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The class of second fundamental forms arising from minimal immersions in a space form

La classe des secondes formes fondamentales résultant d'immersions minimales dans un espace forme

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ABSTRACT. The second fundamental form arising from an oriented minimal immersion of a closed surface in a space form satisfies several constraints. One of them is provided by the Gauss-Codazzi equation that can be rephrased as a semilinear problem on the surface. We discuss some results for these type of nonlinear problems and analyze the behaviors of the solutions when the hyperbolic norm of the second fundamental form is small.
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1 Introduction

In his famous memoirs that settled the foundation of the calculus of variations, Lagrange [23] also raised the question of finding a surface in $\mathbb{R}^3$ that minimizes area among all surfaces that assume a prescribed values on a given closed curve. For the restricted class of graph surfaces, he derived the governing nonlinear equation, but aside observing that the plane provides a trivial solution, he did not investigate further the problem. After realizing that Lagrange nonlinear equation is equivalent to the vanishing of the mean curvature of a surface, Meunier (1776) discovered two new minimal surfaces in the Euclid space: the catenoid and the helicoid.

For many years, those have been the only known complete minimal surfaces in the Euclidean space. A third family were discovered by Scherk [36], and in the middle of the XIXth century the Enneper-Weierstrass representation formula brought suddenly an infinite family of examples of minimal surfaces. Since then, there have been impressive progresses, and for a survey in Euclidean space we refer to [27, 29, 30].

Instead of considering the Euclidean space, there has been extensive studies on the theory of minimal surfaces in a three dimensional space form, namely in a connected Riemannian 3-manifold of constant sectional curvature σ, which after scaling of the metric can be chosen as $\sigma \in \{-1, 0, 1\}$. The simplest space forms are the simply connected complete Riemannian manifold $\mathbb{M}_\sigma$, and by the Killing–Hopf theorem those one are isometric to one of the three spaces: the Euclidean space $\mathbb{R}^3$, the unit "round" sphere $S^3 \subset \mathbb{R}^3$, or the hyperbolic space $\mathbb{H}^3$. By taking the quotient of those spaces by a subgroup $G \subset$ Isom $(\mathbb{M}_\sigma)$ of isometries satisfying:

$$G \text{ acts properly discontinuously on } \mathbb{M}_\sigma, \tag{1.1}$$

$$G \text{ acts freely on } \mathbb{M}_\sigma, \tag{1.2}$$

one obtains all complete space forms ([48]). Note that for subgroup of isometries the condition (1.1) is equivalent to request G is a discrete subgroup of Isom $(\mathbb{M}_\sigma)$ ([28, Prop. 1.5.8]). This leads to the following precise classification of complete space forms (see [48]):

- Complete “Euclidean space forms" of zero curvature. There are, up to conjugacy, only finitely many subgroups of Isom $(\mathbb{R}^3)$ that satisfy (1.1) and (1.2). More specifically, the family of co-compact groups satisfying (1.1), the so-called “crystallographic groups", is finite and only ten of them act freely on $\mathbb{R}^3$. This list is supplemented by eight non co-compact groups satisfying (1.1) and (1.2) and we refer to see [48] for more details.
- The complete “spherical space forms", namely spaces of sectional curvature 1, are given as S^3/G with G a finite group of isometries acting freely. Those include the “Lens space" that are obtained from the action of a finite cyclic group.
- The “complete hyperbolic spaces" of sectional curvature -1 are given by the spaces $\mathbb{H}^3/G$, where G is a subgroup of the group of orientation-preserving isometries of $\mathbb{H}^3$ satisfying (1.1) and (1.2) (the “Kleinian groups").

In $\mathbb{R}^3$ or $\mathbb{H}^3$, the existence of a closed minimal surface is prevented by the strong maximum principle and the possibility of foliating these spaces with planes (a trivial family of minimal surfaces). However, such an obstruction vanishes in other space forms. For instance, by the work of Lawson [24] it is known that, except the projective space, every closed surface can be minimally immersed into S^3, and the oriented one are in fact embedded. This turns out to be a general fact. Indeed the Almgren’s min-max theory that originates in [2] has successfully been applied by Pitts [32] to derive general existence results of critical points for the area functional, and leads in particular to the conclusion that *every closed* 3*-Riemannian manifold contains a smooth embedded closed minimal surface*.

In the light of this abundant class of minimal surfaces, a lot of emphasis has been put on better understanding some features arising from a minimal immersions: genus bound, stability, morse index of the minimal surface $\cdots$, and use them as a constraint to look for a possible classification. Here we follow the seminal paper by Uhlenbeck [45], who looked at the possible induced metric g and second fundamental form h that can arise from an oriented minimal immersion of a closed surface in a 3-Riemannian manifold of sectional curvature -1 (not necessarily complete). Such a pair (g, h) of symmetric $(0, 2)$-tensor must satisfy, aside the positive definiteness of g, several compatibility conditions:

(i) A first type of conditions show that this pair determines in each small tubular neighborhood the Riemannian metric of the ambient manifold. In fact the metric is given as the solution to a second order ODE with initial conditions (g, h).

(ii) Furthermore, by looking at the complex structure on the surface that is associated to the induced metric g, a second condition is provided by the fact that the second fundamental form must be the real part of a holomorphic quadratic differential, a strong restriction on the possible set of $(0, 2)$-tensors that can arise as second fundamental form of a minimal immersion.

(iii) Finally, a third condition on (g, h) arises from the Gauss equation that reflects the interplay along the surface between its Gauss curvature and the sectional curvatures of the ambient space.

Conversely, given a pair (g,h) of $(0,2)$-tensors on the orientable surface Σ with g positive definite, consider the complex structure associated to the conformal class defined by g. Then if h is the real part of a holomorphic quadratic differential required in (ii), and the pair (g,h) satisfies the Gauss condition (iii), then the solution to the ODE in (i) defines a metric of constant sectional curvature in $\Sigma \times (-\varepsilon, \varepsilon)$ which turns out to be a space form (possibly incomplete) in which $\Sigma \times \{0\}$ is minimally embedded.

Therefore, by analogy with the Kazdan-Warner problem raised for the class of curvatures functions [20, 21], we are led to the following question: *By fixing a complex structure on Σ, which holomorphic quadratic differentials are associated to a symmetric $(0,2)$-tensor that arise as the second fundamental form of a minimal immersion of Σ in a space form ?*

Henceforth, we discuss this question on a closed orientable surface Σ that is endowed with a fixed complex structure (a Riemann surface). Denoting by $Q(\Sigma)$ the set of holomorphic quadratic differential on Σ, we are interested in the subset $Q_m(\Sigma,\sigma)$ consisting of those elements that one can get from a minimal immersion in a 3-manifold of sectional curvature σ. From above discussion the elements $h \in Q_m(\Sigma,\sigma)$ are precisely the quadratic differentials for which the following problem admits a solution:

Prescribed Second Fundamental Form Problem (for minimal surfaces):
Given a closed Riemann surface Σ and $h \in Q(\Sigma)$. Find a metric g on Σ such that

(i) The conformal class $[g]$ is compatible with the complex structure;

(ii) g solves the Gauss equation

$$K_g = \sigma - \frac{\|\mathrm{Re}\, h\|_g^2}{2},$$

where $\|h\|_g$ stands for the $(0,2)$-tensor norm with respect to the metric g.

When $\sigma = -1$ this has been studied by Uhlenbeck in [45]. Here we will discuss the general case of a 3-Riemannian manifold of constant sectional curvature, and will highlight some main differences with respect to the hyperbolic case.

When the surface is a sphere or torus, their space of holomorphic quadratic differentials are respectively $\{0\}$ and a space of dimension one. This property combined with Gauss-Bonnet theorem allow to describe precisely the set $Q_m(\Sigma,\sigma)$ in these two cases. For a surface of genus two or higher, the question becomes more challenging. The above Gauss equation can be rewritten as a nonlinear PDE, whose analysis is complicated by the presence of zeros for h. Since the uniqueness and existence of solutions is not yet fully understood, it is crucial to obtain a priori bounds, and understand in which manner the metrics that are solutions to the “Prescribed Second Fundamental Form Problem" can degenerate.

Denoting by $d\mu_g$ the area element with respect to a metric g, the possible degeneration of a sequence of metric will naturally lead to investigate the “bending energy" density $\|\mathrm{Re}\, h\|_g^2 d\mu_g$, one of the variant of the Willmore density. Since the metrics under consideration live in the same conformal class $[e^{2u}g_0]$ with g_0 the unique hyperbolic metric compatible with the prescribed complex structure, the area element and bending energy density can respectively be written in terms of g_0:

$$d\mu_g := e^{2u} d\mu_{g_0}, \qquad \|\mathrm{Re}\, h\|_{g_0}^2 e^{-2u} d\mu_{g_0}.$$

Thus our study will be reduced to an analytical problem involving only the function u.

In the case of minimal surfaces in three dimensional flat spaces, we will show the following:

Theorem 1.1. *(The case $\sigma = 0$) Let Σ be a closed Riemann surface of negative Euler characteristic, $h \in Q_m(\Sigma, 0)$ and ${}_n : \Sigma \to M_n$ a sequence of oriented minimal immersions such that*

(i) *M_n has zero sectional curvature,*

(ii) *the induced metric g_n belongs to the same conformal class $g_n = e^{2u_n} g_0$ where g_0 is hyperbolic,*

(iii) *the second fundamental form ${}_n$ is given by the real part of h.*

If the sequence of induced metric g_n degenerate in the sense that $\min\limits_{\Sigma} e^{2u_n} \to 0$, then there is a finite set $\mathcal{B} := \{p_1, \cdots, p_n\}$ of points in Σ, and non-negative integers $(m_p)_{p\in\mathcal{B}}$ such that the following weak convergence of measures holds:

$$\|\mathrm{Re}\, h\|^2_{g_n} d\mu_{g_n} \rightharpoonup 8\pi \sum_{p\in\mathcal{B}} (1 + m_p)\delta_p, \qquad \sum_{p\in\mathcal{B}} (1 + m(p)) = \textit{ genus}(\Sigma) - 1,$$

and furthermore, $Area_{g_n}(K) \to \infty$ for each relatively compact open set $K \subset\subset \Sigma \setminus \mathcal{B}$.

In fact a more general compactness result could be stated in terms of a sequence h_n. Noting first that a flat metric multiplied by a positive scalar provides again a flat metric, this invariance induces a family of equivalent minimal immersions. After normalizing $\int_\Sigma \|h_n\|^2_{g_n} d\mu_{g_n} = 1$, one selects a solution within this class (see Section 5). However, in that generality the set of zeros of h_n will change, and some more works is needed that will be discussed in a future work. The proof of above theorem will be analytic, and based on a blowup analysis of the solutions of the PDE that transcribes the Gauss equation.

In a hyperbolic space form the situation is quite different. For instance, the Gauss equation combined with the Gauss-Bonnet Theorem show that the area of a minimal surface $|\Sigma|_g$ with respect to the induced metric is bounded from above by $-2\pi\chi(\Sigma)$. Furthermore, Theorem 1.1 is also in contrast with the result obtained in [17], where we have been led to a more complicated alternative, which in geometrical terms reads as follows:

Theorem 1.2. *(The case $\sigma = -1$) Let $h \in Q_m(\Sigma, -1)$, and ${}_n : \Sigma \to M_n$ be a sequence of minimal immersions such that*

(i) *M_n has sectional curvature $\sigma = -1$,*

(ii) *the induced metric g_n belongs to the same conformal class $g_n = e^{2u_n} g_0$ where g_0 is hyperbolic,*

(iii) *the second fundamental form of ${}_n$ is given by the real part of Re $(t_n h)$ for some $t_n \leq 1$.*

Under the assumption $\min\limits_{\Sigma} e^{2u_n} \to 0$, then $t_n \to 0$ and there are three possibilities:

(a) *If $\|\mathrm{Re}\,(t_n h)\|^2_{g_n} d\mu_{g_n}$ is bounded in L^∞, we have*

$$t_n \to 0, \quad \textit{Area}_{g_n}(\Sigma) \to 0 \quad \int\limits_{\Sigma} \|\mathrm{Re}\,(t_n h)\|^2_{g_n} d\mu_{g_n} \to 8\pi\big(\textit{genus}(\Sigma) - 1\big), \tag{1.3}$$

while $-2[u_n - \bar{u}_n]$ converges smoothly to a solution of

$$-\Delta_{g_0} w = 8\pi n \left(\frac{|h|^2_{g_0} e^w}{\int_\Sigma |h|^2_{g_0} e^w d\mu_{g_0}} - \frac{1}{8\pi n} \right), \quad n := \textit{ genus}(\Sigma) - 1; \tag{1.4}$$

Whereas, if $\|\mathrm{Re}\,(t_n h)\|^2_{g_n} d\mu_{g_n}$ *is unbounded in* L^∞*, then there is a non-empty finite blowup set* $\mathcal{B} := \{p_1, \cdots, p_n\}$ *of distinct points in* Σ*, and integers* $(m_p)_{p\in\mathcal{B}}$ *such that*

$$\|\mathrm{Re}\,(t_n h)\|^2_{g_n} d\mu_{g_n} \rightharpoonup 8\pi \sum_{p\in\mathcal{B}} (1+m_p)\delta_p,$$

weakly in the sense of measure, and the following alternative holds:

(b) *either* $Area_{g_n}(\Sigma) \to 0$*, and in such a case*

$$t_n \to 0, \qquad \sum_{p\in\mathcal{B}} (1+m(p)) = \textit{genus}(\Sigma) - 1, \tag{1.5}$$

$$u_n - \bar{u}_n \overset{W^{1,q}}{\rightharpoonup} -4\pi \sum_{p\in\mathcal{B}} (1+m_p) G(p,\cdot), \quad \forall q \in (1,2), \tag{1.6}$$

where $G(p,\cdot)$ *stands for the Green's function of average zero of* $-\Delta_{g_0}$.

(c) *or,* $Area_{g_n}(\Sigma) \geq C > 0$*, and in this case*

$$t_n \to 0, \qquad Area_{g_n}(\Sigma) \to 4\pi N > 0, \quad \sum_{p\in\mathcal{B}} (1+m(p)) + N = \textit{genus}(\Sigma) - 1, \tag{1.7}$$

whereas $u_n \rightharpoonup u$ *weakly in* $W^{1,q}$*, for* $q \in (1,2)$*, with* u *solution of*

$$-\Delta_{g_0} u + e^{2u} = 1 - 4\pi \sum_{p\in\mathcal{B}} (1+m_p)\delta_p. \tag{1.8}$$

In the above result, it is not clear if each of the three alternative can really occur. However, in a work in progress we have highlighted some classes of immersions in *complete* 3-hyperbolic manifolds for which only the alternative **(c)** can potentially occur.

The plan of the paper is as follows.

In Section 2, we collect some notations and recall some known but important results from Differential Geometry. Following [45], Section 3 set up the governing equations that a minimal immersion in a space form must satisfy. Section 4 discusses when a sphere or torus can be realized as minimal surface in a space form. The case of a higher genus surface in a flat or spherical space form is discussed in Section 5. The study of minimal immersions in a flat space is undertaken in Section 6, and the last Section 7 is dedicated to the hyperbolic case.

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2 Preliminaries and notations

Given a Riemannian 3-manifold $(M, \langle\cdot,\cdot\rangle)$ with Levi-Civita connection ∇, our sign convention for the Riemann curvature tensor evaluated on vector fields $X, Y, Z \in \Gamma(TM)$ will be

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,,$$

and the sectional curvature of two linearly independent vector fields (X, Y)

$$\sigma(X,Y) = \frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

The sectional curvature allows to recover the tensor R, and if M has constant sectional curvature σ the following formula holds

$$R(X,Y)Z = \sigma\big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\big). \tag{2.1}$$

Given a differentiable surface Σ, an immersion $: \Sigma \to M$ induces a Riemannian metric on Σ by considering the pull back metric still denoted as $\langle\cdot,\cdot\rangle$. Let $\pi : N\Sigma \to \Sigma$ be the normal bundle of the immersion , whose fiber at each $p \in \Sigma$ is given by the orthogonal complement of $T_p\Sigma$ in $T_{(p)}M$. At each $(p_0, 0) \in N\Sigma$, the normal exponential map

$$(p, n) \mapsto \exp_{(p)}(n(p))$$

defines a local diffeomorphism from an open neighborhood of $(p_0, 0) \in \Sigma \times \mathbb{R}$ to a neighborhood of (p_0) in M. This induces a system of local charts at each point of (p_0) in M, that will be called "Fermi coordinates". In those charts the metric takes the form

$$G_{ij}(x,t)dx^i \otimes dx^j + dt \otimes dt, \tag{2.2}$$

and the Christoffel symbols can be expressed as follows

$$\Gamma^i_{33} = 0, \quad \Gamma^3_{ij} = -\frac{1}{2}\partial_3 G_{ij}, \quad \Gamma^i_{j3} = \frac{1}{2}G^{i\alpha}\partial_3 G_{\alpha j}. \tag{2.3}$$

In particular, in these coordinates we have $\nabla_{\partial_3}\partial_3 = 0$.

Given now an oriented immersion $f : \Sigma \to M$ in 3-manifold M, the second fundamental form associated to a choice of global unit vector field ν on Σ is defined as

$$h(X,Y) = \langle \nabla_X Y, \nu\rangle, \qquad X, Y \in \Gamma(T\Sigma)$$

which gives a symmetric $(0,2)$-tensor on Σ, namely a section of $T^*\Sigma \otimes T^*\Sigma$. For a non-orientable surface Σ, the discussion can be performed on the orientable 2-sheeted cover of Σ. The trace of h with respect to the metric is called the mean curvature, and the immersion is "minimal" if it has mean curvature zero at all points. Note that in Fermi coordinates the second fundamental form of the immersion is given by

$$h_{ij} = \langle \nabla_{\partial_i}\partial_j, \partial_3\rangle = \Gamma^3_{ij} = -\frac{1}{2}\partial_3 G_{ij}, \qquad i, j \in \{1, 2\}. \tag{2.4}$$

Let us also recall a major result that goes back to Gauss which concerns the existence of "isothermal coordinates" on a differentiable surface (see for instance [10]). It allows to select from the differentiable structure a system of local charts in which the components of the metric are given as $g_{ij} = e^{2\varphi}\delta_{ij}$. Furthermore, in these coordinates the transition map are conformal, and for oriented surfaces those are orientation preserving conformal maps (namely holomorphic) which provide a complex structure on Σ. Conversely, given a complex structure on an oriented surface, by "gluing" with a partition of the unity the family of pull back metrics f^*g_{euc} with f a local complex chart, one produces a metric, and therefore a conformal class. Therefore, on an oriented surface there is a one-to-one correspondence between complex structures (Riemann's moduli space) and conformal structures.

Another essential tool that we will need is given by the possibility of finding a canonical metric through the “uniformization theorem".

Uniformization of metric for compact Riemannian surfaces: *Let (Σ, g) be a closed, oriented Riemannian surface. Then, for some $u \in C^\infty(\Sigma)$ the conformal metric $e^{2u}g$ has a constant Gauss curvature.*

When the Euler characteristic $\chi(\Sigma) \leq 0$, the case that will mostly be relevant for us, Berger ([6]) found a variational approach that justifies the “uniformization of the metrics". His method can be summarized as follows: Finding a metric $e^{2u}g$ of constant curvature σ_0 is equivalent to solving the following PDE

$$\frac{-\Delta_g u + K_g}{e^{2u}} = \sigma_0 \tag{2.5}$$

where K_g stands for the Gauss curvature of g, and $\sigma_0 = 0$ if $\chi(\Sigma) = 0$ whereas $\sigma_0 = -1$ if $\chi(\Sigma) < 0$.

In the case, $\chi(\Sigma) = 0$ (surface is topologically a torus), we also have $\int_\Sigma K_g = 0$ (by Gauss-Bonnet), and the above PDE reduces to the Poisson problem $-\Delta_g w = -K_g$, which admits a unique solution of average zero. In the case $\chi(\Sigma) < 0$ and $\sigma_0 = -1$, (2.5) is the Euler-Lagrange equation of

$$J(u) = \frac{1}{2} \int\limits_\Sigma \{|\nabla u|^2 + e^{2u}\} + \int\limits_\Sigma K_g u, \quad u \in H^1(\Sigma),$$

which is smooth and weekly lower semi-continuous by the work of Trudinger ([44]) who derived a fundamental inequality, whose optimal form has been given later by Moser in [31] (see also [14]). Furthermore, one can verify that this functional is coercive, strictly convex and consequently admits a minimizer which is also the unique critical point. Furthermore, similar arguments show that given two conformal metrics of constant Gauss curvature, then,

(a) if $\chi(\Sigma) = 0$, the metrics are equal up to a positive scalar multiple;

(b) if $\chi(\Sigma) < 0$, the metrics are equal.

On the sphere the result holds but is more subtle (see [31]).

3 Governing equations of a minimal immersion

Given an oriented immersion $: \Sigma \to M$, the curvatures equations (2.1) provide several compatibility conditions that we will express using Fermi charts (x, x_3) with $x = (x_1, x_2)$ local coordinates on the surfaceΣ. Writing (2.1) using the coordinate basis vector fields $(\partial_1, \partial_2, \partial_3)$ give, as in [45], three types of constraints on the components of the metric G_{ij}:

(A) $\big\langle R(\partial_i, \partial_3)\partial_j, \partial_3\big\rangle = -\sigma G_{ij}, \;\; i, j \in \{1, 2\},$

(B) $\big\langle R(\partial_1, \partial_2)\partial_i, \partial_3\big\rangle = 0, \;\; i \in \{1, 2\},$

(C) $\big\langle R(\partial_1, \partial_2)\partial_2, \partial_1\big\rangle = \sigma\big(G_{11}G_{22} - G_{12}^2\big),$

and due to the symmetries of the $(0, 4)$-Riemann tensor $\mathrm{Rm}(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle$, the remaining possible relations only lead to trivial identities. The following crucial observation is a consequence of the second Bianchi's identity:

Lemma 3.1. *Assume $G_{ij}dx^i \otimes dx^j + dt \otimes dt$ is a metric on $\Sigma \times (-t_0, t_0)$ fulfilling property* **(A)***, and satisfying* **(B)**–**(C)** *only at $t = 0$ (i.e. on Σ). Then,* **(B)** *and* **(C)** *are satisfied on $\Sigma \times (-t_0, t_0)$.*

Now let us write more explicitly the condition (A), and (B)–(C) restricted on Σ.

(A) Due to the symmetry of the Riemannian tensor, we have

$$(A) \iff \big\langle R(\partial_i, \partial_3)\partial_3, \partial_j \big\rangle = \sigma G_{ij} \overset{(2.3)}{\iff} -\langle \nabla_{\partial_3}\nabla_{\partial_i}\partial_3, \partial_3\rangle = \sigma G_{ij}.$$

which is equivalent to

$$-\big(\partial_3 \Gamma^\alpha_{i3} + \Gamma^\beta_{i3}\Gamma^\alpha_{3\beta}\big)G_{\alpha j} = \sigma G_{ij} \tag{3.1}$$

The expression in the left hand-side is given by

$$\begin{aligned}
\partial_3 \Gamma^\alpha_{i3} + \Gamma^\beta_{i3}\Gamma^\alpha_{3\beta} &= \frac{1}{2}\partial_3[G^{\alpha\beta}\partial_3 G_{\beta i}]G_{\alpha j} + \frac{1}{4}[\partial_3 G_{\beta i}]G^{\alpha\beta}[\partial_3 G_{\alpha j}] \\
&= \frac{1}{2}\partial_3\big(G^{\alpha\beta}[\partial_3 G_{\beta i}]G_{\alpha j}\big) - \frac{1}{2}[\partial_3 G_{\beta i}]G^{\alpha\beta}[\partial_3 G_{\alpha j}] + \frac{1}{4}[\partial_3 G_{\alpha i}]G^{\alpha\beta}[\partial_3 G_{\beta j}] \\
&= \frac{1}{2}\partial_{33} G_{ij} - \frac{1}{4}[\partial_3 G_{\beta i}]G^{\alpha\beta}[\partial_3 G_{\alpha j}]
\end{aligned}$$

Therefore, with respect to the variable $t := x_3$, we see that the matrix function $G(x,t) := (G_{ij}(x,t)) \in M_2(\mathbb{R})$ satisfies the second order ODE

$$-\frac{1}{2}\partial_{tt}G + \frac{1}{4}[\partial_t G]G^{-1}[\partial_t G] = \sigma G. \tag{3.2}$$

Furthermore, the solutions of this ODE are required to be positive definite, and fulfill two initial conditions that result from the fact that $G(x,0)$ (see (2.2)) coincides with the induced metric $g(x)$ of the immersion and from(2.4):

$$G(x,0) = g(x), \quad \partial_3 G(x,0) = -2h(x,0). \tag{3.3}$$

As a matter of fact, likewise the case $\sigma = -1$ considered in [45], the solution to the Cauchy problem defined by (3.2) and (3.3) can be written explicitly. Indeed, from any choice of $M_2(\mathbb{R})$-valued functions $A(x), B(x), C(x)$ with A, B invertible and $BC = CB$, local solutions to the ODE (3.2) are providing by:

$$G(x,t) = \begin{cases} A(x)\big(B(x)\cosh t + C(x)\sinh t\big)^2 & \text{if } \sigma = -1, \\ A(B + tC)^2 & \text{if } \sigma = 0, \\ A(x)\big(B(x)\cos t + C(x)\sin t\big)^2 & \text{if } \sigma = 1. \end{cases}$$

Hence, the unique (local) solution to (3.2) and (3.3) is explicitly given by:

$$G(x,t) = \begin{cases} g(x)\big([\cosh t]\mathrm{I}_2 - [\sinh t][g^{-1}h](x)\big)^2 & \text{if } \sigma = -1, \\ g(x)(\mathrm{I}_2 - t[g^{-1}h](x))^2 & \text{if } \sigma = 0, \\ g(x)\big([\cos t]\mathrm{I}_2 - [\sin t][g^{-1}h](x)\big)^2 & \text{if } \sigma = 1, \end{cases} \tag{3.4}$$

(note that the negative sign in the three above expression depends on the sign convention in the definition of second fundamental form). In particular, it follows from (3.4) that in a small neighborhood of each point of (Σ) the Riemannian structure is uniquely determined by the induced metric and second fundamental form of the immersion.

(B) The second type of equations are

$$\langle R(\partial_1,\partial_2)\partial_1,\partial_3\rangle = 0 \qquad \langle R(\partial_1,\partial_2)\partial_2,\partial_3\rangle = 0, \tag{3.5}$$

and by Lemma 3.1 we only need to see how they can be satisfied on Σ. An interesting relation for the coefficients h_{ij} of the second fundamental form will come out, if he Fermi coordinates is specialized by choosing an isothermal system of coordinates on Σ. Indeed the above equations (3.5) are equivalent to

$$\begin{cases} \partial_1\Gamma^3_{21} - \partial_2\Gamma^3_{11} + \Gamma^\beta_{21}\Gamma^3_{1\beta} - \Gamma^\beta_{11}\Gamma^3_{2\beta} = & 0 \\ \partial_1\Gamma^3_{22} - \partial_2\Gamma^3_{12} + \Gamma^\beta_{22}\Gamma^3_{1\beta} - \Gamma^\beta_{12}\Gamma^3_{2\beta} = & 0 \end{cases}$$

Using (2.4), $\Gamma^3_{i3} = 0$ and writing the remaining Christoffel symbols by using the property that the components of the induced metric are $e^{2\varphi}\delta_{ij}$, on the surface we get

$$\begin{cases} \partial_1 h_{21} - \partial_2 h_{11} + \partial_2\varphi(h_{11}+h_{22}) = & 0, \\ \partial_1 h_{22} - \partial_2 h_{12} - \partial_1\varphi(h_{11}+h_{22}) = & 0. \end{cases}$$

Since the surface is minimal, we also have $e^{2\varphi}(h_{11}+h_{22}) = 0$, which implies

$$\partial_1 h_{11} = -\partial_2 h_{12} \qquad \partial_2 h_{11} = \partial_1 h_{21}$$

which shows that the complex function $h_{11} - ih_{12}$ satisfies the Cauchy-Riemann equations. Hence, by introducing the complex differential form $dz := dx + idy$, and considering the complex valued tensor $(h_{11} - ih_{12})dz \otimes dz$ we deduce that

$$h_{ij}dx^i \otimes dx^j = \text{Real part of } (h_{11} - ih_{12})dz \otimes dz, \quad h_{11} - ih_{12} \text{ holomorphic.} \tag{3.6}$$

Therefore, by looking at the complex structure associated to the metric induced by a minimal immersion on the surface Σ, the second fundamental form must be the real part of what is called a “holomorphic quadratic differential". This is a strong constraint, and it is worth making some additional comments.

Given a Riemann surface Σ (a surface with a complex structure), denote by $T^{(1,0)}(\Sigma)$ and $T^{*(1,0)}(\Sigma)$ the holomorphic tangent bundle and holomorphic cotangent bundle. In local complex coordinates the elements of those two spaces are respectively spanned by $\frac{\partial}{\partial z} := \frac{1}{2}\big(\partial_x - i\partial_y\big)$ and $dz : dx + idy$. A “holomorphic quadratic differential" is a holomorphic section of the holomorphic bundle $T^{(1,0)*}(\Sigma) \otimes T^{(1,0)*}(\Sigma)$, which henceforth will be denoted by $Q(\Sigma)$. In a local complex coordinates, each $\alpha \in Q(\Sigma)$ can be written $\alpha(z) = f(z)dz \otimes dz$ where f is holomorphic. When Σ is compact, as a corollary of the Riemann-Roch theorem, the space $Q(\Sigma)$ is of finite dimension [18], and its complex dimension is given by

$$\dim Q(\Sigma) = \begin{cases} 0 & \text{if genus}(\Sigma) = 0, \\ 1 & \text{if genus}(\Sigma) = 1, \\ 3\big(\text{genus}(\Sigma) - 1\big) & \text{if genus}(\Sigma) \geq 2. \end{cases} \tag{3.7}$$

Furthermore, for genus ≥ 2, each $\alpha \in Q(\Sigma)$ must have a zero. At each point p where $\alpha(p) = 0$, in a suitable local system of coordinates we have

$$\alpha(z) = (z - z_p)^{\gamma_p} dz \otimes dz, \tag{3.8}$$

for some $\gamma_p \in \mathbb{N}$, and setting $\mathcal{Z}(\alpha) := \{p \in \Sigma \,:\, \alpha(p) = 0\}$, the sum of the zeros is a topological invariant:

$$\sum_{p\in\mathcal{Z}(\alpha)} \gamma_p = 4\big(\mathrm{genus}(\Sigma) - 1\big). \tag{3.9}$$

A metric g on Σ compatible with the complex structure induces naturally an inner product on each fiber of the bundle $Q(\Sigma)$, and we will write $|\alpha(z)|_g$ the norm of a quadratic differential with respect to the metric g.

(C) In order for condition (C) to be satisfied on the surface Σ, the sectional curvatures σ of the space form and K_Σ of Σ (Gaussian curvature) together with the second fundamental form h must satisfy a compatibility condition given by the Gauss equation. For a minimal immersions this one reads

$$K_\Sigma = \sigma - \frac{\|h\|_g^2}{2}$$

where $\|\cdot\|_g$ stands for the norm induced by g in the space of $(0,2)$-tensors. Exploiting the part (B), we can write $h := Re(\alpha)$ which leads to the Gauss equation written in one or the other equivalent forms:

$$K_\Sigma = \sigma - \frac{\|Re(\alpha)\|_g^2}{2} \quad \text{or} \quad K_\Sigma = \sigma - \frac{|\alpha(x)|_g^2}{4}. \tag{3.10}$$

Integrating above equation and applying Gauss-Bonnet Theorem, we obtain the identity

$$2\pi\chi(S) = \sigma|S|_g - \int_S \frac{\|\alpha(x)\|_g^2}{4} d\mu_g \tag{3.11}$$

The identity (3.10) can be rephrased as a PDE. Indeed, setting

$$\sigma_0 := \begin{cases} 1 & \text{if } \chi(\Sigma) = 2 \\ 0 & \text{if } \chi(\Sigma) = 0 \\ -1 & \text{if } \chi(\Sigma) < 0 \end{cases} \tag{3.12}$$

from the uniformization Theorem, we can write the induced metric $g = e^{2u} g_0$ with g_0 a metric of constant Gauss curvature σ_0. Under this conformal change, the Gauss curvature of the metrics g, g_0 are related by the formula

$$K_g = \frac{-\Delta_{g_0} u + \sigma_0}{e^{2u}}.$$

Hence, the Gauss equation (3.10) can then be rephrased as the following PDE

$$\frac{-\Delta_{g_0} u + \sigma_0}{e^{2u}} = \sigma - \frac{1}{2}\frac{\|h\|_{g_0}^2}{e^{4u}}. \tag{3.13}$$

Conversely, given a differentiable surface Σ and two symmetric $(0,2)$-tensor $g, h \in \Gamma(T^*\Sigma \otimes T^*\Sigma)$ with g positive definite. Then, the solution $G(x,t)$ to (3.4), (3.3) in some maximal interval $t \in (-t_0, t_0)$ defines a Riemannian metric on $S \times (-t_0, t_0)$ of sectional curvature σ and the second fundamental form of $S \times \{0\}$ is h.

For instance, for a totally geodesic immersion, $h \equiv 0$, the metric (2.2) with expressions (3.4) turn into:

$$\textbf{(i)}\ [\cosh t]^2 g_{ij}(x)dx^i \otimes dx^j + dt \otimes dt \qquad \textbf{(ii)}\ g_{ij}(x)dx^i \otimes dx^j + dt \otimes dt$$

$$\textbf{(iii)}\ [\cos t]^2 g_{ij}(x)dx^i \otimes dx^j + dt \otimes dt,$$

which in case (i) and (iii) are warped products $\Sigma \times_{\cosh t} \mathbb{R}$, $\Sigma \times_{\cos t} \mathbb{R}$, and a direct product in case (ii). In all cases, one can check that as Σ is endowed with respectively a metric of curvature -1, 0, 1, then the respective metrics **(i)–(iii)** are of sectional curvature $-1, 0, 1$.

To summarize, once a complex structure has been assigned on the surface, the possibility of realizing Σ as a minimal surface in a space form with a prescribed second fundamental form is governed by the solvability of the Gauss equation. So the set $Q_m(\Sigma, \sigma)$, brought up in the introduction, that collects the possible tensors that can arise as second fundamental form of a minimal immersion is given by

$$Q_m(\Sigma, \sigma) := \{h \in Q(\Sigma) \ :\ (3.13) \text{ admits a solution}\}, \tag{3.14}$$

and our goal is ideally to characterize this set.

4 Minimal spheres and tori

In this section, we discuss the possibility of realizing a closed surface Σ of genus zero or one as a minimal surface in a 3-Riemannian manifold of constant sectional curvature.

4.1 The genus zero case

It is well known that the sphere S^2 has only one complex structure (*i.e.*, if X is a Riemann surface which is diffeomorphic to S^2, then X is biholomorphic to the Riemann sphere $\mathbb{C}P^1$), and that the only holomorphic quadratic differential is identically zero (See [18, Cor 5.4]). This was already known by Hopf [15] and it plays a fundamental role in his proof that any CMC immersed sphere in $\mathbb{R}^3$ is a round sphere. This property of holomorphic quadratic differential implies that the second fundamental form of any minimal immersion $S^2 \to M$ is given by $h \equiv 0$ (using (3.6)), namely the immersion is totally geodesic. Using this fact with the identity (3.11) we see that $\sigma = 1$ and $|S|_g = 4\pi$. In particular this gives: *a closed surface of genus zero can only be minimally immersed in a* 3*-Riemannian manifold of sectional curvature* 1*, and furthermore such an immersion must be totally geodesic*. Thus,

$$Q_m(S^2, 0) = \emptyset, \qquad Q_m(S^2, -1) = \emptyset, \qquad Q_m(S^2, 1) = \{0\}.$$

Example of such immersion is given by the $\Sigma = S^2 \times \{0\}$ in the warped product $S^2 \times_{\cos t} \mathbb{R}$. Another one, is the “equator" $\Sigma = \{x \in S^3 \subset \mathbb{R}^4 \ :\ x_4 = 0\}$ in the three dimensional round sphere S^3, and by a result of Almgren [3] this is in fact, up to an isometry of S^3, the only possible minimal immersion in S^3 of a closed surface of genus zero.

4.2 The genus one case

Consider now a torus $T^2 = \mathbb{R}^2/\Gamma$ with $\Gamma = \{m_1\vec{e_1} + m_2\vec{e_2} \ :\ m_1, m_2 \in \mathbb{Z}\}$, and assume the existence of a minimal immersion $\ : T^2 \to M$. Integrating the identity (3.11), we get $0 = \sigma|\Sigma|_g - \frac{1}{2}\int_\Sigma \|h\|_g^2 d\mu_g$

which may be possible only if:

(i) $\sigma = 0,\ h \equiv 0$ **(ii)** $or, \sigma = 1$ with $\displaystyle\int_\Sigma \|h\|_g^2 d\mu_g = 2|\Sigma|_g.$

Let us analyze these two cases separately.

Case (i): When $\sigma = 0$, the Gauss equation furthermore tells us that the Gauss curvature K_g of the induced metric g on T^2 is identically zero. Hence, *minimal immersions of T^2 in a flat space ($\sigma = 0$) are totally geodesic, and with respect to the induced metric (T^2, g) is a flat torus.* Such a minimal immersion can easily be constructed: Take any flat torus (T_2, g_0) and consider either $M := T^2 \times \mathbb{R}$ or $M := T^2 \times S^1$ with the product metric. So the set (3.14) is given by:

$$Q_m(T^2, \sigma = 0) = \{0\}.$$

Case (ii): We first emphasize that by fixing a complex structure on the torus, the only holomorphic quadratic differential on a torus are the one with constant coefficients $\alpha(z) = adz \otimes dz$ with $a \in \mathbb{C}$. To see this, one can either invoke the Riemann-Roch Theorem, or observe (like in [7]) that the torus admits a global complex coordinate and conclude with Liouville Theorem. Hence, the norm $\|h\|_{g_0}^2$ of the second fundamental with respect to the flat metric g_0 of any minimal immersion of a torus is a constant $\lambda^2 \neq 0$, and the Gauss equation (5.2) reads:

$$-\Delta u = e^{2u} - \lambda^2 e^{-2u}. \tag{4.1}$$

This PDE plays also a central role in the contiguous problem of constructing an immersed torus in $\mathbb{R}^3$ of constant mean curvature (see [46, 47]). Note that in the limiting case $\lambda = 0$, we obtain the equation studied by Liouville [26], who classified the set of solutions on simply connected domains. For $\lambda > 0$, the situation we are dealing with here, the PDE always admit the trivial constant solutions $u \equiv \frac{1}{4}\ln\lambda^2$, and therefore

$$Q_m(T^2, \sigma = 1) = Q(T^2) \setminus \{0\}.$$

I.e., on a torus with a prescribed complex structure, any non-zero holomorphic quadratic differential arises as the second fundamental form of a minimal immersion of T^2 in a 3-dimensional manifold of sectional curvature 1.

As a matter of fact, the PDE (4.1) also admits non-trivial solutions. Indeed, setting $v := u - \frac{1}{2}\ln\lambda$ the problem (4.1) is equivalent to the elliptic sinh-Gordon equation on the flat torus T^2:

$$-\Delta v = \lambda(e^{2v} - e^{-2v}), \tag{4.2}$$

for which $\{(\lambda, 0)\ :\ \lambda > 0\}$ is a trivial set of solutions. Existence of non-trivial solutions can be derived in several ways:

- Looking at one-dimensional solution, i.e. solutions satisfying $u(x + t\vec{e_1}) = u(x)$ for all $t \in \mathbb{R}$, or $u(x + t\vec{e_2}) = u(x)$ for all $t \in \mathbb{R}$ (see [1]).
- By applying bifurcation theory to (4.1). Indeed the linearization of this variational problem at each $(\lambda, 0)$ is given by the operator $-\Delta - 4\lambda$ which is non-invertible whenever $4\lambda = \Lambda_n$, where $\{\Lambda_n\}$ stands for the set of positive eigenvalues of the Laplacian operator $(-\Delta)$. By the result of [34] each $(\Lambda_n, 0)$ is a bifurcation point, and the description of the local branch can be made more precise if Λ_n is simple [11].

- Similarly to what happens with the sinh-Gordon problem subjected to zero boundary condition [39], one can also look for large solutions. A blowup analysis for the class of solutions to the PDE that do not change sign can rely on the results of [8], [9]. However, on a compact surface, the solutions to (4.2) must change sign since (since $\int_\Sigma e^u = \int_\Sigma e^{-u}$). The specific blowup analysis needed for the sinh-Gordon equation on a compact surface has been done by Jost and *al.* [19].

5 Minimal immersions of higher genus surfaces

For surfaces of genus ≥ 2, note that Gauss-Bonnet Theorem applied to the unique hyperbolic metric g_0 that is compatible with the prescribed complex structure on Σ shows that the hyperbolic area $|\Sigma|(=|\Sigma|_{g_0})$ is given by

$$|\Sigma| = 4\pi(\text{genus}(\Sigma) - 1). \tag{5.1}$$

Furthermore, the Gauss equation (3.10) reads:

$$-\Delta u = 1 + \sigma e^{2u} - \frac{|h|^2}{4} e^{-2u}, \tag{5.2}$$

where our convention is that unless written explicitly the Laplacian and norm are with respect to the hyperbolic metric g_0. Problem (5.2) has a variational structure, and is the Euler-Lagrange equation of the following functional:

$$J(u) = \frac{1}{2} \int_\Sigma \Big\{ |\nabla u|^2 - \sigma e^{2u} - 2u - \frac{|h|^2}{4} e^{-2u} \Big\} d\mu_{g_0}, \quad u \in H^1(\Sigma). \tag{5.3}$$

Furthermore, the critical points must satisfy the following natural constraint that arises by integrating the Gauss equation:

$$\int_\Sigma \Big\{ \frac{|h|^2}{4} e^{-2u} - \sigma e^{2u} \Big\} = |\Sigma|, \tag{5.4}$$

which defines a codimension 1 submanifold in $H^1(\Sigma)$.

Getting a priori estimates is an important aspect that is needed to understand the structure of the solutions to the Problem (5.2). The Stampacchia duality argument gives immediately $W^{1,q}$-estimates for each $1 \leq q < 2$:

(i) if $\sigma \leq 0$, consider $\mathcal{S} := \big\{ (h,u) \,:\, h \in Q(\Sigma),\, u \text{ solves } (5.2) \big\}$. Then, there exists a constant $C_q > 0$ such that

$$\|u - \bar{u}\|_{W^{1,q}} \leq C_q, \quad \forall (h,u) \in \mathcal{S}, \tag{5.5}$$

(ii) if $\sigma = 1$, for each $M > 0$ consider $\mathcal{S}_M := \big\{ (h,u) \,:\, h \in Q(\Sigma),\, u \text{ solves } (5.2), \int_\Sigma e^{2u} \leq M \big\}$, then $\|u - \bar{u}\|_{W^{1,q}} \leq C_q$ for all $(h,u) \in \mathcal{S}_M$.

The main difficulty consists in getting a priori bounds with respect to the H^1-norm $\|\nabla u\|_2 + \|u\|_2$. Note that for some (explicit) constant $C > 0$, there holds

$$\int_\Sigma |\nabla u|^2 + |\Sigma| \bar{u}^2 \;\leq\; \|u\|_{H^1}^2 \;\leq\; C\Big(\int_\Sigma |\nabla u|^2 + \bar{u}^2 \Big), \quad \forall u \in H^1(\Sigma). \tag{5.6}$$

Furthermore, given a set $E \subset \{(h,u) \in Q_m(\Sigma,\sigma) \times C^2(\Sigma) : u \text{ solves } (5.2)\}$, the two following are equivalent:

(i) E is bounded in $Q_m(\Sigma,\sigma) \times H^1(\Sigma)$,

(ii) the first projection $p_1(E)$ is bounded in $Q_m(\Sigma,\sigma)$ and $p_2(E)$ is bounded in $L^\infty(\Sigma)$.

Indeed, the fact that **(i)** implies **(ii)** follows by using u as a test function in the weak formulation of (6.13). Conversely if **(ii)** holds then the right-hand side of (6.13) is bounded in all L^p by the Moser-Trudinger inequality. Thus, by elliptic regularity we obtain $\{u - \bar{u} \ : \ u \in p_2(E)\}$ is bounded in $L^\infty(\Sigma)$ and thereupon the same holds for $p_2(E)$.

By considering the Green's function of (Σ, g_0) defined as

$$-\Delta G(x,\cdot) = \delta_x - \frac{1}{|\Sigma|}, \qquad \int_\Sigma G(x,y)dy = 0,$$

some immediate upper bounds can be obtained for the solutions to (5.2) in the case $\sigma \leq 0$. They will be immediate consequence of the following:

Lemma 5.1. *Let (Σ, g) be a Riemannian surface, and $v \in C^2(\Sigma)$ be a solution to*

$$-\Delta v = f - \bar{f}$$

where $f \geq 0$ is a given function, and $\bar{f} := \frac{1}{|\Sigma|}\int_\Sigma f(y)dy$. Then, $v - \bar{v} \geq \big(\min_{\Sigma\times\Sigma} G\big) \int_\Sigma f(y)dy$.

Proof: The Green's function G can be split as $G(x,y) := \frac{1}{2\pi}\ln\big(\frac{1}{d(x,y)}\big) + H(x,y)$ where d stands for the distance induced by the Riemannian metric g_0, and H the regular part that is a smooth function. Since G is uniformly bounded from below on the compact surface Σ and $f \geq 0$, we derive

$$v(x) - \bar{v} = \int_\Sigma G(x,y)f(y)dy \geq \big(\min_{\Sigma\times\Sigma} G\big)\|f\|_{L^1}.$$

□

The above result readily implies,

Lemma 5.2. *Assume $\sigma \leq 0$ and let $(h,u) \in Q_m(\Sigma,\sigma) \times C^2$ be such that u solves* (5.2)*. Then,*

$$u - \bar{u} \leq (-\min_{\Sigma\times\Sigma} G)|\Sigma|_{g_0}, \qquad e^{-2\bar{u}} \leq \frac{C_0}{\int_\Sigma |h(z)|^2}, \tag{5.7}$$

for some constant $C_0 := C_0(\Sigma, g_0)$

Proof: Lemma 5.1 applied with $f := \frac{|h(z)|^2}{4}e^{2u} - \sigma e^{-2u}$ $(\sigma \leq 0)$ together with (5.4) give the first upper bound in (5.7). Furthermore, from (5.4) we also have

$$|\Sigma| = \int_\Sigma \left\{\frac{|h|^2}{4}e^{-2u} - \sigma e^{2u}\right\} \geq e^{-2\bar{u}}\int_\Sigma \frac{|h|^2}{4}e^{-2[u-\bar{u}]} \geq C_0 e^{-2\bar{u}}\int_\Sigma |h(z)|^2, \tag{5.8}$$

where $C_0 := \frac{1}{4}e^{2\min_{\Sigma\times\Sigma} G}$. □

6 Immersion in a flat space ($\sigma = 0$)

For a surface of genus greater than two, minimally immersed in a flat space, the Gauss equation to be analyzed reads:

$$-\Delta u = 1 - \frac{|h|^2}{4} e^{-2u}. \tag{6.1}$$

Note that the PDE has no solution for $h \equiv 0$, namely using the notation (3.14), it holds $0 \notin Q_m(\Sigma, 0)$. Furthermore, if u solves the equation (6.1) with h, then replacing h by a non-zero multiple th, the new resulting equation (6.1) is solved by $u + \ln t^2$, *i.e.*

$$h \in Q_m(\Sigma, 0) \qquad \Longrightarrow \qquad th \in Q_m(\Sigma, 0) \quad \forall t \neq 0. \tag{6.2}$$

This invariance reflects the geometrical fact that if Σ is immersed minimally in some flat space form $(M, \langle \cdot, \cdot \rangle)$, then $(M, t^2 \langle \cdot, \cdot \rangle)$ is again a flat space form, in which Σ is again minimally immersed.

Each of the many geometrical examples of minimal immersion in flat space form provide existence of solutions to (6.1). For instance by looking at the data (g, h) arising from Schwarz P-minimal surface (genus 3) or Neovius surface (genus 9) in the three-dimensional flat torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, or from the examples given in [37]. Those are by far the only one, since given any lattice G in $\mathbb{R}^3$, there are infinitely many distinct minimal surfaces of genus three that embeds in T^3 ([33]). For related discussions see also [43].

From the analytic point of view, it is a challenging question to understand the space of holomorphic quadratic differential $h(z) dz \otimes dz$ for which solutions to (6.1) exist. From the variational point of view, the functional (5.3) has the small disadvantage that the norm of the Sobolev space does not appear in its expression. One way of overcoming this, like in [31], is to do the change of variable $w := -2(u - \bar{u})$, which solves the equivalent problem

$$-\Delta w = 2|\Sigma| \left(\frac{|h(z)|^2 e^w}{\int_\Sigma |h(z)|^2 e^w} - \frac{1}{|\Sigma|} \right), \quad w \in \mathring{H}^1(\Sigma), \tag{6.3}$$

where $\mathring{H}^1(\Sigma)$ stands for the H^1-function of average zero. Setting $n := \text{genus}(\Sigma) - 1$ and remembering (5.1), the semilinear problem (6.3) can be rewritten as

$$-\Delta w = 8\pi n \left(\frac{|h(z)|^2 e^w}{\int_\Sigma |h(z)|^2 e^w} - \frac{1}{|\Sigma|} \right), \quad w \in \mathring{H}^1(\Sigma), \tag{6.4}$$

whose solutions are in one-to-one correspondence with the critical points of the functional

$$J_\rho(w) = \int_\Sigma \frac{|\nabla w|^2}{2} - \rho \ln \Big(\int_\Sigma |h(z)|^2 e^w \Big), \qquad \rho = 8\pi n. \tag{6.5}$$

Interestingly, equation (6.3) is precisely one of the possible limiting equations on Σ that appears in Section 7. It is known [31] that the functional (6.5) is bounded from below if and only if $\rho \leq 8\pi$, coercive for $\rho < 8\pi$ and weakly lower semicontinuous. Therefore, by applying direct method of calculus we know that the functional J_ρ achieves its minimum whenever $\rho < 8\pi$. At the value $\rho = 8\pi$, which corresponds to having a genus two surface, the existence of a minimizer or critical point is more subtle. If we were dealing with a function $|h(z)| > 0$, then Ding Jost Li Wang [13] showed that

$$\Delta_{g_0} (\ln |h(p_0)|^2) > -4, \qquad \forall p_0 \in \Sigma, \tag{6.6}$$

provides a sufficient condition for the existence of a minimum for $J_{8\pi}$. However in our case, the function $|h(z)|^2$ is the hyperbolic norm of a holomorphic quadratic differential, and at each point p_0 such that $h(p_0) \neq 0$ a computation shows that

$$\Delta_{g_0}(\ln|h(p_0)|^2) = -4.$$

Hence, the functions $|h(z)|^2$ that are relevant for our geometrical problem are on the border line and fails to satisfy (6.6). The question of existence is still a work in progress.

Let us discuss some a priori bound that one can obtain for the set of solutions (6.1). Due to the invariance (6.2), let us consider:

$$(h,u) \in Q_m(\Sigma,0) \times C^2(\Sigma), \qquad \int_\Sigma |h|^2 = 1, \qquad u \text{ solves (6.1).} \tag{6.7}$$

First, note that multiplying (6.1) by e^{2u} and integrating we get

$$2\int_\Sigma |\nabla u|^2 e^{2u} = \int_\Sigma \{e^{2u} - |h|^2\}$$

and therefore $\int_\Sigma e^{2u} \geq \int_\Sigma |h|^2$. Namely, we obtain a lower bound on the area of the minimal surface with respect to the induced metric:

Given a complex structure on Σ (of genus ≥ 2) with associated hyperbolic metric g_0. Then, for any minimal immersion Σ in a flat space form

$$Area_g(\Sigma) \geq \int_\Sigma |h(x)|^2_{g_0} d\mu_{g_0}. \tag{6.8}$$

Under the normalization (6.7), the inequality (6.8) provides a uniform lower bound on the area. While such a statement is quite easy to derive when $\sigma = 0$, we will see that deriving a similar result in the case $\sigma = -1$ is not as straightforward and requires more tools.

Let us now do a blowup analysis and study how the set of solutions may fail to be bounded in L^∞. For this goal, as in [17] we must take into considerations the zeroes of the holomorphic quadratic differential $h(z)dz \otimes dz$, and introduce at each $p \in \Sigma$ the integer

$$m_p := \begin{cases} 0 & \text{if } h(p) \neq 0 \\ \gamma_p & \text{if } h(p) = 0 \text{ and (3.8) holds.} \end{cases} \tag{6.9}$$

We now focus on the proof of Theorem 1.1 which gives information on the possible degeneration of the metric induced by minimal immersions, in flat spaces, that are compatible with the fixed complex structure. This will be a consequence of the following analytic version of Theorem 1.1:

Proposition 6.1. *Let $h \in Q_m(\Sigma,0)$ and consider a sequence u_n of solutions to* (6.1) *such that* $\lim\limits_{n\to\infty} \|u_n\|_\infty = \infty$. *Then, there is a finite set $\mathcal{B} = \{p_1, \cdots p_N\}$ of points in Σ (the blow-up set) such that*

$$\frac{|h|^2}{4} e^{-2u_n} \rightharpoonup 4\pi \sum_{p\in\mathcal{B}} (1+m_p)\delta_p, \qquad \sum_{p\in\mathcal{B}} (1+m(p)) = \textit{genus}(\Sigma) - 1, \tag{6.10}$$

in the sense of the weak convergence of measures. Furthermore, for each $1 < q < 2$ we have

$$u_n - \bar{u}_n \overset{W^{1,q}}{\rightharpoonup} -\sum_{p\in\mathcal{B}} 4\pi(1+m_p)G(p,\cdot), \tag{6.11}$$

and on each relatively open set $K \subset\subset \Sigma \setminus \mathcal{B}$ we have:

$$\int_K e^{2u_n} \to \infty. \tag{6.12}$$

Proof: To fit the format of some blowup analysis that have been done in the literature, it is more convenient to rephrase the nonlinear PDE in terms of the function $v := -2u$

$$-\Delta v = 2\Big(\frac{|h(z)|^2}{4}e^v - 1\Big), \tag{6.13}$$

and in this variable the constraint (5.4) reads,

$$\int_\Sigma \frac{|h|^2}{4}e^v = 4\pi\big(\text{genus}(\Sigma) - 1\big). \tag{6.14}$$

From (5.7) we obtain $\bar{v}_n \leq C$. So, there are three possibilities:

(i) $\|v_n - \bar{v}_n\|_\infty \leq C$ and $|\bar{v}_n| \leq C$,

(ii) $\|v_n - \bar{v}_n\|_\infty \leq C$ and $\bar{v}_n \to -\infty$,

(iii) $\|v_n - \bar{v}_n\|_\infty \to \infty$.

The case **(i)** cannot holds, since it would imply $\|v_n\|_\infty \leq C$ in contradiction with the blowup assumption. The alternative **(ii)** cannot either hold, since it would imply $|h|^2 e^{v_n} = e^{\bar{v}_n}|h|^2 e^{v_n - \bar{v}_n} \to 0$, in contradiction with (6.14). Hence, only **(iii)** holds, which by Lemma 5.2 (after paying attention to the sign $v := -2u$) is in fact equivalent to $\max_\Sigma\{v_n - \bar{v}_n\} \to +\infty$.

As a consequence, $\max_\Sigma v \to \infty$ otherwise from the elliptic estimates applied to (6.13) we would get that v_n satisfies the alternative (ii) which cannot hold. Hence, we deduce that

$$\exists p \in \Sigma \text{ and } x_n \to p \text{ such that } v(x_n) \to +\infty. \tag{6.15}$$

If the blow up point p in (6.15) is such that $h(p) \neq 0$, then (6.14) gives a uniform bound $\int_{B_r(p)} e^{v_n} \leq C$ in a small ball $B_r(p)$. Therefore, after writing the PDE in local coordinates, applying the results of Brezis-Merle [8] and Li-Shafrir [25] we deduce that in a small ball $B_r(p)$ we have

$$2\frac{|h|^2}{4}e^{v_n} \rightharpoonup 8\pi N\delta_p.$$

Furthermore, it is well known that through a Pohozaev identity we must have $N = 1$.

If $h(p) = 0$, by the results of [5] we obtain $2\frac{|h|^2}{4}e^{v_n} \rightharpoonup 8\pi(1+m_p)\delta_p$, where m_p is the multiplicity of the zero of h. Hence, combining all those results we obtain:

$$\frac{|h|^2}{4}e^{v_n} \rightharpoonup 4\pi\sum_{p\in\mathcal{B}}(1+m_p)\delta_p, \tag{6.16}$$

and in particular, this weak converges gives

$$4\pi(\text{genus}(\Sigma)-1) \overset{(6.14)}{=} \int\limits_{\Sigma} \frac{|h|^2}{4} e^{v_n} \to \sum_{p\in\mathcal{B}} 4\pi(1+m_p).$$

Coming back to the original variable u, the conclusion (6.10) follows.

Since $w_n := v_n - \bar{v}_n$ $(= 2[u_n - \bar{u}_n])$ is bounded in each $W^{1,q}$ for $1 < q < 2$ (see (5.5)), this sequence converges weakly in all $W^{1,q}$ to some w. Using the equation (6.13), we have $-\Delta w_n = 2\big(\frac{|h(z)|^2}{4} e^{v} - 1\big)$ and as a result of the convergence (6.16) we obtain in the limit

$$-\Delta\Big(\frac{w}{2}\Big) = \sum_{p\in\mathcal{B}} 4\pi(1+m_p)\delta_p - 1, \qquad \int\limits_{\Sigma} w = 0.$$

Recalling (5.1) this equation can be written as

$$-\Delta\Big(\frac{w}{2}\Big) = \sum_{p\in\mathcal{B}} 4\pi(1+m_p)\Big\{\delta_p - \frac{1}{|\Sigma|}\Big\}, \qquad \int\limits_{\Sigma} w = 0,$$

and as a result we have $\displaystyle \frac{w}{2} = \sum_{p\in\mathcal{B}} 4\pi(1+m_p)G(p,\cdot)$. Thus, the conclusion (6.11) follows.

Finally (6.12) follows by noting that on each relatively compact set $K \subset\subset \Sigma \setminus \{p_1, \cdots, p_N\}$, on the one hand we have

$$\int\limits_K e^{-2u_n} = \int\limits_K e^{v_n} \to 0,$$

and on the other hand

$$0 < |K| = \int\limits_K 1 d\mu_{g_0} = \int\limits_K e^{u_n} e^{-u_n} \le \Bigg(\int\limits_K e^{2u_n}\Bigg)^{1/2} \Bigg(\int\limits_K e^{-2u_n}\Bigg)^{1/2} = \Bigg(\int\limits_K e^{2u_n}\Bigg)^{1/2} o(1).$$

Hence, $\int_K e^{2u_n} \to \infty$. $\square$

An interesting question is to understand if a blowup point can be or not a zero of the holomorphic quadratic differential. From the relation in (6.10) one obtain for instance the following

- **(i)** If Σ has genus two, then (6.10) can only be satisfied if the blowup set $\mathcal{B} = \{p\}$ with $h(p) \neq 0$.
- **(ii)** If Σ has genus three, the relation (6.10) can only be satisfied if: $\mathcal{B} = (p_1, p_2)$ with $h(p_i) \neq 0$, or $\mathcal{B} = \{q\}$ with $h(q) = 0$ of multiplicity two.
- **(iii)** If $h \in Q(\Sigma)$ has only one zero $\{q\}$, this one must have multiplicity $4(\text{genus}\,(\Sigma) - 1)$ (see (3.9)) and (6.10) can only be satisfied if $\mathcal{B} \cap \{q\} = \emptyset$.

7 Minimal immersion in hyperbolic spaces

Given a surface Σ of negative Euler characteristic, we look now at the problem of realizing Σ as a minimal surface in a hyperbolic space form by assigning the complex structure and second fundamental form. The set $Q_m(\Sigma,-1)$ of holomorphic quadratic differentials for which this is possible is given by the elements h for which the following Gauss equation admits a solution:

$$-\Delta u = 1 - e^{2u} - \frac{|h|^2}{4}e^{-2u}. \tag{7.1}$$

The solutions are subjected to the natural constraint (5.4) that reads

$$\int_\Sigma \frac{|h|^2}{4}e^{-2u} + \int_\Sigma e^{2u} = 4\pi(\text{genus } - 1), \tag{7.2}$$

which states that the"bending energy" added to the area measured with the induced metric is determined by the genus of the surface. Furthermore, we can exploit the fact that the above problem is the Euler Lagrange equation of the functional (5.3) which for convenience we write explicitly

$$J(h;u) = \frac{1}{2}\int_\Sigma \Big\{|\nabla u|^2 + e^{2u} - 2u - \frac{|h|^2}{4}e^{-2u}\Big\}d\mu_{g_0}, \quad u \in H^1(\Sigma). \tag{7.3}$$

The main difference with respect to the flat or spherical space forms considered before is the fact that $h \equiv 0 \in Q_m(\Sigma)$. Indeed, we have seen at the end of Section 3 that Σ (of negative Euler characteristic), endowed with the hyperbolic metric, can be realized as a totally geodesic surface in the space form $\Sigma \times_{\cosh t} \mathbb{R}$. From a PDE point of view, when $h \equiv 0$, Problem (7.1) admits $u \equiv 0$ as a solution, which is the unique one. This is in contrast with the case $\sigma \in \{0, 1\}$ for which no such totally geodesic immersion is possible when the surface Σ has negative Euler characteristic. Let us observe the following

(i) By applying the maximum principle we deduce that, if $h \in Q_m(\Sigma,-1) \setminus \{0\}$, then each solution u to (7.1) satisfies $u < 0$.

(ii) Using in (7.2) the inequality $2ab \leq a^2 + b^2$, which is strict unless $a = b$, we obtain

$$\int_\Sigma |h| < 4\pi(\text{genus } - 1). \tag{7.4}$$

Note that the equality is strict unless $e^{2u} = \frac{|h|}{2}$ a.e., which due to the presence of zeros for h (see (3.9)) would imply u is unbounded, which is not the case.

The fact that $Q_m(\Sigma,-1)$ is not reduced to $h \equiv 0$ is proved in [45], where by fixing $h \in Q(\Sigma)$ an application of the implicit function theorem shows that $th \in Q_m(\Sigma,-1)$ when t is small. More specifically, consider the smooth map

$$F : Q(\Sigma) \times W^{2,2}(\Sigma) \to L^2(\Omega), \qquad (h,u) \mapsto -\Delta u - 1 + e^{2u} + \frac{|h|^2}{4}e^{-2u}, \tag{7.5}$$

where $Q(\Sigma)$ is considered as a set of parameter, a finite dimensional space isomorphic to $\mathbb{R}^{2n}$ for some $n \geq 3$ (as mentioned in (3.7)). Then,

Proposition 7.1. *There exists an open set $U_1 \times U_2 \subset Q_m(\Sigma,-1) \times W^{2,2}(\Sigma)$ containing $(0,0)$, and a smooth map $f : U_1 \to U_2$ such that*

$$F^{-1}(0) \cap \big(U_1 \times U_2\big) = \big\{(h, f(h)) \,:\, h \in U_1\big\},$$

and in particular the set $Q_m(\Sigma,-1)$ contains an open neighborhood of $h\equiv 0$.

Furthermore, $Q_m(\Sigma,-1)$ is a bounded star-shaped set with respect to $h\equiv 0$.

Proof: At $(h,u)=(0,0)$, the partial derivative with respect to the variable u is given by the bounded invertible linear operator

$$D_2F_{(0,0)}: W^{2,2}(\Sigma)\to L^2(\Sigma), \qquad \xi\mapsto -\Delta\xi+2\xi,$$

and therefore the conclusion of the first part of the proposition follows by applying the implicit function.

Concerning the property to be star-shaped, choose $h\in Q_m(\Sigma,-1)$ and let u_h be a solution to the Gauss equation (7.1). Looking at th for some $t\in[0,1]$ we note that

$$F(th,u_h)=-\Delta u_h-1+e^{2u_h}+t^2\frac{|h|^2}{4}e^{-2u_h}\le F(u_h,h)=0$$

namely u_h is a subsolution for the Gauss equation (7.1) with the quadratic differential th. On the other hand, $F(th,0)=t^2\frac{|h|^2}{4}>0$, namely we have a supersolution for (7.1) with the quadratic th. Then, by considering the following closed and convex set of $H^1(\Sigma)$

$$M:=\Big\{\xi\in H^1(\Sigma)\,:\,\underline{u}:=u_h\le\xi\le\overline{u}:=0 \text{ a.e. }\Big\},$$

we can follow, up to some minor modifications, the Perron's method in its variational form given in Struwe [40]. His arguments show that the functional $J_{th}\big|_M$ restricted to M admits a minimizer, which is shown to be a critical point of J_{th}. This shows that if $h\in Q_m(\Sigma,-1)$ then $th\in Q_m(\Sigma,-1)$ for each $t\in[0,1]$. Hence $Q_m(\Sigma,-1)$ is star-shaped with respect to the origin. The boundedness of $Q_m(\Sigma,-1)$ is a consequence of (7.4). $\square$

Before saying more on the existence of solutions (7.1), we state the following general known facts:

Lemma 7.2. *Let (Σ,g) be a closed Riemannian surface, $f\in C^2(\Sigma\times\mathbb{R},\mathbb{R})$ and assume that the problem*

$$-\Delta u=f(\cdot,u), \qquad u\in H^1(\Sigma), \tag{7.6}$$

admits a solution u_0. Then the following hold:

(a) *If u_0 is a maximal solution[1], and $u_0<\bar{u}$ for some supersolution $\bar{u}\in H^1(\Sigma)$, then u_0 is stable in the sense that $\lambda_1(-\Delta_g-\partial_2 f(\cdot,u_0))\ge 0$ or equivalently,*

$$\int_\Sigma \big\{|\nabla^g\xi|_g^2-\partial_2 f(\cdot,u_0)\xi^2\big\}d\mu_g\ge 0, \quad \forall\xi\in H^1(\Sigma) \tag{7.7}$$

Furthermore, if $s\mapsto f(\cdot,s)$ is concave, then any stable solution u_0 of (7.6) satisfies

(b) *u_0 is a maximal solution, and in particular there is only one stable solution.*

(c) *For any subsolution $\underline{u}$ we have $\underline{u}\le u_0$.*

(d) *If $\lambda_1(-\Delta_g-\partial_2 f(\cdot,u_0))=0$, then u_0 is the unique solution to (7.6).*

1 u_0 is called a maximal solution if for any solution $\tilde{u}$ of (7.6) we have $\tilde{u}\le u_0$

Proof: **(a)** Set $F(s) := \int_0^s f(\tau)d\tau$ and $I(u) := \int_\Sigma \left\{\frac{|\nabla^g u|_g^2}{2} - F(\cdot, u)\right\} d\mu_g$ whose critical points are the solutions to (7.6). Then, working with the set $M := \left\{\xi \in H^1(\Sigma) \,:\, u_0 \le \xi \le \overline{u} \text{ a.e.}\right\}$, as in the proof of Prop. 7.1, the functional $I|_M$ achieves its minimum in M, which is also a critical point of I. By the maximality property of u_0, the minimum of $I|_M$ must coincide with u_0. Therefore, given $0 \le \xi \in C^1(\Sigma)$, for some $t_\xi > 0$ we have

$$u_0 + t\xi \in M, \quad I(u_0 + t\xi) - I(u_0) \ge 0, \quad \forall t \in [0, t_\xi].$$

Doing a second-order expansion at $t = 0$ we obtain:

$$0 \le I(u_0 + t\xi) - I(u_0) = \frac{t^2}{2} D^2 I_{(u_0)}(\xi, \xi) + o(t^2),$$

and therefore $D^2 I_{(u_0)}(\xi, \xi) \ge 0$. It follows that

$$\int\limits_\Sigma \left\{|\nabla^g \xi|_g^2 - \partial_2(\cdot, u_0)\xi^2\right\} \ge 0, \quad \forall \xi \in C^1(\Sigma),$$

and by density the previous inequality holds for all $\xi \in H^1(\Sigma)$. This concludes the proof of (7.7).

(b) Let $\tilde{u}$ be a solution to (7.6), and assume that the open set $\Omega_0 := \{x \in \Sigma : \tilde{u} - u_0 > 0\}$ is not empty. Then, the concavity assumption on f implies

$$-\Delta_g(\tilde{u} - u_0) = f(\cdot, \tilde{u}) - f(\cdot, u_0) < \partial_2 f(\cdot, u_0)(\tilde{u} - u_0), \quad \text{in } \Omega_0.$$

Hence, using $(\tilde{u} - u_0)^+$ as test function in (7.7) we obtain

$$\int\limits_\Sigma \left\{|\nabla^g(\tilde{u} - u_0)^+|_g^2 - \partial_2 f(\cdot, u_0)|(\tilde{u} - u_0)^+|^2\right\} < 0,$$

in contradiction with the stability assumption (7.7).

(c) and **(d)** The argument for (c) can be found in [12, Lemma 2.18] and are already mentioned in [22]. Applying (c) with a solution u_1 we obtain $u_1 \le u_0$ and since u_0 is maximal by (b) we deduce that $u_1 \equiv u_0$. $\square$

The above result is useful to clarify the stability of the solutions obtained so far in Prop. 7.1. Let $(h, u_h) \in Q_m(\Sigma, -1) \times C^2(\Sigma)$ with u_h solution of (7.1). Given $t \in (0, 1)$, consider the function:

$$U_{th} := \sup\left\{u \in H^1(\Sigma) : u \le 0,\ F(th, u) \le 0\right\} \tag{7.8}$$

where F is the map (7.5). Since $F(th, u_h) \le 0$ (as already used in the proof of 7.1), the above set is non-empty and so U_{th} is well defined. Now a classical argument used in the Perron method shows that U_{th} is a maximal solution for the (7.1), and by construction for $h \not\equiv 0$ the following strict monotonicity property holds: for each $0 < t_1 < t_2 < 1$ we have

$$U_{t_2 h} < U_{t_1 h} < 0. \tag{7.9}$$

Furthermore, Lemma 7.6 applied with the function

$$f(\cdot, u) := 1 - e^{2u} - \frac{|th|^2}{4} e^{-2u} \tag{7.10}$$

shows that U_{th} is stable, and it is the unique one since the nonlinearity f is concave in the variable u. In the same spirit as the arguments used by Crandall-Rabinowitz in [12], it turns out that U_{th} is a strict local minimum of the associated functional. More specifically,

Lemma 7.3. *Let $(h, u_h) \in Q_m(\Sigma) \times C^2(\Sigma)$ with u_h solution of* (7.1)*, and consider for each $t \in [0,1)$ the maximal solution U_{th} defined by* (7.8)*. Then, at the crititcal point U_{th} the second variation of $J := J(th;\cdot)$ defined in* (7.3) *is coercive, i.e. for some constant $C_t > 0$ it holds:*

$$D^2 J_{(U_{th})}(\xi,\xi) \geq C_t \|\xi\|^2_{H^1}, \quad \forall \xi \in H^1(\Sigma). \tag{7.11}$$

Proof: Choose $t_0 \in (t,1)$, and set $K := \frac{|h|^2}{4}$. Then, the stability property of $U_{t_0 h}$ combined with the monotonicity give:

$$\begin{aligned}
\int_\Sigma \left\{|\nabla\xi|^2 + 2e^{2U_{th}}\xi^2\right\} &\overset{(7.9)}{\geq} \int_\Sigma \left\{|\nabla\xi|^2 + 2e^{2U_{t_0 h}}\xi^2\right\} \\
&\geq \int_\Sigma t_0^2 K e^{-2U_{t_0 h}}\xi^2 \\
&\overset{(7.9)}{\geq} t_0^2 \int_\Sigma K e^{-2U_{th}}\xi^2,
\end{aligned}$$

namely,

$$-t_0^2 \int_\Sigma K e^{-2U_{th}}\xi^2 \geq -\int_\Sigma \left\{|\nabla\xi|^2 + 2e^{2U_{th}}\xi^2\right\}. \tag{7.12}$$

Therefore,

$$\begin{aligned}
D^2 J_{(U_{th})}(\xi,\xi) &= \int_\Sigma \left\{|\nabla\xi|^2 + 2e^{2U_{th}}\xi^2 - t^2 K e^{-2U_{th}}\xi^2\right\} \\
&\overset{(7.12)}{\geq} \int_\Sigma \left\{|\nabla\xi|^2 + 2e^{2U_{th}}\xi^2\right\} - \frac{t^2}{t_0^2}\int_\Sigma \left\{|\nabla\xi|^2 + 2e^{2U_{th}}\xi^2\right\} \\
&\geq (1 - \frac{t^2}{t_0^2}) \int_\Sigma \left\{|\nabla\xi|^2 + 2e^{2U_{th}}\xi^2\right\} \\
&\geq C\|\xi\|^2_{H^1}.
\end{aligned}$$

□

From the above discussion, we have some additional information on the set $Q_m(\Sigma,-1)$. Consider the Gauss equation (7.1) and denote by $f(\cdot,u)$ the nonlinearity given by (7.5).

Proposition 7.4. **(a)** *The interior of the set $Q_m(\Sigma,-1)$ consists of the element $h \in Q(\Sigma)$ for which the Gauss equation* (7.1) *admits a solution U_h with $\lambda_1\big(-\Delta - \partial_2 f(\cdot,U_h)\big) > 0$.*

(b) *The boundary of $Q_m(\Sigma,-1)$ consists of the elements $h \in Q(\Sigma)$ for which the Gauss equation* (7.1) *admits a solution U_h with $\lambda_1\big(-\Delta - \partial_2 f(\cdot,U_h)\big) = 0$.*

If $h \in \partial Q_m(\Sigma,-1)$, then Lemma 7.2 shows that the associated Gauss equation admits a unique solution. On the other hand, for each h picked up in the interior $Q_m(\Sigma,-1)$ the Gauss equation admits a solution which is a local minimizer, and in [16] we showed that in this case the functional J_h exhibits a mountain pass geometry. Furthermore, we also proved that the functional J_h satisfies the "Palais-Smale condition", a property that ensures the compactness of the flow-line defined by the negative gradient of the functional. This leads to the following result:

Theorem 7.5 (Unstable Solution, [16])**.** *Consider an element h in the interior of $Q_m(\Sigma,-1)$. Then, the Gauss equation* (7.1) *admits a unique stable solution U_h, and an unstable one u_h.*

Furthermore, as $\|h\| \to 0$ we have $U_h \to 0$, whereas e^{2u_h} converges to zero at some points on the surface Σ.

This multiplicity results shows that once the complex structure is fixed, there are at least two geometrically distinct minimal immersions of Σ (in some hyperbolic space) with same second fundamental form h. Their induced metrics behave very differently for small h: one converges to the hyperbolic metric g_0 associated with the complex structure, and the second one degenerates. Understanding the way how in this second case the induced metric can degenerate as $\|h\| \to 0$ is the purpose of Theorem 1.2. We study this by fixing $h \in Q(\Sigma)$ and studying as $t \to 0$ the behavior of (th, u_{th}) with u_{th} solution of the Gauss equation (7.1) satisfying $\|u_{th}\|_{L^\infty} \to \infty$.

Proof of Theorem 1.2: The result is in [17], but we have here restructured the proof to highlight the possible behaviors of $\mathrm{Area}_{g_n}(\Sigma)$. First, the induced metrics and second fundamental form $(e^{2u_n}g_0, t_n h)$ satisfy the Gauss equation (7.1) and the natural constraint (7.2). After setting $v := -2u_n \geq 0$ those relations read

$$-\Delta v_n = 2\Big(\frac{t_n^2|h|^2}{4}e^{v_n} + e^{-v_n} - 1\Big), \quad v_n \geq 0, \tag{7.13}$$

$$\int_\Sigma \frac{t_n^2|h|^2}{4}e^{v_n} + \int_\Sigma e^{-v_n} = 4\pi(\text{genus}\,(\Sigma) - 1). \tag{7.14}$$

By assumption $\max\limits_\Sigma v_n \to \infty$ and we note that the area of Σ with respect to the induced metric is given by $\mathrm{Area}_{g_n}(\Sigma) = \int_\Sigma e^{2u_n} d\mu_{g_0} = \int_\Sigma e^{-v_n} d\mu_{g_0}$. There are three possible behaviors for the sequence v_n:

(a) $t_n^2|h|^2 e^{v_n} \leq C$;

(b) $t_n^2|h|^2 e^{v_n}$ is unbounded in $L^\infty(\Sigma)$ and $\int_\Sigma e^{-v_n} \to 0$;

(c) $t_n^2|h|^2 e^{v_n}$ is unbounded in $L^\infty(\Sigma)$ and $\int_\Sigma e^{-v_n} \geq C > 0$.

As a preliminary remark, note that (5.7) (with $v := -2u$) and Jensen's inequality give

$$C_0 \int_\Sigma |t_n h(z)|^2 \overset{(5.7)}{\leq} e^{-\bar v_n} \leq \fint_\Sigma e^{-v_n} = e^{-\bar v_n} \fint_\Sigma e^{-[v_n - \bar v_n]} \overset{(5.7)}{\leq} e^{-\bar v_n} e^{-C}, \tag{7.15}$$

which shows $\int_\Sigma e^{-v_n} \to 0$ (namely $\mathrm{Area}_{g_n}(\Sigma) \to 0$) if and only if $\bar v_n \to +\infty$, and when this occurs we also have $t_n \to 0$.

Case (a): When $t_n^2|h|^2 e^{v_n} \leq C$, from the PDE (7.13) we deduce that $\|v_n - \bar v_n\|_\infty \leq C$. Since $\max_\Sigma v_n \to +\infty$ we must necessarily have $\bar v_n \to +\infty$. Furthermore, (7.15) give

$$t_n \to 0 \quad \text{and} \quad \int_\Sigma e^{-v_n} \to 0.$$

Therefore, by taking also into consideration (7.14), we deduce that (1.3) holds.

Now the sequence $w_n := v_n - \bar{v}_n$ satisfies

$$-\Delta w_n = 2\big(e^{\bar{v}_n}\frac{t_n^2|h|^2}{4}e^{w_n} + e^{-v_n} - 1\big)$$

and after integrating we obtain $\frac{e^{\bar{v}_n}}{4} := \frac{|\Sigma|+o(1)}{\int_\Sigma t_n^2|h|^2e^{w_n}}$. So, since the right-hand side of the above PDE is bounded in L^∞, by elliptic regularity we deduce that w_n converges in all $W^{2,p}$ to a function w that solves the PDE (1.4).

In the case where $t_n^2|h|^2e^{v_n}$ is unbounded in $L^\infty(\Sigma)$, like in the proof of Prop. 6.1 we can rely on the results of [8], [25], [5] and deduce that there is a finite set $\mathcal{B} \subset \Sigma$

$$2\frac{|t_n h|^2}{4}e^{v_n} \rightharpoonup \sum_{p\in\mathcal{B}} 8\pi(1+m_p)\delta_p, \tag{7.16}$$

where the integers m_p are given by (6.9). The further asymptotic properties of v_n depends now on whether the integral $\int_\Sigma e^{-v_n}$ does or does not tend to zero.

Case (b): $\int_\Sigma e^{-v_n} \to 0$ (equivalent to $\mathrm{Area}_{g_n}(\Sigma) \to 0$).
Then by (7.15), we have $\bar{v}_n \to +\infty$, $t_n \to 0$, and by using also (7.14) the conclusion (1.5) holds. Now since sequence $w_n := v_n - \bar{v}_n \rightharpoonup w$ weakly in $W^{1,q}$ (by (5.5)), from the PDE (7.13) and (7.16) we deduce that the limiting equation for w is

$$-\Delta w = \sum_{p\in\mathcal{B}} 8\pi(1+m_p)\delta_p - 1, \qquad \int\limits_\Sigma w = 0,$$

which can be rewritten (recalling (5.1)) as

$$-\Delta\big(\frac{w}{2}\big) = \sum_{p\in\mathcal{B}} 4\pi(1+m_p)\Big\{\delta_p - \frac{1}{|\Sigma|}\Big\}, \qquad \int\limits_\Sigma w = 0.$$

Hence, (1.6) follows.

Case (c) If $\int_\Sigma e^{-v_n} \geq C > 0$.

By (7.15) we have that $\bar{v}_n \leq C$. Thus from (5.5) the sequence is bounded in $W^{1,q}$ for each $q \in (1,2)$ and as a result $v_n \overset{W^{1,q}}{\rightharpoonup} v$. Furthermore, since $e^{-v_n} \leq 1$, by the Lebesgue dominated convergence we have $e^{-v_n} \to e^{-v}$ in all L^p space. Since on each compact set $K \subset \Sigma \setminus \mathcal{B}$ we have $t_n \int_K |h|^2 e^{v_n} \to 0$ we must have $t_n \to 0$. Furthermore, (7.14) and (7.16) give

$$4\pi\sum_{p\in\mathcal{B}}(1+m_p) + \int\limits_\Sigma e^{-v} = 4\pi\big(\mathrm{genus}(\Sigma) - 1\big).$$

Hence, (1.7) holds. The conclusion (1.8) follows from the convergences $v_n \overset{W^{1,q}}{\rightharpoonup} v$ and (7.16). $\square$

Interestingly, the limiting equation (1.4) is exactly the same as the Gauss equation governing minimal immersions in a flat space written in the equivalent form (6.3). Based on the known existence results of minimal immersions of surfaces of genus higher than three in a flat three dimensional torus, we know that for some h those two problems admit solution for each $n \geq 3$. It would be useful to understand those two equations from an analytical point of view, and clarify the class of h for which such existence result holds.

Concerning the limiting equation (1.8) arising as third possible alternative in Theorem 1.2, it does always have a solution. It is a consequence of the following more general result.

Proposition 7.6. *Let (Σ, g) be a closed Riemannian surface, $\mathcal{P} \subset \Sigma$ a finite set and $(\alpha_p)_{p\in\mathcal{P}}$ real numbers such that*

$$|\Sigma|_g - \sum_{p\in\mathcal{P}} \alpha_p > 0$$

Then the problem

$$-\Delta_g u + e^{2u} = 1 - \sum_{p\in\mathcal{P}} \alpha_p \delta_p \tag{7.17}$$

admits a unique solution.

Proof: The problem (7.17) with measures can be rewritten as a semilinear problem with a spatial weight. Indeed, by considering the Green's function $G(p,\cdot)$ at each point $p \in \mathcal{P}$ and setting

$$\tilde{u} := u + \sum_{p\in\mathcal{P}} \alpha_p G(p,\cdot)$$

$$K := e^{-2\sum_{p\in\mathcal{P}} \alpha_p G(p,\cdot)} \in C^0(\Sigma) \quad \text{ and } \quad A := 1 - \frac{\sum_{p\in\mathcal{P}} \alpha_p}{|\Sigma|_g} > 0,$$

then Problem (7.17) is equivalent to

$$-\Delta_g \tilde{u} + K e^{2\tilde{u}} = A, \qquad \tilde{u} \in H^1(\Sigma).$$

The associated functional is given by

$$J(f) = \frac{1}{2}\int_\Sigma \{|\nabla f|^2 + K e^{2f} - 2Af\}, \qquad f \in H^1(\Sigma),$$

and we claim that it admits a minimizer. Indeed if f_n is a minimizing sequence, we have

$$\int_\Sigma |\nabla(f-\bar{f})|^2 \leq C, \quad e^{2\bar{f}} \int_\Sigma K e^{2[f-\bar{f}]} \leq C, \quad -\bar{f} \leq C. \tag{7.18}$$

Observe now that by applying Jensen, Hölder and Poincaré's inequality we obtain

$$\int_\Sigma K e^{2[f-\bar{f}]} \geq \|K\|_1 e^{\frac{2}{\|K\|_1}\int_\Sigma K[f-\bar{f}]} \geq \|K\|_1 e^{-2\frac{2}{\|K\|_1}\|K\|_2\|f-\bar{f}\|_2} \geq C.$$

Using this lower bound in (7.18) we obtain $\int_\Sigma |\nabla(f_n - \bar{f}_n)|^2 + |\bar{f}_n| \leq C$, from which we easily see that f_n is bounded in$H^1(\Sigma)$ and so it converges weakly in this space to some f. By standard argument we deduce that f is a minimizer. The uniqueness follows for instance from the fact that J is strictly convex. □

In some situations, we are able to show that only the third type of blowup behaviors stated in Theorem 1.2 can occur. This will be discussed in a later work.

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