scriptstyle \mu_\sigma} & \searrow^{\sigma \times_{\mathcal{W}_l[\mathsf{Top}]} 1_Y} & \\ X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y & \xrightarrow[h]{} & \mathsf{colim} J_X \times_{\mathsf{Top}_B} Y & \xrightarrow[\tilde{\theta}]{} & X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y \end{array} \tag{35}$$

By (33), $(\sigma \times_{\mathcal{W}_l[\mathsf{Top}_B]} 1_Y)\alpha_f = \lambda_f$. Therefore $\tilde{\theta} h = 1_{X \times_{\mathcal{W}_l[\mathsf{Top}_B]} Y}$. The maps $\tilde{\theta}$ and h induce isomorphisms on the underlying sets, therefore, we also have $h\tilde{\theta} = 1_{\mathsf{colim} J_X \times_{\mathsf{Top}_B} Y}$. It follows that $\tilde{\theta}$ is an isomorphism and condition 3 of Definition 9.3 is fulfilled. Condition 4 results from Lemma 10.4 below. □

Lemma 10.3. *Let $W, Y \in \mathsf{Top}_B$ with W exponentiable in Top_B and $Hom(W, .) : \mathsf{Top}_B \longrightarrow \mathsf{Top}_B$ a right adjoint of the functor $W \times_{\mathsf{Top}_B} . : \mathsf{Top}_B \longrightarrow \mathsf{Top}_B$. Then*

$$|Hom(W, Y)_b| \cong \mathsf{Top}(W_b, Y_b), \quad \forall b \in B.$$

Proof. Let $b \in B$, B^b be the fibrewise space over B defined by Example 8.3. Then

$$|\mathrm{Hom}(W, Y)_b| \cong \mathsf{Top}_B(B^b, \mathrm{Hom}(W, Y)) \cong \mathsf{Top}_B(B^b \times_{\mathsf{Top}_B} W, Y) \cong \mathsf{Top}(W_b, Y_b).$$

□

Lemma 10.4. *Assume that B is T_1, $\mathcal{W}$ is a suitable subcategory of Top_B and that every object of $\mathcal{W}$ is exponentiable in Top_B. Let $Y, Z \in \mathcal{W}_l[\mathsf{Top}_B]$, then the functor*

$$\begin{array}{rccc} S^Y_Z : & \mathcal{W}/Y & \longrightarrow & \mathsf{Top}_B \\ & (W \xrightarrow{\sigma} Y) & \longmapsto & Hom(W, Z) \end{array}$$

has a limit.

Proof. Let $T_Y : \mathcal{W}/Y \longrightarrow \mathsf{Top}_B$ be as in (16). Then $\mathsf{colim} T_Y \cong Y$. Let $b \in B$ and let

$$\pi^s_b : \mathsf{Set}_{|B|} \longrightarrow \mathsf{Set} \quad \text{and} \quad \pi^t_b : \mathsf{Top}_B \longrightarrow \mathsf{Top}$$

be the functors defined by (63) and (73) respectively.

$$\begin{aligned}
\mathsf{Top}(Y_b, Z_b) &\cong \mathsf{Top}(\pi_b^t(\mathsf{colim} T_Y), Z_b) \\
&\cong \mathsf{Top}(\mathsf{colim}\pi_b^t T_Y, Z_b) && \text{(by Lemma C.7.2)} \\
&\cong \mathsf{Top}(\underset{(W\xrightarrow{\sigma}Y)\in\mathcal{W}|Y}{\mathsf{colim}} W_b, Z_b) \\
&\cong \underset{(W\xrightarrow{\sigma}Y)\in\mathcal{W}|Y}{\mathsf{lim}} \mathsf{Top}(W_b, Z_b) \\
&\cong \underset{(W\xrightarrow{\sigma}Y)\in\mathcal{W}|Y}{\mathsf{lim}} |\mathrm{Hom}(W, Z)|_b && \text{(by Lemma 10.3)} \\
&\cong \underset{(W\xrightarrow{\sigma}Y)\in\mathcal{W}|Y}{\mathsf{lim}} \left|S_Z^Y(\sigma)\right|_b \\
&\cong \lim \pi_b^s \left|S_Z^Y\right|
\end{aligned}$$

By Lemma B.3.1, $\left|S_Z^Y\right|$ has a limit. Therefore by Lemma 9.2, S_Z^Y has a limit. □

Remark 10.5. *Let $\mathcal{W}$ be as in Theorem 10.2 and*

$$\hom(.,.) : \mathcal{W}_l[\mathsf{Top}_B]^{op} \times \mathcal{W}_l[\mathsf{Top}_B] \longrightarrow \mathsf{Top}_B \tag{36}$$

be the functor defined by (23). Let $Y, Z \in \mathcal{W}_l[\mathsf{Top}_B]$.

$$\begin{aligned}
|\hom(Y, Z)|_b &\cong \pi_b^s(|\hom(Y, Z)|) \\
&\cong \pi_b^s(\left|\lim S_Z^Y\right|) \\
&\cong \pi_b^s(\lim \left|S_Z^Y\right|) && (|\ |\ \textit{preserves limits}) \\
&\cong \lim \pi_b^s(\left|S_Z^Y\right|) && \textit{(by Lemma B.3.1)}
\end{aligned}$$

That is, $\lim \pi_b^s(\left|S_Z^Y\right|) \cong \mathsf{Top}(Y_b, Z_b)$. It follows from Lemma C.4.2 that $\hom(Y, Z)$ is the topological space whose underlying set is $\coprod_{b\in B} \mathsf{Top}(Y_b, Z_b)$ and whose topology is the initial topology induced from the spaces $Hom(W, Z)$ by the maps $\coprod_{b\in B} \sigma_b : \coprod_{b\in B} \mathsf{Top}(Y_b, Z_b) \longrightarrow \coprod_{b\in B} \mathsf{Top}(W_b, Z_b) = |Hom(W, Z)|$, where $(W \xrightarrow{\sigma} Y) \in \mathcal{W}/Y$.

By substituting Pt for B, Theorem 5.1 corresponds under the isomorphism P of (10) to the following celebrated theorem of Day.

Corollary 10.6. *([11, Theorem 3.1])*

Assume that:

1. *The subcategory $\mathcal{W}$ of Top is suitable.*
2. *Every space in $\mathcal{W}$ is exponentiable as an object of Top.*
3. *For every $V, W \in \mathcal{W}$, the space $V \times_{\mathsf{Top}} W$ is $\mathcal{W}$-generated.*

Then $\mathcal{W}$ is left Kan extendable. Furthermore, $\mathcal{W}_l[\mathsf{Top}]$ is a cartesian closed subcategory of Top.

Remark 10.7. *Let $\mathcal{W}$ be as in Corollary 10.6 and $Y, Z \in \mathcal{W}_l[\mathsf{Top}]$. By Remark 10.5, $\mathsf{lim}\left|S_Z^Y\right|$ exists and is isomorphic to $\mathsf{Top}(Y, Z)$. Therefore by Lemma C.1, $\hom(Y, Z) = \mathsf{lim} S_Z^Y$ is the topological space whose underlying set is $\mathsf{Top}(Y, Z)$ and whose topology is the initial topology defined by the functions*

$$\mathsf{Top}(Y, Z) \xrightarrow{\mathsf{Top}(\sigma, Z)} \left|S_Z^Y(\sigma)\right| = \mathsf{Top}(W, Z) \tag{37}$$

By (25), the exponential object Z^Y is given by $Z^Y \cong F_L(\hom(Y,Z))$, where $F_L : \mathsf{Top} \longrightarrow \mathcal{W}_l[\mathsf{Top}]$ is the coreflector.

Examples 10.8. *Let Comp be the subcategory of Top of compact Hausdorff spaces.*

1. *By Corollary 10.6, Comp is left Kan extendable and $\mathsf{Comp}_l[\mathsf{Top}]$ is a cartesian closed coreflective subcategory of Top. The Comp-generated objects of Top are precisely the compactly generated spaces so that we recover ([11, Theorem 3.1] and [31, page 49]). Let $\mathsf{kTop} = \mathsf{Comp}_l[\mathsf{Top}]$ and $k : \mathsf{Top} \longrightarrow \mathsf{kTop}$ a coreflector. By Corollary 8.7, kTop contains every locally compact Hausdorff space. We next give a description of the internal hom functor of kTop.*

   Recall that if K is compact Hausdorff and Z is any space, then the exponential object Hom(K,Z) is the topological space whose underlying set is $\mathsf{Top}(K,Z)$ and whose topology is generated by the subsets

   $$(C,V) = \{f \in \mathsf{Top}(K,Z) \mid f(C) \subset V\} \tag{38}$$

   where C is closed in K and V is open in Z [17, Proposition A.14.].

   For $\sigma : K \longrightarrow Y$ continuous, the pull back of the subsets (C,V) of $\mathsf{Top}(K,Z)$ by the maps

   $$\mathsf{Top}(Y,Z) \xrightarrow{\mathsf{Top}(\sigma,Z)} \mathsf{Top}(K,Z) \tag{39}$$

   are the subsets

   $$(C,\sigma,V) = \{f \in \mathsf{Top}(Y,Z) \mid f\sigma(C) \subset V\} \tag{40}$$

   where C is any compact Hausdorff space, V is any open subset of Z and $\sigma : C \longrightarrow Z$ is any continuous map. Let $\hom(Y,Z)$ be the topological space whose underlying set is $\mathsf{Top}(Y,Z)$ and whose topology is generated by the subsets (C,σ,V). By Remark 10.7, the exponential object Z^Y in the cartesian closed category kTop is given by

   $$Z^Y = k(\hom(Y,Z)) \tag{41}$$

2. *Assume that B is Hausdorff. The category Comp/B is suitable. By Theorems 6.2 and 6.15, every object in Comp/B is exponentiable in Top_B. The base space B is Hausdorff, therefore the diagonal of B is closed. It follows that the product, in Top_B, of two objects of Comp/B is again in Comp/B. By Theorem 10.2, the subcategory Comp/B of Top_B is left Kan extendable and $(\mathsf{Comp}/B)_l[\mathsf{Top}_B]$ is cartesian closed. By Proposition C.2.2, $(\mathsf{Comp}/B)_l[\mathsf{Top}_B] = \mathsf{kTop}/B$. Thus kTop/B is cartesian closed. We therefore recover a theorem of Booth ([6, Theorem 1.1]).*

We next use the terminology developed in this paper to state another result due to Day and compare it to Theorem 10.2.

Theorem 10.9. *(Day [11, Theorem 3.4]).*

Let $\mathcal{E}$ be a subcategory of Top such that:

1. *The subcategory $\mathcal{E}$ contains the one point space.*

2. *Each object of $\mathcal{E}$ is an exponentiable object of Top.*

3. *For any two fibrewise spaces $p : V \longrightarrow B$ and $q : W \longrightarrow B$ in $\mathcal{E}/B$, the domain of the product $p \times_{Top_B} q$ (in Top_B) is closed in $V \times_{Top} W$.*

Then $\mathcal{E}/B$ is left Kan extendable in Top_B and $(\mathcal{E}/B)_l[Top_B]$ is cartesian closed.

Theorems 10.2 and 10.9 do overlap. Actually, the proof of [6, Theorem 1.1] given in Example 10.8.2, and which uses Theorem 10.2, can also be derived from Theorem 10.9. There are however some essential differences:

1. The subcategory $\mathcal{W}$ of Top_B in Theorem 10.2 has the form $\mathcal{E}/B$ in Theorem 10.9. Not any subcategory of Top_B has this form.
2. Objects of $\mathcal{W}$ in Theorem 10.2 are assumed to be exponentiable in Top_B, while the objects of $\mathcal{E}$ in Theorem 10.9 are assumed to be exponentiable in Top. For instance, Theorem 10.2 can be used to prove that the category of fibrewise compactly generated spaces over a T_1-space is cartesian closed as shown in a latter section. Theorem 10.9 does not apply to prove this fact.
3. In Theorem 10.2, B is assumed to be T_1. Theorem 10.9 uses a different separation condition (condition 3).

Observe that Theorem 10.9 can be derived from Theorem 10.2 when B is Hausdorff.

11 The category kTop_B of fibrewise compactly generated spaces

Our objective in this section is to prove that the category of fibrewise compactly generated spaces over a T_1-base is cartesian closed.

Let Comp_B be the subcategory of Top_B of fibrewise compact, fibrewise Hausdorff spaces over B.

Proposition 11.1. *Assume that B is a T_1-space. Then Comp_B is left Kan extendable in Top_B.*

Proof. B is a T_1-space. Thus Comp_B contains the fibrewise spaces B^b of Example 8.3 for all $b \in B$. Therefore Comp_B is a suitable subcategory of Top_B. By Theorem 8.2, Comp_B is left Kan extendable in Top_B. ☐

Assume that B is T_1. Then $\mathsf{kTop}_B = (\mathsf{Comp}_B)_l[\mathsf{Top}_B]$ is a coreflective subcategory of Top_B. Let

$$k : \mathsf{Top}_B \longrightarrow \mathsf{kTop}_B \tag{42}$$

be a coreflector. An object in kTop_B is called a fibrewise compactly generated space over B.

Proposition 11.2. *Assume that B is a T_1-space and let X be a fibrewise Hausdorff space over B. Then the following properties are equivalent:*

1. *The fibrewise space X is fibrewise compactly generated.*
2. *If a subset A of X is such that $A \cap K$ is open in K for any subspace K of X which is fibrewise compact over B, then A is open in X.*
3. *If a subset A of X is such that $A \cap K$ is closed in K for any subspace K of X which is fibrewise compact over B, then A is closed in X.*

Proof. Let $u : K \to X$ be a continuous fibrewise map with K fibrewise compact over B. By Proposition 6.7.2, $u(K)$ is fibrewise compact fibrewise Hausdorff. The result then follows from Theorem 8.2.3. □

Recall that if X is a fibrewise space over B with projection $p : X \longrightarrow B$ and $W \subset B$, then the subspace $p^{-1}(W)$ of X is denoted by X_W.

Definition 11.3. *([25, Definition 10.1])*

Let X be fibrewise space over B. Then a subset A of X is said to be quasi-open (resp. quasi-closed) if the following condition is satisfied:

For each point $b \in B$ and each neighborhood V of b, there exists a neighborhood $W \subset V$ of b such that whenever $K \subset X_W$ is fibrewise compact over W, then $A \cap K$ is open (resp. closed) in K.

Lemma 11.4. *Let X be a topological space and $(V_i)_{i \in I}$ a family of subsets of X whose interiors cover X. Then a subset A of X is open (resp.closed) iff $A \cap V_i$ is open (resp.closed) in V_i for all $i \in I$.*

Proof. Clear. □

Corollary 11.5. [4] *Assume that B is a T_1-space. Let X be a fibrewise compactly generated fibrewise Hausdorff space over B. Then every quasi-open (resp. quasi-closed) subset of X is open (resp.closed).*

Proof. The two claims concerning quasi-open sets and quasi-closed sets are equivalent. We therefore only need to prove one of them.

Let O be a quasi-open subset of X. For each $b \in B$, there exists a neighborhood W_b of b such that given any subspace K of X_{W_b} which is fibrewise compact over W_b, $O \cap K$ is open in K.

Let K be any subspace of X which is fibrewise compact over B and let $b \in B$. The fibrewise subspace $K \cap X_{W_b}$ is fibrewise compact over W_b. Therefore $K \cap X_{W_b} \cap O$ is open in $K \cap X_{W_b}$. The family $(K \cap X_{W_b})_{b \in B}$ is a family of subsets of K whose interiors in K cover K. By Lemma 11.4, $K \cap O$ is open in K. By Proposition 11.2, O is open in X. □

Recall that a topological space X is said to be regular if for every $x \in X$ and every neighborhood V of x, there exists a closed neighborhood W of x which is contained in V. Observe that a regular T_1-space is Hausdorff.

4 I would like to greatly thank the first referee for explicitly stating this result to me.

Proposition 11.6. *Let B be a regular Hausdorff space and X a fibrewise Hausdorff space over B. Assume that every quasi-open (resp.quasi-closed) subset of X is open (resp.closed) in X. Then X is fibrewise compactly generated.*

Proof. Again, we only need to prove the proposition under the quasi-open hypothesis.

Let $O \subset X$ be such that $O \cap K$ is open in K for any subspace K of X which is fibrewise compact over B. We want to show that O is quasi-open.

Let $b \in B$ and let V be any neighborhood of b. The space B is regular. There exists a closed neighborhood W of b which is contained in V. Let K be any subspace of X_W which is fibrewise compact over W. The subspace W of X is closed. By Theorem 6.2, K is fibrewise compact over B. Therefore $O \cap K$ is open in K. It follows that O is a quasi-open subset of X, and is therefore open in X. By Proposition 11.2, X is fibrewise compactly generated. □

Remark 11.7. *We next compare our notion of fibrewise compactly generated space to the equally named notion considered by James in [25, Definition 10.3].*

1. *Our notion of fibrewise compactly generated spaces is defined only when the base space B is T_1.*
2. *A fibrewise space X over B is fibrewise compactly generated in the sense of James iff:*
   (a) *X is fibrewise Hausdorff.*
   (b) *Every quasi-open subset of X is open, or equivalently, if every quasi-closed subset of X is closed.*
3. *Assume that B is a T_1-space and X is a fibrewise Hausdorff space over B.*
   (a) *By Corollary 11.5, if X is fibrewise compactly generated in our sense, then it is so in the sense of James.*
   (b) *Assume further that B is a regular space. Then by Corollary 11.5 and Proposition 11.6, X is fibrewise compactly generated in our sense iff it is so in the sense of James.*

To fit our purposes, we give a definition of fibrewise locally compact spaces which is slightly stronger than the one given by James in [25, Definition 3.12.].

Definition 11.8. *A fibrewise space X over B is said to be fibrewise locally compact if for each $x \in X$, there exists a neighborhood K of x which is fibrewise compact over B.*

Proposition 11.9. *Assume that B is T_1. Then every fibrewise locally compact, fibrewise Hausdorff space is fibrewise compactly generated.*

Proof. This is a consequence of Corollary 8.7. □

Assume that B is T_1 and let Lcomp_B be the subcategory of Top_B of fibrewise locally compact, fibrewise Hausdorff spaces. Lcomp_B contains the suitable subcategory Comp_B of Top_B. Therefore Lcomp_B is suitable. By Theorem 8.2, Lcomp_B is left Kan extendable in Top_B. Let $\mathsf{lkTop}_B = (\mathsf{Lcomp}_B)_\iota[\mathsf{Top}_B]$.

Corollary 11.10. *Assume that B is T_1. Then* $\mathsf{lkTop}_B = \mathsf{kTop}_B$.

Proof. Comp_B is a subcategory of Lcomp_B and by Proposition 11.9, Lcomp_B is a subcategory of kTop_B. Therefore by Corollary 3.9, $\mathsf{lkTop}_B = \mathsf{kTop}_B$. □

The next result generalizes Proposition 11.9 and is a fibrewise version of [37, Proposition 2.6].

Proposition 11.11. *Assume that B be is T_1. Let X be a fibrewise locally compact fibrewise Hausdorff space and Y a fibrewise compactly generated space. Then the product $X \times_{\mathsf{Top}_B} Y$ is fibrewise compactly generated.*

Proof. This follows from Lemma 9.2 and Corollary 11.10. □

Theorem 11.12. *Assume that B is T_1. Then kTop_B is cartesian closed.*

Proof. B is T_1, thus Comp_B is a suitable subcategory of Top_B. By Theorem 6.15, every fibrewise compact fibrewise Hausdorff space is exponentiable in Top_B. By Corollary 6.4, the product of two fibrewise compact spaces is fibrewise compact. By Examples 5.2, the subcategory of fibrewise Hausdorff spaces over B is reflective. Therefore by Proposition 1.5.1.(a), the product (in Top_B) of two fibrewise Hausdorff spaces is fibrewise Hausdorff. It follows from Theorem 10.2 that kTop_B is cartesian closed. □

We next give a description of the internal $\hom$ functor of kTop_B.

Let $K \in \mathsf{Comp}_B$ and $Z \in \mathsf{Top}_B$. By Theorem 6.15, K is exponentiable in Top_B and the exponential object $\mathrm{map}_B(K, Z)$ is the topological space whose underlying set is $\coprod_{b\in B} \mathsf{Top}(K_b, Z_b)$ and whose topology is generated by the subsets

$$(C, O, \Omega) = \coprod_{b\in\Omega} \{\gamma \in \mathsf{Top}(K_b, Z_b) \mid \gamma(C_b) \subset O_b\} \tag{43}$$

where C is closed in K, O is open in Z and Ω is open in B.

Let

$$\hom(.,.) : \mathsf{kTop}_B^{op} \times \mathsf{kTop}_B \longrightarrow \mathsf{Top}_B \tag{44}$$

be the functor defined as in (36) and let $Y, Z \in k\mathsf{Top}_B$. By Remark 10.5, $\hom(Y, Z)$ is the topological space whose underlying set is $\coprod_{b\in B} \mathsf{Top}(Y_b, Z_b)$ and whose topology is generated by the subsets

$$(\sigma, C, O, \Omega) = \coprod_{b\in\Omega} \{\gamma \in \mathsf{Top}(Y_b, Z_b) \mid \gamma\sigma_b(C_b) \subset O_b\} \tag{45}$$

Where $(\sigma : K \longrightarrow Y) \in \mathsf{Comp}_B/Y$, C closed in K and O open in Z. By Theorem 9.6, the composite functor

$$(.)^{(.)} : \mathsf{kTop}_B^{op} \times \mathsf{kTop}_B \xrightarrow{\hom} \mathsf{Top}_B \xrightarrow{k} \mathsf{kTop}_B \tag{46}$$

is an internal $\hom$ functor for the cartesian closed category kTop_B, where $\mathsf{Top}_B \xrightarrow{k} \mathsf{kTop}_B$ is a coreflector.

Proposition 11.13. *Assume that B is T_1 and let top_B be one of the reflective subcategories*

$$\mathsf{fTop}_B, \mathsf{hTop}_B, \mathsf{uTop}_B, \mathsf{h_k Top}_B \text{ or } \mathsf{h_w Top}_B. \tag{47}$$

Then

1. Comp_B *is left Kan extendable as a subcategory of* top_B.
2. $(\mathsf{Comp}_B)_l[\mathsf{top}_B] = \mathsf{top}_B \cap \mathsf{kTop}_B$.
3. *A reflection of* Top_B *on* top_B *induces a reflection of* kTop_B *on* $\mathsf{Comp}_l[\mathsf{top}_B]$.
4. *The coreflection of* Top_B *on* kTop_B *given by Proposition 3.3 induces a coreflection of* top_B *on* $(\mathsf{Comp}_B)_l[\mathsf{top}_B]$.

Proof. top_B is reflective, closed under subobjects subcategory of Top_B containing the suitable subcategory Comp_B. Properties 1-4 are then consequences of Proposition 8.8. □

Corollary 11.14. *Let* top *be one of the reflective subcategories*

$$\mathsf{fTop}, \mathsf{hTop}, \mathsf{uTop}, \mathsf{h_c Top}, \mathsf{h_k Top} \text{ or } \mathsf{h_w Top} \tag{48}$$

of Top. *Then*

1. Comp *is left Kan extendable as a subcategory of* top.
2. *A reflection of* Top *on* top *induces a reflection of* kTop *on* $\mathsf{Comp}_l[\mathsf{top}]$.
3. $\mathsf{Comp}_l[\mathsf{top}] = \mathsf{top} \cap \mathsf{kTop}$.
4. *The coreflection of* Top *on* kTop *given by Proposition 3.3 induces a coreflection of* top *on* $\mathsf{Comp}_l[\mathsf{top}]$.

Notation 11.15.

1. *Let* $\mathsf{kfTop}_B = \mathsf{kTop}_B \cap \mathsf{fTop}_B$, $\mathsf{khTop}_B = \mathsf{kTop}_B \cap \mathsf{hTop}_B$, $\mathsf{kuTop}_B = \mathsf{kTop}_B \cap \mathsf{uTop}_B$ *and* $\mathsf{kh_k Top}_B = \mathsf{kTop}_B \cap \mathsf{h_k Top}_B$.
2. *Similarly, let* $\mathsf{kfTop} = \mathsf{kTop} \cap \mathsf{fTop}$, $\mathsf{khTop} = \mathsf{kTop} \cap \mathsf{hTop}$, $\mathsf{kuTop} = \mathsf{kTop} \cap \mathsf{uTop}$, $\mathsf{kh_c Top} = \mathsf{kTop} \cap \mathsf{h_c Top}$, $\mathsf{kh_k Top} = \mathsf{kTop} \cap \mathsf{h_k Top}$ *and* $\mathsf{kh_w Top} = \mathsf{kTop} \cap \mathsf{h_w Top}$.

12 Cartesian closed subcategories of kTop_B and kTop

In this section, we prove that kfTop_B is a cartesian closed subcategory of kTop_B. We also prove that the subcategories kfTop, khTop, kuTop, $\mathsf{kh_c Top}$, and $\mathsf{kh_w Top}$ are cartesian closed.

Proposition 12.1. *Assume that B is T_1. Then the reflective subcategory* kfTop_B *of* kTop_B *is cartesian closed with internal* hom *functor induced by that of* kTop_B.

Proof. By Proposition 11.13 , kfTop_B is a reflective subcategory of kTop_B. By Remark 1.12, we just need to prove that if $Y, Z \in \mathsf{kfTop}_B$, then the exponential object Z^Y in kTop_B defined by (46) is again an object of kfTop_B.

So let $Y, Z \in \mathsf{kfTop}_B$.

$$\hom(Y, Z) \cong \lim_{(K \xrightarrow{\sigma} Y) \in \mathsf{Comp}_B | Y} \mathrm{map}_B(K, Z). \tag{49}$$

By Proposition 6.16, the spaces $\mathrm{map}_B(K, Z)$ in (49) are T_1. The subcategory fTop_B is a reflective, by Proposition 1.5.1.(a), $\hom(Y, Z)$ is T_1. Let ϵ be the counit of the coreflection of Top_B on kTop_B. The $\hom(Y, Z)$-component

$$Z^Y = k(\hom(Y, Z)) \xrightarrow{\epsilon_{\hom(Y,Z)}} \hom(Y, Z) \tag{50}$$

of ϵ is monic. The category fTop_B is closed under subobjects, therefore $Z^Y \in \mathsf{fTop}_B$. The space $Z^Y \in \mathsf{kTop}_B$, thus $Z^Y \in \mathsf{kTop}_B \cap \mathsf{fTop}_B = \mathsf{kfTop}_B$. □

Remark 12.2.

1. *Assume that B be a T_1-space. Let K be a fibrewise compact, fibrewise Hausdorff space and $Z \in \mathsf{khTop}_B$. Then as observed in Remark 6.17, the space $\mathrm{map}_B(K, Z)$ may not be fibrewise Hausdorff. Therefore the argument used in the proof of Proposition 12.1 cannot be used to prove that khTop_B is cartesian closed. In fact, this does not seem to be true.*

2. *Let $\mathcal{K}$ be the subcategory of Top_B of compactly generated spaces in the sense of James ([25, Definition 10.3] and Remark 11.7.2). Then $\mathcal{K}$ is cartesian with binary product $\times'_B$ defined in ([25, Page 83]. For any $X \in \mathcal{K}$ that is locally sliceable [25, Definition 1.16], the functor*

$$- \times'_B X : \mathcal{K} \longrightarrow \mathcal{K}$$

has a right adjoint which is the functor

$$\mathit{map}'_B(X, -) : \mathcal{K} \longrightarrow \mathcal{K}$$

defined in [25, Page 84]. This follows from the fact that the evaluation functions

$$\mathit{map}'_B(X, Z) \times'_B X \longrightarrow Z$$

are continuous [25, Page 85] and that the adjoint of a continuous function

$$h : Y \times'_B X \longrightarrow Z$$

can be regarded as a continuous function

$$k_B(\widehat{h}) : Y \longrightarrow \mathit{map}'_B(X, Z)$$

as in [25, Lemma 10.16].

Proposition 12.3. *[35, Proposition 11.4.]*

Every compactly generated, k-Hausdorff space is weak Hausdorff. That is,

$$\mathsf{kTop} \cap \mathsf{h_k Top} \subset \mathsf{h_w Top}.$$

Proof. Let X be a compactly generated k-Hausdorff space. Let $f : K \longrightarrow X$ be continuous where K is compact Hausdorff. Let $g : L \longrightarrow X$ be a continuous map from a compact Hausdorff space L to X, $f \times_{\mathsf{Top}} g : K \times_{\mathsf{Top}} L \longrightarrow X \times_{\mathsf{Top}} X$ and $pr_L : K \times_{\mathsf{Top}} L \longrightarrow L$ is the projection. The map pr_L is closed, therefore $g^{-1}(f(K)) = pr_L((f \times_{\mathsf{Top}} g)^{-1})(\Delta_X)$ is closed. It follows that $f(K)$ is closed and X is weak Hausdorff. □

Remark 12.4. *Observe that by Propositions 12.3 and 7.10,* $\mathsf{kh_kTop} = \mathsf{kh_wTop}$.

Proposition 12.5. *The reflective subcategories*

$$\mathsf{kfTop}, \mathsf{khTop}, \mathsf{kuTop}, \mathsf{kh_cTop}, \mathsf{kh_kTop} \textit{ and } \mathsf{kh_wTop} \tag{51}$$

of kTop *are cartesian closed with internal* hom *functor induced by that of* kTop.

Proof. Let top be one of the categories in (51). By Proposition 11.14, top is a reflective subcategory of kTop. By Remark 1.12, we just need to prove that if $Y, Z \in$ top, then the exponential object Z^Y in kTop defined by Examples 10.8.1 is again in top. So let $Y, Z \in$ top. For $y \in Y$, the evaluation map $Ev_y : Z^Y \longrightarrow Z$ at y and is continuous.

- top = kfTop:
  Let $f_0 \in Z^Y$. Ev_y is continuous, Z is Fréchet, thus $Ev_y^{-1}(f(y))$ is closed in Z^Y. It follows that $\{f_0\} = \bigcap_{y \in Y} Ev_y^{-1}(f_0(y))$ is closed in Z^Y and Z^Y is a Fréchet space.

- top = khTop:
  Let $f, g \in \mathsf{khTop}$ with $f \neq g$. Let $y_0 \in Y$ be such that $f(y_0) \neq g(y_0)$. Let U, V be disjoint open neighborhoods of $f(y_0)$ and $g(y_0)$. Then $Ev_{y_0}^{-1}(U)$ and $Ev_{y_0}^{-1}(V)$ are disjoint open neighborhoods of f and g. It follows that Z^Y is Hausdorff.

- top = kuTop:
  Let $f, g \in \mathsf{khTop}$ with $f \neq g$. Let $y_0 \in Y$ be such that $f(y_0) \neq g(y_0)$. Let A, B be disjoint closed neighborhoods of $f(y_0)$ and $g(y_0)$. Then $Ev_{y_0}^{-1}(A)$ and $Ev_{y_0}^{-1}(B)$ are disjoint closed neighborhoods of f and g. It follows that Z^Y is Urysohn.

- top = $\mathsf{kh_cTop}$:
  Let $f, g \in \mathsf{kh_cTop}$ with $f \neq g$. Let $y_0 \in Y$ be such that $f(y_0) \neq g(y_0)$. Z is completely Hausdorff, thus there exists a continuous fonction $\psi : Z \longrightarrow [0, 1]$ such that $\psi(f(y_0)) = 0$ and $\psi(g(y_0)) = 1$. Ev_{y_0} is continuous, thus $\psi Ev_{y_0} : Z^Y \longrightarrow [0, 1]$ is continuous. $\psi Ev_{y_0}(f) = 0$ and $\psi Ev_{y_0}(g) = 1$. It follows that Z^Y is completely Hausdorff.

- top = $\mathsf{kh_kTop} = \mathsf{kh_wTop}$:
  For a topological space X, let Δ_X denote the diagonal of X. Let

  $$f : K \longrightarrow Z^Y \times_{\mathsf{Top}} Z^Y$$

  be a continuous map, where K is compact Hausdorff. Then

  $$f^{-1}(\Delta_{Z^Y}) = \bigcap_{y \in Y} ((Ev_y \times_{\mathsf{Top}} Ev_y)f)^{-1}(\Delta_Z) \tag{52}$$

  is closed in K. It follows that Z^Y is k-closed.

□

Proposition 12.6. *Assume that B is Hausdorff. Then* $\mathsf{kTop}/B \subset \mathsf{kTop}_B$.

Proof. B is Hausdorff, by Theorem 6.2, $\mathsf{Comp}/B \subset \mathsf{Comp}_B$. By Examples 10.8.2, Comp/B is left Kan extendable and $(\mathsf{Comp}/B)_l[\mathsf{Top}_B] = \mathsf{kTop}/B$. It follows from Corollary 3.9 that

$$\mathsf{kTop}/B = (\mathsf{Comp}/B)_l[\mathsf{Top}_B] \subset (\mathsf{Comp}_B)_l[\mathsf{Top}_B] = \mathsf{kTop}_B.$$

□

Proposition 12.7. *Assume that B is locally compact Hausdorff space. Then* $\mathsf{kTop}_B = \mathsf{kTop}/B$.

Proof. By Proposition 12.6, $\mathsf{kTop}/B \subset \mathsf{kTop}_B$.

Let $(X \xrightarrow{p} B) \in \mathsf{Comp}_B$, let $x_0 \in K$, $b_0 = p(x_0)$ and K a compact neighborhood of b_0. Then $p^{-1}(K)$ is a neighborhood of x_0 which is Hausdorff. By Proposition 6.10, $p^{-1}(K)$ is compact. Therefore $(p^{-1}(K) \xrightarrow{p_/} B) \in \mathsf{Comp}/B$. By Corollary 8.7, $(X \xrightarrow{p} B) \in (\mathsf{Comp}/B)_l[\mathsf{Top}_B]$. It follows that $\mathsf{Comp}_B \subset (\mathsf{Comp}/B)_l[\mathsf{Top}_B]$. By Corollary 3.9,

$$\mathsf{kTop}_B = (\mathsf{Comp}_B)_l[\mathsf{Top}_B] \subset (\mathsf{Comp}/B)_l[\mathsf{Top}_B] = \mathsf{kTop}/B.$$

Therefore $\mathsf{kTop}_B = \mathsf{kTop}/B$. □

Assume that B is locally compact Hausdorff. Then by Example 10.8.1, $B \in \mathsf{kTop}$ and by Proposition 12.7, the category kTop_B is just the slice category kTop/B. The adjunction given by lemma A.2 yields an adjunction

$$\mathsf{kTop}_B \underset{. \times_{\mathsf{kTop}} B}{\overset{P_B}{\rightleftarrows}} \mathsf{kTop}. \tag{53}$$

13 Fibrewise sequential spaces

It is a well known fact that the category of sequential spaces is cartesian closed. We here show that this fact extends to the fibrewise setting, provided that the base space B is Hausdorff.

Let $\mathbb{N}$ be the discrete space of non-negative integers, $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$ its one point compactification. Let N_B be the subcategory of Top_B whose objects are continuous maps $\mathbb{N}^+ \longrightarrow B$.

Proposition 13.1. *The subcategory* N_B *is a left Kan extendable in* Top_B.

Proof. The subcategory N_B is suitable. By Theorem 8.2, N_B is left Kan extendable. □

We will call N_B-generated objects fibrewise sequential spaces. The category $(\mathsf{N}_B)_l[\mathsf{Top}_B]$ of fibrewise sequential spaces will be denoted by Seq_B.

Remark 13.2. *Let* Seq *be the subcategory of* Top *that corresponds to* Seq_B *under the isomorphism P of (10). Then* N *is a dense subcategory of* Seq*. The objects of* Seq *are called sequential spaces.*

Proposition 13.3. *A fibrewise topological space $X \longrightarrow B$ is fibrewise sequential iff its domain X is a sequential space.*

Proof. This is a consequence of Corollary 3.5 and lemma A.2.1. □

Proposition 13.4. *Assume that B is T_2. Then:*

1. Seq_B *is cartesian closed.*
2. Seq_B *is a coreflective subcategory of* kTop_B*. Furthermore, the inclusion functor* $\mathsf{Seq}_B \hookrightarrow \mathsf{kTop}_B$ *preserves finite products.*

Proof.

1. The space B is T_2. By Theorem 6.2, objects of N_B are fibrewise compact, fibrewise Hausdorff. By Theorem 6.15 they are exponentiable in Top_B. The space $\mathbb{N}^+$ is a metric space. Let $\mathbb{N}^+ \xrightarrow{p} B$ and $\mathbb{N}^+ \xrightarrow{q} B$ be two objects in N_B. The domain of the product $p \times_{\mathsf{Top}_B} q$ is a subspace of $\mathbb{N}^+ \times_{\mathsf{Top}} \mathbb{N}^+$ and is therefore a metric space. It follows that the domain of $p \times_{\mathsf{Top}_B} q$, which is a subspace of $\mathbb{N}^+ \times_{\mathsf{Top}} \mathbb{N}^+$, is a metric space. A metric space is sequential, thus by Proposition 13.3, $p \times_{\mathsf{Top}_B} q \in \mathsf{Seq}_B$. By Theorem 10.2, Seq_B is cartesian closed.
2. The base space B is T_2, therefore N_B is a subcategory of Comp_B. By Corollary 3.9.1, Seq_B is a coreflective subcategory of kTop_B, and by Corollary 9.7, the inclusion functor $\mathsf{Seq}_B \hookrightarrow \mathsf{kTop}_B$ preserves finite products.

□

Remark 13.5.

1. *Assume that B is a sequential space. Then by Proposition 13.3,* Seq_B *is just the slice category* Seq/B *and the adjunction given by Lemma A.2.2 yields an adjunction*

$$\mathsf{Seq}_B \underset{.\times_{\mathsf{Seq}} B}{\overset{P_B}{\rightleftarrows}} \mathsf{Seq}. \tag{54}$$

2. *Let s :* $\mathsf{Top} \longrightarrow \mathsf{Seq}$ *be a coreflector. Then the functor* $\mathsf{Seq}_B \longrightarrow \mathsf{Seq}_{s(B)}$ *which takes a fibrewise sequential space $X \xrightarrow{p} B$ to $X \xrightarrow{s(p)} s(B)$ is an isomorphism of categories.*
3. *By the previous points, for any topological space B, the functor P_B :* $\mathsf{Seq}_B \longrightarrow \mathsf{Seq}$ *is left adjoint. Its right adjoint takes a sequential space X to the fibrewise sequential space $X \times_{\mathsf{Seq}} s(B)$ whose projection is the composite map*

$$X \times_{\mathsf{Seq}} s(B) \xrightarrow{pr} s(B) \xrightarrow{\epsilon_B} B,$$

where s : $\mathsf{Top} \longrightarrow \mathsf{Seq}$ *is a coreflector and ϵ_B is the B-component of the counit ϵ of the coreflection of* Top *on* Seq.

14 Fibrewise Alexandroff spaces

The category of Alexandroff space is known to be equivalent to the cartesian category of preorders and is therefore cartesian closed (Escardó, Lawson [15, Examples (2), page 114]). Our objective in this section is to extend this fact to the fibrewise setting.

For $b \in B$, let

$$\pi_b^t : \mathsf{Top}_B \longrightarrow \mathsf{Top}$$

be the functor which takes a fibrewise space over B to its fibre over b as defined by (73), and let

$$i_b : \mathsf{Top} \longrightarrow \mathsf{Top}_B$$

be the functor which takes a space X to the fibrewise space whose domain is X and whose projection $X \longrightarrow B$ is constant at b.

Lemma 14.1. *Let $b \in B$. Then*

1. *The functor i_b is left adjoint to π_b^t.*
2. *Assume that $\{b\}$ is closed in B. Then π_b^t is left adjoint to the functor*

$$\begin{array}{rccc} map(B^b, i_b(.)) : & \mathsf{Top} & \longrightarrow & \mathsf{Top}_B \\ & Y & \longmapsto & map_B(B^b, i_b(Y)). \end{array} \tag{55}$$

where B^b is the fibrewise space defined by Example 8.3.

Proof.

1. Let $X \in \mathsf{Top}$ and $Y \in \mathsf{Top}_B$. Then $\mathsf{Top}_B(i_b(X), Y) \cong \mathsf{Top}(X, \pi_b^t(Y))$.
2. Let $X \in \mathsf{Top}_B$ and $Y \in \mathsf{Top}$. Then

$$\begin{array}{rcll} \mathsf{Top}(\pi_b^t(X), Y) & \cong & \mathsf{Top}(X_b, Y) & \\ & \cong & \mathsf{Top}_B(X \times_{\mathsf{Top}_B} B^b, i_b(Y)) & \\ & \cong & \mathsf{Top}_B(X, \mathrm{map}_B(B^b, i_b(Y))) & \text{(by Theorem 6.15)} \end{array}$$

□

Proposition 14.2. *Let $E \in \mathsf{Top}$ be an exponentiable space and let $b \in B$ be such that $\{b\}$ is closed. Then $i_b(E)$ is an exponentiable object of Top_B.*

Proof. Let

$$(.)^E : \mathsf{Top} \longrightarrow \mathsf{Top}$$

be a right adjoint of the functor

$$. \times_{\mathsf{Top}} E : \mathsf{Top} \longrightarrow \mathsf{Top}$$

and let $X, Y \in \mathsf{Top}_B$. We have

$$X \times_{\mathsf{Top}_B} i_b(E) = i_b(\pi_b^t(X) \times_{\mathsf{Top}} E).$$

Therefore

$$\begin{array}{rcll} \mathsf{Top}_B(X \times_{\mathsf{Top}_B} i_b(E), Y) & \cong & \mathsf{Top}_B(i_b(\pi_b^t(X) \times_{\mathsf{Top}} E), Y) & \\ & \cong & \mathsf{Top}(\pi_b^t(X) \times_{\mathsf{Top}} E, Y_b) & \text{(by Lemma 14.1.1)} \\ & \cong & \mathsf{Top}(\pi_b^t(X), Y_b^E) & \\ & \cong & \mathsf{Top}_B(X, \mathrm{map}_B(B^b, i_b(Y_b^E))) & \text{(by Lemma 14.1.2)} \end{array}$$

Thus $i_b(E)$ is exponentiable in Top_B. □

The Sierpinski space is the topological space denoted by $\mathbb{S}$, whose underlying set is $\{0,1\}$ and whose set of open sets is $\mathcal{O}(\mathbb{S}) = \{\emptyset, \{1\}, \mathbb{S}\}$. Let Sier be the subcategory of Top having $\mathbb{S}$ as its unique object and Sier_B the subcategory of Top_B whose objects are all continuous maps $\mathbb{S} \to B$.

Proposition 14.3. Sier_B *is a left Kan extendable subcategory of* Top_B.

Proof. Sier_B is a suitable subcategory of Top_B. By Theorem 8.2, Sier_B is left Kan extendable. □

The subcategory Sier of Top corresponds to the subcategory $\mathsf{Sier}_{\mathsf{pt}}$ of $\mathsf{Top}_{\mathsf{pt}}$ under the isomorphism P of (10). We therefore have the following.

Corollary 14.4. Sier *is left Kan extendable subcategory in* Top.

Proposition 14.5. *A fibrewise topological space* $X \longrightarrow B$ *is* Sier_B*-generated iff its domain* X *is* Sier*-generated.*

Proof. This is a consequence of Proposition 3.3.1 and lemma A.2.1. □

Recall that an Alexandroff space is a topological space in which arbitrary intersections of open subsets are open. Equivalently, an Alexandroff space is a topological space for which arbitrary unions of closed subsets are closed. Let Alex be the subcategory of Top of Alexandroff spaces. A finite topological space has only finitely many open sets, and is therefore an Alexandroff space.

Let B be a subset of an Alexandroff space X and let $\overline{B}$ denote the topological closure of B. The subspace $\bigcup_{b\in B} \overline{\{b\}}$ is a closed subset of X containing B. It follows that $\overline{B} = \bigcup_{b\in B} \overline{\{b\}}$.

We next provide a simple proof of the following result which is given (without proof) in [15, Examples (2), page 114].

Proposition 14.6. *A topological space* X *is* Sier*-generated iff it is an Alexandroff space. That is* $\mathsf{Sier}_l[\mathsf{Top}] = \mathsf{Alex}$.

Proof. Let X be a Sier-generated topological space, $(O_i)_{i\in I}$ a family of open sets in X and $f : \mathbb{S} \longrightarrow X$ a continuous map. Then $f^{-1}(\bigcap_{i\in I} O_i) = \bigcap_{i\in I} f^{-1}(O_i)$ which open in $\mathbb{S}$ since $\mathbb{S}$ is an Alexandroff space. By Corollary 8.4.2, $\bigcap_{i\in I} O_i$ is open in X and X is an Alexandroff space. Conversely, assume that X is an Alexandroff space. Let $B \subset X$ be such that $f^{-1}(B)$ is closed for every continuous map $f : \mathbb{S} \longrightarrow X$ and let $a \in \overline{B}$. There exists $b \in B$ such that $a \in \overline{\{b\}}$. If $a = b$ then $a \in B$, if $a \neq b$, define $g : \mathbb{S} \longrightarrow X$ by $g(0) = a$ and $g(1) = b$. Then

$$g(\overline{\{1\}}) = g(\mathbb{S}) = \{a, b\} \subset \overline{\{b\}} \subset \overline{g(\{1\})}$$

Therefore g is continuous and $g^{-1}(B)$ is a closed subset of $\mathbb{S}$ containing 1. It follows that $g^{-1}(B) = \mathbb{S}$, in particular, $a = g(0) \in B$ and B is closed in X. By Corollary 8.4.2, X is Sier-generated. □

It follows that $(\mathsf{Sier}_B)_l[\mathsf{Top}_B]$ is the subcategory Alex_B of fibrewise spaces $X \to B$ whose domain X is an Alexandroff space. Objects of Alex_B are called fibrewise Alexandroff spaces over B.

Corollary 14.7.

1. Alex *is a coreflective subcategory of* Top *containing* Sier *as a dense subcategory.*

2. Alex_B *is a coreflective subcategory of* Top_B *containing* Sier_B *as a dense subcategory.*

Proof. This follows from Proposition 14.6, Proposition 3.3, Proposition 3.3.1 and Proposition 14.5. □

We next generalize [15, Lemma 4.6.].

Proposition 14.8.

1. *The Sierpinski space* $\mathbb{S}$ *is sequential.*

2. *The category* Alex *is a coreflective subcategory of* Seq.

3. *The category* Alex_B *is a coreflective subcategory of* Seq_B.

Proof. Define $q : \mathbb{N}^+ \longrightarrow \mathbb{S}$ by $q(\infty) = 0$ and $q(n) = 1$ for all $n \in \mathbb{N}$. The map q is a quotient map, thus by Proposition 8.6.1, $\mathbb{S}$ is sequential. By Corollary 3.9, Alex is a coreflective subcategory of Seq. Similarly, Alex_B is a coreflective subcategory of Seq_B. □

Proposition 14.9.

1. *The subcategory* Alex *of* Top *is cartesian closed.*

2. *If B is* T_1*, then the subcategory* Alex_B *of* Top_B *is cartesian closed.*

Proof. We just need to prove 2.

Being finite, $\mathbb{S}$ is a core-compact space. It is therefore an exponentiable object of Top. Let $\mathbb{S} \xrightarrow{p} B$ be continuous. Assume the space B is T_1, therefore p is constant. By Proposition 14.2, $\mathbb{S} \xrightarrow{p} B$ is an exponentiable object of Top_B. The product in Top_B of two fibrewise Sierpinski spaces is a fibrewise Alexandroff space. By Theorem 10.2, Alex_B is cartesian closed. □

Remark 14.10.

1. *Assume that B is an Alexandroff space. Then* Alex_B *is just the slice category* Alex/B *and the adjunction given by Lemma A.2.2 yields an adjunction*

$$\mathsf{Alex}_B \underset{\cdot \times_{\mathsf{Alex}} B}{\overset{P_B}{\rightleftarrows}} \mathsf{Alex}. \tag{56}$$

2. *Let $a : \mathsf{Top} \longrightarrow \mathsf{Alex}$ be a coreflector. Then the functor* $\mathsf{Alex}_B \longrightarrow \mathsf{Alex}_{a(B)}$ *which takes a fibrewise Alexandroff space $X \xrightarrow{p} B$ to $X \xrightarrow{a(p)} a(B)$ is an isomorphism of categories.*

3. *By the previous two points, for any topological space B, the functor $P_B : \mathsf{Alex}_B \longrightarrow \mathsf{Alex}$ is left adjoint. Its right adjoint takes an Alexandroff space X to the fibrewise Alexandroff space $X \times_{\mathsf{Alex}} a(B)$ with projection the composite*

$$X \times_{\mathsf{Alex}} a(B) \xrightarrow{pr} a(B) \xrightarrow{\epsilon_B} B.$$

Where $a : \mathsf{Top} \longrightarrow \mathsf{Alex}$ is a coreflector and ϵ_B is the B-component of the counit ϵ of the of the coreflection of Top *on* Sier.

Appendices

A Limits in a slice category

The aim of this section is to prove that if $\mathcal{C}$ is a bicomplete category and $b \in \mathcal{C}$, then the slice category $\mathcal{C}/b$ of $\mathcal{C}$ over b is bicomplete.

Let $F : \mathcal{A} \longrightarrow \mathcal{C}$ an $G : \mathcal{B} \longrightarrow \mathcal{C}$ be two functors. The comma category F/G is defined to be the category whose objects are arrows $F(a) \xrightarrow{\alpha} G(b)$ and whose morphisms from $F(a) \xrightarrow{\alpha} G(b)$ to $F(a') \xrightarrow{\alpha'} G(b')$ are pairs of morphisms $(f, g) \in \mathcal{A}(a, a') \times_{\mathsf{Set}} \mathcal{B}(b, b')$ rendering commutative the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(a') \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ G(b) & \xrightarrow[G(g)]{} & G(b') \end{array} \tag{57}$$

We have functors

$$P : F/G \longrightarrow \mathcal{A} \quad \text{and} \quad Q : F/G \longrightarrow \mathcal{B} \tag{58}$$

defined as follows: if $F(a) \xrightarrow{\alpha} G(b) \in F/G$, then $P(\alpha) = a$ and $Q(\alpha) = b$. If (f, g) is a morphism from $F(a) \xrightarrow{\alpha} G(b)$ to $F(a') \xrightarrow{\alpha'} G(b')$ as in (57), then $P((f, g)) = f$ and $Q((f, g)) = g$.

Notations A.1. *Let $F : \mathcal{A} \longrightarrow \mathcal{C}$ an $G : \mathcal{B} \longrightarrow \mathcal{C}$ be two functors.*

1. *If $\mathcal{A}$ is a subcategory of $\mathcal{C}$ and $F : \mathcal{A} \longrightarrow \mathcal{C}$ is the inclusion functor, then F/G is also denoted by $\mathcal{A}/G$.*

2. *If $\mathcal{B}$ is a subcategory of $\mathcal{C}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ is the inclusion functor, then F/G is also denoted by $F/\mathcal{B}$.*

3. *If $\mathcal{A}, \mathcal{B}$ are subcategories of $\mathcal{C}$ and $F : \mathcal{A} \longrightarrow \mathcal{C}$, $G : \mathcal{B} \longrightarrow \mathcal{C}$ are the inclusion functors, then F/G is denoted by $\mathcal{A}/\mathcal{B}$.*

4. *If $\mathcal{A}$ has just one object $*$ and just one morphisms id_*, then F/G is denoted by c/G where $c = F(*)$. If further $\mathcal{B} = \mathcal{C}$ and G is the identity functor, then c/G is called the slice category under c and is denoted by $c/\mathcal{C}$.*

5. *If $\mathcal{B}$ has just one object $*$ and just one morphisms id_*, then F/G is denoted by F/c where $c = G(*)$. If further $\mathcal{A} = \mathcal{C}$ and F is the identity functor, then F/c is called the slice category over c and is denoted by $\mathcal{C}/c$. Observe that this notation is consistent with the one previously used.*

Let $\mathcal{C}$ be category, $b \in \mathcal{C}$, $\mathcal{C}/b$ the slice category of $\mathcal{C}$ over b and define

$$P_b : \mathcal{C}/b \longrightarrow \mathcal{C} \tag{59}$$

to be the functor which takes an arrow-object $c \to b$ to its domain c.

Lemma A.2.

1. *The functor P_b creates colimits. In particular, if $\mathcal{C}$ is cocomplete, then so is $\mathcal{C}/b$.*

2. *Assume that the categorical product $c \times_{\mathcal{C}} b$ exists for every $c \in \mathcal{C}$, then P_b is left adjoint. In particular P_b preserves colimits.*

Proof.

1. This follows from the dual of a straightforward generalization of [32, Lemma, page 121].

2. The functor $\mathcal{C} \longrightarrow \mathcal{C}/b$ which takes an object $c \in \mathcal{C}$ to the arrow $c \times_{\mathcal{C}} b \to b$ is a right adjoint of P_b. Thus P_b preserves colimits.

□

Let Cat be the category of small categories. A poset carries a category structure in the standard way. Thus the ordinal numbers $1 = \{0\}$ and $2 = \{0, 1\}$ may be viewed as small categories. The small category 1 is a terminal object in Cat. The cone $\mathcal{I}^{\triangleright}$ of $\mathcal{I} \in \text{Cat}$ is defined in [36, Exercice 3.5.iv] to be the pushout in Cat:

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\;*\;} & 1 \\ {\scriptstyle i_1}\downarrow & & \downarrow \\ \mathcal{I} \times_{\mathsf{Cat}} 2 & \longrightarrow & \mathcal{I}^{\triangleright} \end{array}$$

Let i be the composite functor $\mathcal{I} \overset{i_0}{\hookrightarrow} \mathcal{I} \times_{\mathsf{Cat}} 2 \to \mathcal{I}^{\triangleright}$. Then $i : \mathcal{I} \hookrightarrow \mathcal{I}^{\triangleright}$ is fully faithful and $\mathcal{I}$ may be viewed as a full subcategory of $\mathcal{I}^{\triangleright}$. Furthermore

1. The category $\mathcal{I}^{\triangleright}$ contains one more object than $\mathcal{I}$, it is denoted by $*$.

2. The set $\mathcal{I}^{\triangleright}(i, *)$ contains precisely one morphism denoted by σ_i, $\forall i \in \mathcal{I}$.

3. The set $\mathcal{I}^{\triangleright}(*, *)$ contains solely the identity morphism.

4. The set $\mathcal{I}^{\triangleright}(*, i)$ is empty, $\forall i \in \mathcal{I}$.

Let $X : \mathcal{I} \longrightarrow \mathcal{C}/b$ be any functor. Define $X^{\triangleright} : \mathcal{I}^{\triangleright} \longrightarrow \mathcal{C}$ to be the unique functor satisfying the following properties:

1. The functor $X^{\triangleright}$ extends $P_b X$ over the category $\mathcal{I}^{\triangleright}$. That is the following diagram commutes

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{X} & \mathcal{C}/b \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle P_b} \\ \mathcal{I}^{\triangleright} & \xrightarrow[X^{\triangleright}]{} & \mathcal{C} \end{array}$$

2. $X^{\triangleright}(*) = b$.

3. $X^{\triangleright}(\sigma_i)$ is the arrow $X(i)$ in $\mathcal{C}$, $i \in \mathcal{I}$.

Then one has the following result.

Lemma A.3.

1. *The functor X has a limit if and only if $X^{\triangleright}$ has a limit. Furthermore, a limiting cone $l \overset{\lambda}{\Longrightarrow} X^{\triangleright}$ induces a limiting cone from $\lambda_* : l \longrightarrow b$ to X, where λ_* is the $*$-component of the cone λ.*

2. *If $\mathcal{C}$ is complete, then so is $\mathcal{C}/b$.*

Proof. Clear. □

Examples A.4. *Assume that $\mathcal{C}$ is complete*

1. *Let $x \xrightarrow{\sigma} b, y \xrightarrow{\tau} b \in \mathcal{C}/b$, $p_1 : x \times_{\mathcal{C}} y \longrightarrow x$, $p_2 : x \times_{\mathcal{C}} y \longrightarrow y$ be the projections and let*

$$e \xrightarrow{i} x \times_{\mathcal{C}} y \underset{\tau p_2}{\overset{\sigma p_1}{\rightrightarrows}} b$$

be the equalizer (in $\mathcal{C}$) of the maps σp_1 and τp_2. The diagram

$$\begin{array}{ccc} e & \xrightarrow{p_2 i} & y \\ {\scriptstyle p_1 i}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \tau} \\ x & \xrightarrow[\sigma]{} & b \end{array}$$

is a pullback diagram. By Lemma A.3, the composite $\sigma p_1 i = \tau p_2 i : e \longrightarrow b$ is the product of the objects σ and τ of $\mathcal{C}/b$.

2. *Using generalized equalizers, the previous example may be extended to the case where one has a family of arrow-objects $x_i \xrightarrow{\sigma_i} b$ of $\mathcal{C}/b$ indexed be a small set I.*

B Limits in a slice category of sets

The aim of this section is to establish certain properties of limits and colimits in a slice category of sets.

The category Set of (small) sets is a bicomplete category. For $X, Y, Z \in \mathsf{Set}$, there is a natural isomorphism

$$\mathsf{Set}(X \times_{\mathsf{Set}} Y, Z) \cong \mathsf{Set}(X, \mathsf{Set}(Y, Z)) \tag{60}$$

so that Set is cartesian closed.

Let $E \in \mathsf{Set}$. The slice category of Set over E is denoted by Set_E. An object of Set_E is called a set over E. It consists of a set X together with a function $p : X \longrightarrow E$ called projection. A set $X \xrightarrow{p} E$ over E is often identified with its domain X. Let

$$P_E : \mathsf{Set}_E \longrightarrow \mathsf{Set} \tag{61}$$

be the functor defined as in (59).

Proposition B.1.

1. *The category* Set_E *is bicomplete.*
2. *The functor* P_E *creates and preserves colimits.*

Proof. This follows from Lemma A.2, Lemma A.3 and the fact that Set is bicomplete. □

Let $F \subset E$, $J^s_F : F \to E$ the inclusion map and

$$J^{s,*}_F : \mathsf{Set}_E \to \mathsf{Set}_F \tag{62}$$

the functor given by pulling back along the inclusion map J^s_F.

Lemma B.2. *Let $F \subset E$. Then the functor $J^{s,*}_F : \mathsf{Set}_E \longrightarrow \mathsf{Set}_F$ preserves both limits and colimits.*

Proof. Clearly, $J^{s,*}_F$ is at once a right adjoint and a left adjoint. Therefore it preserves both limits and colimits. □

Let $e \in E$. For $X \in \mathsf{Set}_E$, define the fibre of X over e to be the set $X_e = p^{-1}(e)$, where p is the projection of the set X over E. One has a functor

$$\pi^s_e : \mathsf{Set}_E \longrightarrow \mathsf{Set} \tag{63}$$

defined as follows: For $X \in \mathsf{Set}_E$, $\pi^s_e(X) = X_e$ and for $f \in \mathsf{Set}_E(X, Y)$, $\pi^s_e(f)$ is the map $f_e : X_e \longrightarrow Y_e$ induced by f.

Lemma B.3.

1. *The functors π^s_e, $e \in E$, preserve limits and a functor $X : \mathcal{I} \longrightarrow \mathsf{Set}_E$ has a limit iff the composite functor $\pi^s_e X$ has a limit for all $e \in E$.*

2. *The functors π^s_e, $e \in E$, preserve colimits and a functor $X : \mathcal{I} \longrightarrow \mathsf{Set}_E$ has a colimit iff the composite functor $\pi^s_e X$ has a colimit for all $e \in E$.*

Proof. By Lemma B.2, π^s_e preserves limits and colimits. The other properties are easy to verify. □

Let $X, Y \in \mathsf{Set}_E$. Then

$$P_E(X) \cong \coprod_{\mathsf{Set}}^{e \in E} X_e \tag{64}$$

and a map $f : X \longrightarrow Y$ can be written as

$$f = \coprod_{\mathsf{Set}}^{e \in E} f_e : \coprod_{\mathsf{Set}}^{e \in E} X_e \longrightarrow \coprod_{\mathsf{Set}}^{e \in E} Y_e \tag{65}$$

so that one has a natural isomorphism

$$\mathsf{Set}_E(X, Y) \cong \mathsf{Set}_E(\coprod_{\mathsf{Set}}^{e \in E} X_e, \coprod_{\mathsf{Set}}^{e \in E} Y_e) \cong \prod_{\mathsf{Set}}^{e \in E} \mathsf{Set}(X_e, Y_e) \tag{66}$$

Define

$$\begin{array}{rrcl} (.)^{(.)} : & \mathsf{Set}_E^{op} \times \mathsf{Set}_E & \longrightarrow & \mathsf{Set}_E \\ & (Y, Z) & \mapsto & Z^Y = \coprod_{\mathsf{Set}}^{e \in E} \mathsf{Set}(Y_e, Z_e) \end{array} \tag{67}$$

That is, Z^Y is the set over E whose fibre over $e \in E$ is $\mathsf{Set}(Y_e, Z_e)$.

Let $X, Y, Z \in \mathsf{Set}_E$. By Lemma B.3.1, $(X \times_{\mathsf{Set}_E} Y)_e = X_e \times_{\mathsf{Set}} Y_e$. Therefore

$$\begin{array}{rcll} \mathsf{Set}_E(X \times_{\mathsf{Set}_E} Y, Z) & \cong & \prod_{\mathsf{Set}}^{e \in E} \mathsf{Set}(X_e \times_{\mathsf{Set}} Y_e, Z_e) & \text{(by (66))} \\ & \cong & \prod_{\mathsf{Set}}^{e \in E} \mathsf{Set}(X_e, \mathsf{Set}(Y_e, Z_e)) & \text{(by (60))} \\ & \cong & \mathsf{Set}_E(X, Z^Y) & \text{(by (66) and (67))} \end{array}$$

It follows that Set_E is cartesian closed.

Remark B.4. *(Category of elements, [36, Definition 2.4.1.] and [27, (3.35)])*

1. *Let Set be the category of (small) sets and $T : \mathcal{I} \longrightarrow \mathsf{Set}$ be a functor. Recall that*

   (a) *The category $\int T$ of elements of T is the category whose objects are pairs (i, s) where $i \in \mathcal{I}$ and $s \in T(i)$ and morphisms from (i, s) to (j, t) are morphisms f from i to j satisfying $T(f)(s) = t$.*

   (b) *The functor T has a colimit iff the connected components of the category $\int T$ form an object $\pi_0(\int T) \in \mathsf{Set}$, i.e. a small set . In this case, the cone $T \overset{\lambda}{\Longrightarrow} \pi_0(\int T)$ whose i-component is the map*

$$\begin{array}{rrcl} \lambda_i : & T(i) & \longrightarrow & \pi_0(\int T) \\ & t & \mapsto & [(i, t)] \end{array}$$

   is a colimiting cone.

(c) *It follows from the previous point that if a cone* $T \overset{\lambda}{\Longrightarrow} S$ *from the functor* T *to* $S \in \mathsf{Set}$ *is such that:*

- $\forall s \in T(i), \forall t \in T(j), \lambda_i(s) = \lambda_j(t) \Leftrightarrow$ *the objects* (i,s) *and* (j,t) *of* $\int T$ *are in the same connected component.*
- $\bigcup_{i \in \mathcal{I}} \lambda_i(T(i)) = S$.

Then λ *is a colimiting cone.*

2. *Assume now that* $T : I \longrightarrow \mathsf{Set}_E$ *is a functor. Let* $p_i : T(i) \longrightarrow E$ *be the projection of the set* $T(i)$ *over* E *and* $P_E : \mathsf{Set}_E \longrightarrow \mathsf{Set}$ *the functor given by (61). Then by Lemma A.2 and Remark B.4.1.(b),* T *has a colimit iff the connected components of the category* $\int P_E T$ *form a (small) set. When this is the case, then the colimit of* T *is*

$$\begin{array}{rcl} \pi_0(\int P_E T) & \longrightarrow & E \\ [(i,t)] & \mapsto & p_i(t) \end{array}$$

Lemma B.5. *Let* $X : \mathcal{I} \longrightarrow \mathsf{Set}$ *be a functor and* $Y \in \mathsf{Set}$*. Assume that* $X \overset{\lambda}{\Longrightarrow} Y$ *is a colimiting cone and let* $Y' \subset Y$*. Then the functor* X *induces a functor* $X' : \mathcal{I} \longrightarrow \mathsf{Set}$ *given by* $X'(i) = \lambda_i^{-1}(Y')$*,* $i \in \mathcal{I}$*. Furthermore, the cone* $\lambda' : X' \Longrightarrow Y'$ *induced by* λ *is a colimiting cone.*

Proof. Clearly, the functor X induces a functor $X' : \mathcal{I} \longrightarrow \mathsf{Set}$ and the cone λ induces a cone $\lambda' : X' \Longrightarrow Y'$. Two objects (i_1, x_1) and (i_2, x_2) in the category $\int X'$ are in the same path-component iff they are in the same path-component of the category $\int X$. That is iff $\lambda'_{i_1}(x_1) = \lambda_{i_1}(x_1) = \lambda_{i_2}(x_2) = \lambda'_{i_2}(x_2)$. Furthermore, $\bigcup_{i \in \mathcal{I}} \lambda'_i(X'(i)) = Y'$. By Remark B.4.3, λ' is a colimiting cone. □

C Limits in the category of fibrewise spaces

Define $| \; | : \mathsf{Top} \longrightarrow \mathsf{Set}$ to be the underlying set functor. $|.|$ has a left adjoint which is the discrete functor

$$\mathsf{Disc} : \mathsf{Set} \longrightarrow \mathsf{Top}$$

and has a right adjoint which is the codiscrete functor

$$\mathsf{Codisc} : \mathsf{Set} \longrightarrow \mathsf{Top}$$

In particular, the underlying functor $|.|$ preserves limits and colimits.

For any functor $T : \mathcal{I} \longrightarrow \mathsf{Top}$, define the underlying set functor of T to be the composite functor

$$|T| : \mathcal{I} \overset{T}{\longrightarrow} \mathsf{Top} \overset{|.|}{\longrightarrow} \mathsf{Set}. \tag{68}$$

Lemma C.1. *Let* $\mathcal{I}$ *be a (not necessarily small) category and* $T : \mathcal{I} \longrightarrow \mathsf{Top}$ *a functor. Then:*

1. T *has a limit (resp. colimit) iff* $|T|$ *has a limit (resp. colimit).*
2. *Suppose that* $|T|$ *has a limit (resp. colimit). Then* $\lim T$ *(resp.* $\operatorname{colim} T$*) is the topological space whose underlying set is* $\lim |T|$ *(resp.* $\operatorname{colim} |T|$*) and whose topology is the initial (resp. final) topology defined by the limiting (resp. colimiting) cone components* $\lim |T| \longrightarrow |T(i)|$ *(resp.* $|T(i)| \longrightarrow \operatorname{colim} |T|$*).*

Proof. Clear. □

It follows from the above lemma that functor $|.| : \mathsf{Top} \longrightarrow \mathsf{Set}$ preserves and lifts limits and colimits. Set is bicomplete, then so is Top.

The slice category of Top over B is denoted by Top_B. An object of Top_B is called a fibrewise topological space over B. It consists of a topological space X together with a continuous map $p : X \longrightarrow B$ called projection. A fibrewise topological space $p : X \longrightarrow B$ is often identified with its domain X. Let

$$P_B : \mathsf{Top}_B \longrightarrow \mathsf{Top} \tag{69}$$

be the functor defined as in (59).

Proposition C.2.

1. *Top_B is bicomplete.*
2. *The functor P_B creates and preserves colimits.*

Proof. This follows from Lemma A.2 and Lemma A.3. □

If X is a fibrewise space over B, then $|X|$ is a set over $|B|$ so that one has an underlying "fibrewise" set functor

$$|.| : \mathsf{Top}_B \longrightarrow \mathsf{Set}_{|B|}.$$

The functor $|.|$ has a left adjoint which is the ordinary discrete functor

$$\mathrm{Disc} : \mathsf{Set}_{|B|} \longrightarrow \mathsf{Top}_B,$$

and a right adjoint which is the codiscrete functor

$$\mathrm{Codisc} : \mathsf{Set}_{|B|} \longrightarrow \mathsf{Top}_B.$$

It associates to a fibrewise set S over $|B|$ the topological space whose underlying set is S and whose topology is the initial topology defined by the projection $p : S \longrightarrow |B|$ of the fibrewise set S on $|B|$.

It follows that the underlying fibrewise set functor $|.|$ preserves both limits and colimits. We have a commutative diagram of colimit preserving functors

$$\begin{array}{ccc} \mathsf{Top}_B & \xrightarrow{|.|} & \mathsf{Set}_{|B|} \\ {\scriptstyle P_B}\downarrow & & \downarrow{\scriptstyle P_{|B|}} \\ \mathsf{Top} & \xrightarrow[|.|]{} & \mathsf{Set} \end{array} \tag{70}$$

where P_B and $P_{|B|}$ are the functors defined by (69) and (61) respectively. For any functor $T : \mathcal{I} \longrightarrow \mathsf{Top}_B$, define the underlying functor of T to be the composite functor

$$|T| : \mathcal{I} \xrightarrow{T} \mathsf{Top}_B \xrightarrow{|.|} \mathsf{Set}_{|B|} \tag{71}$$

Lemma C.3. *Let $\mathcal{I}$ be a (not necessarily small) category and $T : \mathcal{I} \longrightarrow \mathsf{Top}_B$ a functor.*

1. *If one of the functors T, $|T|$, $P_B T$, $|P_B T| = P_{|B|}|T|$ has a colimit, then so do the others.*
2. *Assume that T has a colimit, then $P_B(\mathsf{colim} T)$ is the topological space whose underlying set is $\mathsf{colim}\,|P_B T|$ and whose topology is the final topology induced by the components of the colimiting cone $|P_B T| \Rightarrow \mathsf{colim}\,|P_B T|$.*

Proof.

1. The functors in diagram (70) are left adjoints, they are therefore colimit preserving, we just need to prove that if $|P_B T|$ has a colimit, then so is T. Assume then that $|P_B T|$ has a colimit. By Lemma C.1, $P_B T$ has a colimit. P_B creates colimits, therefore T has a colimit as desired.
2. This follows immediately the first property, Lemma C.1 and the fact that P_B preserves colimits.

□

Lemma C.4. *Let $\mathcal{I}$ be a (not necessarily small) category and $T : \mathcal{I} \longrightarrow \mathsf{Top}_B$ a functor. Then:*

1. *T has a limit iff $|T|$ has a limit.*
2. *Assume that S is a set over $|B|$ and $S \overset{\lambda}{\Rightarrow} |T|$ is a limiting cone. Let L be the topological space whose underlying set is S and whose topology is the initial topology defined by the components of λ. Then L is a fibrewise space over B and the cone $L \Rightarrow T$ induced by λ is a limiting cone.*

Proof. These are consequences of Lemmas A.3 and C.1. □

Lemma C.5. *Let $X : \mathcal{I} \longrightarrow \mathsf{Top}$ be a functor and $Y \in \mathsf{Top}$. Assume that $X \overset{\lambda}{\Longrightarrow} Y$ is a colimiting cone and let $Y' \subset Y$. Let $X' : \mathcal{I} \longrightarrow \mathsf{Top}$ be the functor given by $X'(i) = \lambda_i^{-1}(Y')$, $i \in \mathcal{I}$. Then the cone $\lambda' : X' \Longrightarrow Y'$ induced by λ is a colimiting cone, provided that Y' is either open or closed in Y.*

Proof. By Lemma B.5, the cone $|\lambda'| : |X'| \Longrightarrow |Y'|$ is a colimiting cone in Set. Assume that Y' is closed, then $X'(i)$ is closed in $X(i)$, all $i \in \mathcal{I}$. Let $C \subset Y'$ such that $\lambda_i'^{-1}(C)$ is closed in $X'(i)$ for all $i \in I$. The subset $\lambda_i^{-1}(C) = \lambda_i'^{-1}(C)$ is closed in $X(i)$ for all $i \in I$. By Lemma C.1, C is closed Y and therefore C is closed Y'. It follows that Y' has the final topology defined by the components of the cone $|\lambda'|$. By Lemma C.1, λ' is a colimiting cone. A similar argument can be used in the case where Y' is open. □

Let $A \subset B$, $J_A^t : A \to B$ the inclusion map and

$$J_A^{t,*} : \mathsf{Top}_B \longrightarrow \mathsf{Top}_A \tag{72}$$

The functor given by pulling back along the inclusion map J_A^t.

Lemma C.6.

1. *The functor $J_A^{t,*}$ preserves limits.*
2. *The functor $J_A^{t,*}$ preserves colimits provided that A is either open or closed in B.*

Proof. The first point results from the fact that $J_A^{t,*}$ is a right adjoint. The second is a consequence of Lemma C.5 and Proposition C.2. □

Let $b \in B$. For $X \in \mathsf{Top}_B$, define the fibre of X over b to be the subspace $X_b = p^{-1}(b)$, where p is the projection of the fibrewise space X. One has a functor

$$\pi_b^t : \mathsf{Top}_B \longrightarrow \mathsf{Top} \tag{73}$$

defined as follows: For $X \in \mathsf{Top}_B$, $\pi_b^t(X) = X_b$ and for $f \in \mathsf{Top}_B(X, Y)$, $\pi_b^t(f)$ is the map $f_b : X_b \longrightarrow Y_b$ is the map induced by f.

Lemma C.7. *Let $X : \mathcal{I} \longrightarrow \mathsf{Top}_B$ be a functor.*

1. *The functors π_b^t, $b \in B$, preserve limits and the functor X has a limit iff the composite functor $\pi_b^t X$ has a limit for all $b \in B$.*
2. *Assume B is a T_1-space. Then the functors π_b^t, $b \in B$, preserve colimits and the functor X has a colimit iff the composite functor $\pi_b^t X$ has a colimit for all $b \in B$.*

Proof.

1. By Lemma C.6.1, the functors π_b^t preserve limits. The fact that $\pi_b^t X$ has a limit for all $b \in B$ implies that X has a limit is a consequence of Lemmas C.1.1, B.3.1 and C.4.1.
2. By Lemma C.6.2, the functors π_b^t preserve colimits. The fact that $\pi_b^t X$ has a colimit for all $b \in B$ implies that X has a colimit is a consequence of Lemmas C.1.1, B.3.2 and C.3.1.

□

References

[1] J. Adámek, H. Herrlich, G. Strecker, Abstract and Concrete Categories, Wiley 1990, reprinted as: Reprints in Theory and Applications of Categories, 17 (2006) 1-507.

[2] Adámek, J., Rosický, J.: Intersections of reflective subcategories. Proc. Amer. Math. Soc. 103(3), 710–712 (1988)

[3] Barr, M., Kennison, J.F., Raphael, R.: On the reflective and coreflective hulls. Géom. Différ. Catég. Vol.LVI(3), 163–208 (2015)

[4] Booth, P.I., Brown, R.: Spaces of partial maps, fibred mapping spaces and the compact-open topology. Gen. Top. Appl. 8, 181–195 (1978)

[5] Booth, P.I.: The exponential law of maps i. Proc. London Math. Soc. 20(3), 179–192 (1970)

[6] Booth, P.I.: The exponential law of maps ii. Math Z 121, 311–319 (1971)

[7] Borceaux, F.: Handbook of categorical algebra 2, categories and structures. In: Rota, G.C. (ed.) Encyclopedia Math. Appl, vol. 51. Cambridge University Press, Cambridge (1994)

[8] Bourbaki, N.: Topologie Générale. Hermann, Paris (1965)

[9] Casacuberta, C., Frei, A.: Localizations as idempotent approximations to com- pletions. J. Pure Appl. Algebra 142, 25–33 (1999)

[10] Crabb, M.C., James, I.M.: A Concise Course in Algebraic Topology. Springer, Oxford (2012)

[11] Day, B.J.: A reflection theorem for closed categories. J. Pure Appl. Algebra 2(1), 1–11 (1972)

[12] Day, B.J., Kelly, G.M.: On topological quotient maps preserved by pullbacks or products. Math. Proc. Cambridge Philos. Soc. 67(3), 553–558 (1970)

[13] Dubuc, E.J.: Kan extensions in enriched category theory. In: Zaimis, E. (ed.) Lecture Notes in Mathematics, vol. 145. Springer, Berlin–Heidelberg–New York (1970)

[14] Escardó, M.H., Heckmann, R.: Topologies on spaces of continuous functions. Topology Proc. 26(2), 545–564 (2001)

[15] Escardó, M.H., Lawson, J., A., S.: Comparing cartesian closed categories of (core) compactly generated spaces. Topology Appl. 143, 105–145 (2004)

[16] Gillman, L., Jerison, M.: Rings of Continuous Functions. Van Nostrand, Princeton (1960)

[17] Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)

[18] Herrlich, H.: Categorical topology. Gen. topol. appl. (1), 1–15 (1970)

[19] Herrlich, H., Strecker, G.: Categorical topology-its origins, as exemplified by the unfolding of the theory of topological reflections and coreflections before 1971. In: Aull, C.E., Lowen, R. (eds.) Handbook of the History of General Topology, vol. 1, pp. 255–341. Kluwer Acad. Publ., Dordrecht (1971)

[20] Herrlich, H., Strecker, G.: Coreflective subcategories in general topology. Fundam. Math. 73, 199–218 (1972)

[21] Herrlich, H., Strecker, G.: Coreflective subcategories. Trans. Amer. Math. Soc. 157, 205–226 (1971)

[22] Isbell, J.: Adequate subcategories. Illinois J. Math. 4, 541–552 (1960)

[23] Isbell, J.: General function spaces. products, continuous lattices. Math. Proc. Cambridge Philos. Soc. 100, 193–205 (1986)

[24] James, I.M.: Fibrewise compactly-generated spaces. Publ. Res. Inst. Math. Sci. 31, 45–61 (1995)

[25] James, I.M.: Fibrewise Topology. Cambridge University Press, Cambridge (1990)

[26] Joyal, A., Tierney, M.: Notes on simplicial homotopy theory. Preprint at https://mat.uab.cat/ kock/crm/hocat/advanced-course/Quadern47.pdf

[27] Kelly, G.M.: Basic concepts of enriched category theory. Reprints in Theory Appl. Categ. 10, 1–136 (2005)

[28] Kelly, G.M.: On the ordered set of reflective subcategories. Bull. Aust. Math. Soc. 36(1), 137–152 (1987)

[29] Kelly, G.M., Stephen, L.: On property-like structures. Theory Appl. Categ. 3(9), 213–250 (1997)

[30] Lang, S.: Algebra. In: Axler, S., Gehring, F.W., Ribet, K.A. (eds.) Grad. Texts in Math., vol. Vol 211. Springer, New York (2002)

[31] May, P.: Fibrewise Homotopy Theory. University of Chicago Press, Chicago (1999)

[32] Mac Lane, S.: Categories for the working mathematician, 2nd ed. In: Axler, S., Gehring, F.W., Ribet, K.A. (eds.) Grad. Texts in Math., vol. Vol 5. Springer, New York (1998)

[33] Munkres, J.: Topology, 2nd Ed. Pearson Education Limited, Pearson (2014)

[34] Niefield, S.B.: Cartesianness, topological spaces and affine schemes. J. Pure and Appl. Algebra 23, 147–168 (1982)

[35] Rezk, C.: Compactly generated spaces. Preprint at https://rezk.web.illinois.edu/cg-spaces-better.pdf

[36] Riehl, E.: Category Theory in Context. Courier Dover Publications, New York (2017)

[37] Strickland, N.P.: The category of CGWH spaces. Preprint at https://ncatlab.org/ nlab/files/StricklandCGHWSpaces.pdf (2009)

[38] Tholen, W.: Reflective subcategories. Topology Appl. 27, 201–212 (1987)