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Energy scattering for a 2D Hartree type INLS

Diffusion d'énergie pour un INLS de type HARTREE 2

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ABSTRACT. This paper studies the asymptotic behavior of energy solutions to the focusing non-linear generalized Hartree equation

$$iu_t + \Delta u = -|x|^{-\varrho}|u|^{p-2}(\mathcal{J}_\alpha * |\cdot|^{-\varrho}|u|^p)u, \quad \varrho > 0, \quad p \geq 2.$$

Here, $u := u(t,x)$, where the time variable is $t \in \mathbb{R}$ and the space variable is $x \in \mathbb{R}^2$. The source term is inhomogeneous because $\varrho > 0$. The convolution with the Riesz-potential $\mathcal{J}_\alpha := C_\alpha|\cdot|^{\alpha-2}$ for certain $0 < \alpha < 2$ gives a non-local Hartree type non-linearity. Taking account of the standard scaling invariance, one considers the inter-critical regime $1+\frac{2-2\varrho+\alpha}{2} < p < \infty$. It is the purpose to prove the scattering under the ground state threshold. This naturally extends the previous work by the first author for space dimensions greater than three (Scattering Theory for a Class of Radial Focusing Inhomogeneous Hartree Equations, Potential Anal. (2021)). The main difference is due to the Sobolev embedding in two space dimensions $H^1(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2)$, for all $2 \leq r < \infty$. This makes any exponent of the source term be energy sub-critical, contrarily to the case of higher dimensions. The decay of the inhomogeneous term $|x|^{-\varrho}$ is used to avoid any radial assumption. The proof uses the method of Dodson-Murphy based on Tao's scattering criteria and Morawetz estimates.
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1 Introduction

This note considers the scattering of energy global solutions to the focusing inhomogeneous generalized Hartree problem

$$\begin{cases} iu_t + \Delta u + |x|^{-\varrho}|u|^{p-2}(\mathcal{J}_\alpha * |\cdot|^{-\varrho}|u|^p)u = 0; \\ u_{|t=0} = u_0. \end{cases} \tag{1.1}$$

Here and hereafter, the wave function is $u := u(t,x) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$. The above problem is said inhomogeneous because of the singular quantity $|\cdot|^{-\varrho}$ for a certain positive real number $\varrho > 0$. The convolution with the above Riesz-potential

$$\mathcal{J}_\alpha : x \mapsto \frac{\Gamma(\frac{2-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi 2^\alpha |x|^{2-\alpha}}, \quad 0 \neq x \in \mathbb{R}^2, \quad 0 < \alpha < 2,$$

gives a non-local source term of Hartree type. In order to avoid an eventual singular term $|u|^{p-2}$, one assumes that $p \geq 2$. In all this work, one assumes the following natural conditions done in [1],

$$\min\{\alpha, \varrho, 2-\alpha, 2-\varrho, 2+\alpha-2\varrho\} > 0. \tag{1.2}$$

The equation (1.1) models various domains of mathematical physics. In the homogeneous regime $\varrho = 0$, the problem (1.1) for $p = 2$ is called Hartree type equation arises in atomic and nuclear physics

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and is related to the mean-field theory with respect to wave functions describing boson systems [5, 20]. The inhomogeneous term $|x|^{-\varrho}$ represents some inhomogeneity in the medium [8, 2]. Also, the problem (1.1) is of interest in the meanfield limit of large systems of non-relativistic bosonic atoms and molecules in a regime where the number of bosons is very large, but the interactions between them are weak [7, 9, 16].

The inhomogeneous Hartree equation (1.1) is invariant under the time-space scaling $u_\delta = \delta^{\frac{2-2\varrho+\alpha}{2(p-1)}} u(\delta^2\cdot, \delta\cdot)$, for $\delta > 0$. Moreover, the identity $\|u_\delta(t)\|_{\dot H^s} = \delta^{s-s_c}\|u(\delta^2 t)\|_{\dot H^s}$ gives the unique Sobolev norm invariant under the above scaling, which corresponds to the critical Sobolev exponent $s_c := 1 - \frac{2-2\varrho+\alpha}{2(p-1)}$.

This work is concerned with the mass super-critical and energy sub-critical regime, called also inter-critical one: $0 < s_c < 1$. Indeed, for the Schrödinger equation (1.1), two regimes are of particular interest in the literature. The first one, called L^2-critical or mass-critical, corresponds to $s_c = 0$ and is related to the mass conservation law

$$M[u(t)] := \int_{\mathbb{R}^2} |u(t,x)|^2\, dx = M[u_0].$$

The second one, called energy-critical or $\dot H^1$-critical, corresponds to $s_c = 1$ and is related to the energy conservation law

$$E[u(t)] := \int_{\mathbb{R}^2} |\nabla u(t,x)|^2\, dx - \frac{1}{p}\int_{\mathbb{R}^2} |x|^{-\varrho}|u|^p(\mathcal{J}_\alpha * |\cdot|^{-\varrho}|u|^p)\, dx = E[u_0].$$

The inhomogeneous generalized Hartree equation was treated first by the first author, where the ground state threshold dichotomy was investigated using a sharp adapted Gagliargo-Nirenberg type estimate [1]. After that, the first author treated the local well-posedness in $\dot H^1 \cap \dot H^{s_c}$, $0 < s_c < 1$, see [17]. The scattering under the ground state threshold with spherically symmetric data, was proved by the first author [19] and extended to the non-radial regime in [22, 18]. The well-posedness in the energy-critical regime was investigated recently [14, 13]. The energy critical scattering was treated in [10].

It is the aim of this note to establish the energy scattering of global solutions to (1.1) in the inter-critical regime and under the ground state threshold. This naturally extends the previous paper by the first author [19], where the scattering was proved for space dimensions larger than three. The main difference is due to the Sobolev embedding in two space dimensions $H^1(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2)$, for all $2 \le r < \infty$. This makes any exponent of the source term be energy sub-critical, contrarily to the case of higher dimensions. The decay of the inhomogeneous term $|x|^{-\varrho}$ is used to avoid any radial assumption. The scattering is obtained by using the new approach of Dodson-Murphy [4] which is based on Tao's scattering criteria [21] and Morawetz estimates.

The rest of this paper is organized as follows. The next section contains the main result and some useful estimates. Sections 3 proves the main result.

Here and hereafter, one denotes for simplicity, the Lebesgue and Sobolev spaces $L^r := L^r(\mathbb{R}^2)$, $H^1 := H^1(\mathbb{R}^2)$ equipped with the usual norms

$$\|\cdot\|_r := \|\cdot\|_{L^r}, \quad \|\cdot\| := \|\cdot\|_2, \quad \|\cdot\|_{H^1} := \Big(\|\cdot\|^2 + \|\nabla\cdot\|^2\Big)^{\frac{1}{2}}.$$

Finally, $T^* > 0$ denotes the lifespan of an eventual solution to (1.1).

2 Background and main results

In this section, one gives the contribution of this manuscript and some estimates to be used later.

2.1 Preliminary

Here and hereafter one denotes the mass critical exponent $p_m := 1 + \frac{\alpha+2-2\varrho}{2}$ and real numbers $\gamma := 2p - 2 - \alpha + 2\varrho$ and $\rho := 2p - \gamma$. Let us denote the source term $\mathcal{N}[u] := \mathcal{N}(x,u) := |x|^{-\varrho}|u|^{p-2}(\mathcal{J}_\alpha * |\cdot|^{-\varrho}|u|^p)u$. Define also the potential energy $\mathcal{P}[u] := \int_{\mathbb{R}^2} \bar{u}\mathcal{N}[u]\, dx$ and the quantity $\mathcal{I}[u] := \|\nabla u\|^2 - \frac{\gamma}{2p}\mathcal{P}[u]$.

The next Gagliardo-Nirenberg type estimate related to the inhomogeneous Hartree problem (1.1) was proved in [1, Theorem 4.1].

Proposition 2.1. *Assume that* (1.2) *is satisfied and take* $1 + \frac{\alpha}{2} < p < \infty$. *Thus,*

1. *a sharp constant* $C_{p,\varrho,\alpha} > 0$ *exists, such that for all* $u \in H^1$,

$$\mathcal{P}[u] \le C_{p,\varrho,\alpha}\|u\|^{\rho}\|\nabla u\|^{\gamma}; \tag{2.1}$$

2. *moreover, there exists* φ *satisfying*

$$\varphi - \Delta\varphi = \mathcal{N}[\varphi], \quad 0 \neq \varphi \in H^1, \tag{2.2}$$

$$C_{p,\varrho,\alpha} = \frac{2p}{\rho}\left(\frac{\rho}{\gamma}\right)^{\frac{\gamma}{2}}\|\varphi\|^{-2(p-1)}; \tag{2.3}$$

3. *furthermore, one has the following Pohozaev identities*

$$\mathcal{P}[\varphi] = \frac{2p}{\rho}M[\varphi] = \frac{2p}{\gamma}\|\nabla\varphi\|^2. \tag{2.4}$$

Here and hereafter, φ denotes a solution to (2.2), which satisfies (2.3). Since, we are interested on the inter-critical regime, one defines the positive real number $\kappa_c := \frac{1-s_c}{s_c}$ and the scale invariant quantities, which are independent of the choice of φ, by use of Pohozaev identities

$$\mathcal{ME}[u] := \left(\frac{E[u]}{E[\varphi]}\right)\left(\frac{M[u]}{M[\varphi]}\right)^{\kappa_c}, \quad \mathcal{MG}[u] := \left(\frac{\|\nabla u\|}{\|\nabla\varphi\|}\right)\left(\frac{\|u\|}{\|\varphi\|}\right)^{\kappa_c};$$
$$\mathcal{MP}[u] := \left(\frac{\mathcal{P}[u]}{\mathcal{P}[\varphi]}\right)\left(\frac{M[u]}{M[\varphi]}\right)^{\kappa_c}.$$

It is proved in [1, Theorem 5.2] that the inhomogeneous Hartree problem (1.1) is locally well-posed in the energy space, for ϱ, α satisfying (1.2) and $2 \le p < \infty$. Moreover, the solution satisfies the mass and energy conservation laws. Note that, letting $e^{i\cdot\Delta}$ be the operator associated to the free Schrödinger equation $\Big(i\partial_t + \Delta\Big) = 0$. Then, by Duhamel integral formula, energy solutions to the problems (1.1) are fix point of the function

$$f(u) := e^{i\cdot\Delta}u_0 + i\int_0^{\cdot} e^{i(\cdot-s)\Delta}\Big[\mathcal{N}[u(s)]\Big]\, ds\,. \tag{2.5}$$

2.2 Main result

In this manuscript, one proves mainly the following scattering result.

Theorem 2.2. *Let ϱ, α satisfying* (1.2) *and* $\max\{2, p_m\} < p < \infty$*. Take* $u \in C_{T^*}(H^1)$ *be a maximal solution to* (1.1)*. Then,* u *is global and scatters if*

$$\sup_{t\in[0,T^*)} \mathcal{MP}[u(t)] < 1. \tag{2.6}$$

or if the datum satisfies

$$\max\Big\{\mathcal{ME}[u_0], \mathcal{MG}[u_0]\Big\} < 1. \tag{2.7}$$

Remarks 2.3. *1. The threshold is expressed in terms of the non-conserved potential energy in* (2.6)*;*

2. *the threshold is expressed in terms of the conserved mass and energy in* (2.7)*. This condition is more simple to check, but stronger than* (2.6)*;*
3. *the assumption* $p > 2$ *avoids an singular quantity in the source term;*
4. *following lines in [18, Theorem 2.3 and Theorem 2.4], the solution concentrates if* $\sup_{[0,T^*)} \mathcal{I}[u(t)] < 0$ *or if the datum satisfies* $\mathcal{ME}[u_0] < 1 < \mathcal{MG}[u_0]$*;*
5. *the proof of the second point is omitted because it follows like in [18, Theorem 2.3];*
6. *the scattering is obtained by using the new approach of Dodson-Murphy [4] which is based on Tao's scattering criteria [21] and Morawetz estimates.*

2.3 Useful estimates

Let us give a standard estimate in the Schrödinger context.

Definition 2.4. *A couple of real numbers* (q, r) *is said to be* μ *admissible* ($admissible$ *if* $\mu = 0$) *if* $\frac{1}{2} - \frac{1}{r} = \frac{1}{q} + \frac{\mu}{2}$ *and* $\frac{2}{1-\mu} < r \leq \Big((\frac{2}{1-\mu})^+\Big)'$*. Here,* $0 < a^+ - a << 1$ *and* $(a^+)' = \frac{a^+ a}{a^+ - a}$*. Moreover,* (q, r) *is said to be* $-\mu$ *admissible if* $\frac{1}{2} - \frac{1}{r} = \frac{1}{q} - \frac{\mu}{2}$ *and*

$$\Big(\frac{2}{1-\mu}\Big)^+ < r \leq \Big((\frac{2}{1+\mu})^+\Big)'. \tag{2.8}$$

For simplicity, one denotes Λ_μ *the set of* μ *admissible pairs. Let also*

$$\|\cdot\|_{S^\mu(I)} := \sup_{(q,r)\in\Lambda_\mu} \|\cdot\|_{L^q(I,L^r)}, \quad \|\cdot\|_{(S^{-\mu})'(I)} := \inf_{(q,r)\in\Lambda_{-\mu}} \|\cdot\|_{L^{q'}(I,L^{r'})}.$$

A standard tool to control solutions of (1.1) is the Strichartz estimate [6, 11, 12].

Proposition 2.5. *There exists* $C > 0$ *such that*

1. $\|e^{i\cdot\Delta}u\|_{S^\mu} \leq C\||\nabla|^\mu u\|$;
2. $\|\int_0^\cdot e^{i(\cdot-s)\Delta} f(s)\, ds\|_{S^\mu} \leq C\|f\|_{(S^{-\mu})'}$.

Let us recall also a classical dispersive estimate [3, Proposition 3.2.1].

Proposition 2.6. *Let $2 \leq r \leq \infty$. There exists $C > 0$ such that*

$$\|e^{it\Delta}u\|_{r'} \leq C\frac{\|u\|_r}{|t|^{2(\frac{1}{r}-\frac{1}{2})}}. \tag{2.9}$$

Let $\zeta : \mathbb{R}^2 \to \mathbb{R}$ be a convex smooth function. Define the variance potential and the Morawetz action

$$V_\zeta := \int_{\mathbb{R}^2} \zeta(x)|u(\cdot, x)|^2 \, dx; \tag{2.10}$$

$$M_\zeta = 2\Im \int_{\mathbb{R}^2} \bar{u}(\nabla\zeta \cdot \nabla u) \, dx := 2\Im \int_{\mathbb{R}^2} \bar{u}(\zeta_j u_j) \, dx, \tag{2.11}$$

where here and in the sequel, repeated index are summed. Let us give a Morawetz type estimate for the Schrödinger equation [19].

Proposition 2.7. *Take $u \in C_T(H^1)$ be a local solutions to* (1.1). *Let $\zeta : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Then, the following equality holds on $[0, T]$,*

$$\begin{aligned} & V_\zeta''[u] = M_\zeta'[u] \\ = \; & 4\int_{\mathbb{R}^2} \partial_l\partial_k\zeta\Re(\partial_k u \partial_l \bar{u}) \, dx - \int_{\mathbb{R}^2} \Delta^2\zeta|u|^2 \, dx \\ + \; & 2(\frac{2}{p} - 1)\int_{\mathbb{R}^2} \Delta\zeta\bar{u}\mathcal{N}[u] \, dx + \frac{4}{p}\int_{\mathbb{R}^2} \nabla\zeta \cdot \nabla(|x|^{-\varrho})|u|^p(\mathcal{J}_\alpha * |\cdot|^{-\varrho}|u|^p) \, dx \\ + \; & \frac{4}{p}(\alpha - 2)\int_{\mathbb{R}^2} |x|^{-\varrho}|u|^p\nabla\zeta(\frac{\cdot}{|\cdot|^2}\mathcal{J}_\alpha * |\cdot|^{-\varrho}|u|^p) \, dx. \end{aligned}$$

The next Hardy-Littlewood-Sobolev inequality will be useful [15].

Lemma 2.8. *Let $0 < \alpha < 2$.*

1. *Let $r \geq 1$ and $s > 1$ such that $\frac{1}{r} = \frac{1}{s} + \frac{\alpha}{2}$. Then,*
$$\|\mathcal{J}_\alpha * g\|_s \leq C_{s,\alpha}\|g\|_r, \quad \forall g \in L^r.$$
2. *Let $t \geq 1$ and $1 < s, r < \infty$ be such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{t} + \frac{\alpha}{2}$. Then,*
$$\|f(\mathcal{J}_\alpha * g)\|_t \leq C_{N,s,\alpha}\|f\|_r\|g\|_s, \quad \forall (f, g) \in L^r \times L^s.$$

From now, one hides the time variable t for simplicity, spreading it out only when necessary. Moreover, one denotes the centered ball of $\mathbb{R}^2$ with radius $R > 0$ and its complementary, respectively $B(R)$ and $B^c(R)$. Furthermore $C(R, R')$ is the centered annulus of $\mathbb{R}^2$ with small radius R and large radius R'. In what follows, one proves the main result of this note.

3 Proof of Theorem 2.2

In this section, one proves the the scattering of energy global solutions. The proof is divided to several steps.

3.1 Global existence

The global existence follows by the conservation laws via the next coercivity result.

Lemma 3.1. *Let $u \in H^1$ satisfying*

$$\mathcal{MP}[u] < \nu < 1. \tag{3.1}$$

Then,

$$\|\nabla u\|^2 \lesssim_{\nu,\varphi} E[u] \lesssim_{\nu,\varphi} \mathcal{I}[u]. \tag{3.2}$$

Remark 3.2. *The last inequality is because $p > p_m$ implies that $\gamma > 2$.*

Proof. A direct computation gives the useful identities

$$2(p-1)s_c = \gamma - 2; \tag{3.3}$$
$$\alpha_c(\gamma - 2) = \rho. \tag{3.4}$$

Using the Gagliardo-Nirenberg inequality (2.1) via Pohozaev identities (2.4), the explicit expression (2.3) and the equalities (3.3)-(3.4), one writes

$$\begin{aligned}
[\mathcal{P}[u]]^{\frac{\gamma}{2}} &\leq C_{p,\varrho,\alpha}\Big(\|u\|^{2\alpha_c}\mathcal{P}[u]\Big)^{\frac{\gamma}{2}-1}\|\nabla u\|^{\gamma}\\
&\leq \frac{2p}{\rho}(\frac{\rho}{\gamma})^{\frac{\gamma}{2}}\|\varphi\|^{-2(p-1)}\Big(M[u]^{\alpha_c}\mathcal{P}[u]\Big)^{\frac{\gamma}{2}-1}\|\nabla u\|^{\gamma}\\
&\leq \frac{2p}{\rho}(\frac{\rho}{\gamma})^{\frac{\gamma}{2}}M[\varphi]^{\frac{\rho-2(p-1)}{2}}[\mathcal{P}[\varphi]]^{\frac{\gamma}{2}-1}\Big(\mathcal{MP}[u]\Big)^{\frac{\gamma}{2}-1}\|\nabla u\|^{\gamma}\\
&\leq \Big(\frac{\rho}{\gamma}\frac{\mathcal{P}[\varphi]}{M[\varphi]}\Big)^{\frac{\gamma}{2}}\Big(\mathcal{MP}[u]\Big)^{\frac{\gamma}{2}-1}\|\nabla u\|^{\gamma}\\
&\leq \Big(\mathcal{MP}[u]\Big)^{\frac{\gamma}{2}-1}\Big(\frac{2p}{\gamma}\|\nabla u\|^2\Big)^{\frac{\gamma}{2}}.
\end{aligned}$$

Thus,

$$\mathcal{P}[u] \leq \frac{2p}{\gamma}\Big(\mathcal{MP}[u]\Big)^{\frac{\gamma-2}{\gamma}}\|\nabla u\|^2. \tag{3.5}$$

Moreover, by (3.5) and (3.1), one proves (3.2) as follows

$$\begin{aligned}
\mathcal{I}[u] &= \|\nabla u\|^2 - \frac{\gamma}{2p}\mathcal{P}[u]\\
&\geq \|\nabla u\|^2\Big(1 - \Big(\mathcal{MP}[u]\Big)^{\frac{\gamma-2}{\gamma}}\Big)\\
&\gtrsim \|\nabla u\|^2.
\end{aligned}$$

□

3.2 Scattering criteria

Here and hereafter, one denotes a smooth function $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\psi = 1$ on $B(\frac{1}{2})$, $\psi = 0$ on $B^c(1)$ and $0 \leq \psi \leq 1$. Take also $\psi_R := \psi(\frac{\cdot}{R})$. In this sub-section, one proves the next scattering criteria.

Proposition 3.3. *Take the assumptions of Theorem 2.2. Let $u \in C(\mathbb{R}, H^1)$ be a global solution to* (1.1). *Assume that*

$$0 < \sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} := E < \infty. \tag{3.6}$$

There exist $R, \varepsilon > 0$ depending on E, p, ϱ such that u scatters if

$$\liminf_{t\to\infty} \int_{B(R)} |u(t,x)|^2 \, dx < \varepsilon^2. \tag{3.7}$$

It is sufficient to prove that $u \in S^{s_c}(\mathbb{R})$. Moreover, by continuity argument, Strichartz estimate and Sobolev embedding, the key of the proof of the scattering criterion is the next result.

Proposition 3.4. *Take the assumptions of Proposition 3.3. Then, for any $\varepsilon > 0$, there exist $T, \mu > 0$ satisfying*

$$\|e^{i(\cdot-T)\Delta}u(T)\|_{S^{s_c}(T,\infty)} \lesssim \varepsilon^\mu. \tag{3.8}$$

Proof. By the integral formula, one writes for $\beta > 0$ to pick later

$$\begin{aligned} e^{i(t-T)\Delta}u(T) &= e^{it\Delta}u_0 + i\int_0^T e^{i(t-s)\Delta}\Big[\mathcal{N}[u]\Big]\, ds \\ &= e^{it\Delta}u_0 + i\Big(\int_0^{T-\varepsilon^{-\beta}} + \int_{T-\varepsilon^{-\beta}}^T\Big)e^{i(t-s)\Delta}\Big[\mathcal{N}[u]\Big]\, ds \\ &:= e^{it\Delta}u_0 + i\Big(\int_{J_1} + \int_{J_2}\Big)e^{i(t-s)\Delta}\Big[\mathcal{N}[u]\Big]\, ds \\ &:= e^{it\Delta}u_0 + F_1 + F_2. \end{aligned} \tag{3.9}$$

Now, one estimates the three different parts in (3.9).

• The linear term. By by the Dominated convergence Theorem via Strichartz estimates, one may choose $T_0 > \varepsilon^{-\beta} > 0$, such that

$$\|e^{i\cdot\Delta}u_0\|_{S^{s_c}(T_0,\infty)} \leq \varepsilon^2. \tag{3.10}$$

• The term F_1. First, the integral formula (2.5) gives

$$F_1(t) = e^{it\Delta}\Big(e^{-i(T-\varepsilon^{-\beta})\Delta}u(T-\varepsilon^{-\beta}) - u_0\Big). \tag{3.11}$$

So, using Strichartz estimate via (3.11) and an interpolation argument, one writes

$$\begin{aligned}\|F_1\|_{S^{s_c}(T,\infty)} &\leq \|F_1\|^{s_c}_{L^\infty((T,\infty),L^\infty)}\|F_1\|^{1-s_c}_{S(T,\infty)}\\ &\leq c\|F_1\|^{s_c}_{L^\infty((T,\infty),L^\infty)}.\end{aligned} \tag{3.12}$$

Let us prove the next claim:

$$\int_{\mathbb{R}^2}\mathcal{N}[u]\,dx \lesssim \|u\|^{2p-1}_{H^1}. \tag{3.13}$$

One decomposes the integral on the unit ball and it's complementary as follows:

$$\begin{aligned}\int_{B(1)}\mathcal{N}[u]\,dx &\leq c\||x|^{-\varrho}|u|^{p-1}\|_{L^\gamma(B(1))}\Big(\||x|^{-\varrho}u^p\|_{L^\beta(B(1))}+\||x|^{-\varrho}u^p\|_{L^\beta(B^c(1))}\Big)\\ &\leq c\|u\|^{2p-1}_b\Big(\||x|^{-\varrho}\|_{L^{a_1}(B(1))}\||x|^{-\varrho}\|_{L^c(B(1))}+\||x|^{-\varrho}\|_{L^{a_2}(B(1))}\||x|^{-\varrho}\|_{L^d(B^c(1))}\Big)\\ &\leq c\|u\|^{2p-1}_{H^1}.\end{aligned}$$

Here, one uses Lemma 2.8 and Hölder estimate, so that

$$\begin{cases}1+\frac{\alpha}{2}=\frac{1}{\gamma}+\frac{1}{\beta}=\frac{1}{a_1}+\frac{2p-1}{b}+\frac{1}{c}=\frac{1}{a_2}+\frac{2p-1}{b}+\frac{1}{d};\\ \frac{2}{d}<\varrho<\min\{\frac{2}{a_i},\frac{2}{c}\};\\ 0<\frac{1}{b}\leq\frac{1}{2}.\end{cases} \tag{3.14}$$

Thus,

$$\begin{cases}1+\frac{\alpha}{2}-\frac{2p-1}{b}=\frac{1}{a_1}+\frac{1}{c}>\varrho;\\ 1+\frac{\alpha}{2}-\frac{2p-1}{b}=\frac{1}{a_2}+\frac{1}{d};\\ \frac{2}{d}<\varrho<\min\{\frac{2}{a_i},\frac{2}{c}\};\\ 0<\frac{1}{b}\leq\frac{1}{2}.\end{cases} \tag{3.15}$$

This reads

$$0<\frac{2p-1}{b}<\frac{2+\alpha-2\varrho}{2}. \tag{3.16}$$

This is possible because $2-2\varrho+\alpha>0$. Moreover,

$$\begin{aligned}\int_{B^c(1)}\mathcal{N}[u]\,dx &\leq \|u\|^{2p-1}_{b_1}\Big(\||x|^{-\varrho}\|_{L^{e_1}(B^c(1))}\||x|^{-\varrho}\|_{L^{c_1}(B(1))}+\||x|^{-\varrho}\|_{L^{e_2}(B^c(1))}\||x|^{-\varrho}\|_{L^{d_1}(B^c(1))}\Big)\\ &\leq c\|u\|^{2p-1}_{H^1}.\end{aligned}$$

Here, by Lemma 2.8,

$$\begin{cases}1+\frac{\alpha}{2}=\frac{1}{e_1}+\frac{2p-1}{b_1}+\frac{1}{c_1}=\frac{1}{e_2}+\frac{2p-1}{b_1}+\frac{1}{d_1};\\ \max\{\frac{2}{e_i},\frac{2}{d_1}\}<\varrho<\frac{2}{c_1};\\ 0<\frac{1}{b_1}\leq\frac{1}{2}.\end{cases} \tag{3.17}$$

Thus,

$$\begin{cases} 1+\frac{\alpha}{2}-\frac{2p-1}{b_1}=\frac{1}{e_2}+\frac{1}{d_1}<\varrho; \\ 1+\frac{\alpha}{2}-\frac{2p-1}{b_1}=\frac{1}{e_1}+\frac{1}{c_1}; \\ \max\{\frac{2}{d_1},\frac{2}{e_i}\}<\varrho<\frac{2}{c_1}; \\ 0<\frac{1}{b_1}\leq\frac{1}{2}. \end{cases} \tag{3.18}$$

This reads

$$\frac{2+\alpha-2\varrho}{2}<\frac{2p-1}{b_1}<\frac{2p-1}{2}. \tag{3.19}$$

This is satisfied because

$$p>p_m>\frac{1}{2}+\frac{2+\alpha-2\varrho}{2}. \tag{3.20}$$

Now, (3.12) and (3.13) via (2.9), imply that

$$\begin{aligned} \|F_1\|_{S^{s_c}(T,\infty)} &\leq c\sup_{\{t\geq T\}}\Big(\int_0^{T-\varepsilon^{-\beta}}|t-s|^{-1}\|\mathcal{N}[u]\|_1\,ds\Big)^{s_c} \\ &\leq c\sup_{\{t\geq T\}}\Big((t-T+\varepsilon^{-\beta})^{-1}\|u\|_{H^1}^{2p-1}\Big)^{s_c} \\ &\leq c\varepsilon^{\beta s_c}. \end{aligned} \tag{3.21}$$

• The term F_2. By the assumption (3.7), one has for $T>\varepsilon^{-\beta}$ large enough,

$$\int_{\mathbb{R}^2}\psi_R(x)|u(T,x)|^2\,dx<\varepsilon^2.$$

Moreover, a computation with use of (1.1) and Hölder estimate gives

$$\begin{aligned} \Big|\frac{d}{dt}\int_{\mathbb{R}^2}\psi_R|u|^2\,dx\Big| &= \Big|-2\Im\int_{\mathbb{R}^2}\psi_R\bar{u}\Delta u\,dx\Big| \\ &= \Big|2\Im\int_{\mathbb{R}^2}\bar{u}\nabla\psi_R\cdot\nabla u\,dx\Big| \\ &\lesssim \frac{1}{R}. \end{aligned}$$

Then, for any $T-\varepsilon^{-\beta}\leq t\leq T$ and $R>\varepsilon^{-(2+\beta)}$, yields

$$\|\psi_R u(t)\|\leq\Big(\int_{\mathbb{R}^2}\psi_R(x)|u(T,x)|^2\,dx+C\frac{T-t}{R}\Big)^{\frac{1}{2}}\leq C\varepsilon.$$

This gives

$$\|\psi_R u\|_{L^\infty((T-\varepsilon^{-\beta},T),L^2)}\leq C\varepsilon. \tag{3.22}$$

Using Strichartz estimates in Proposition 2.5, one writes for $(q,r)\in\Lambda_{-s_c}$,

$$\begin{aligned}\|F_2\|_{S^{s_c}(T,\infty)} &\leq c\|\mathcal{N}[u]\,\|_{\Lambda'_{-s_c}(T,\infty)}\\ &\leq c\|\mathcal{N}[u]\,\|_{L^{q'}(J_2,L^{r'})}\\ &\leq \|\psi_R\mathcal{N}[u]\,\|_{L^{q'}(J_2,L^{r'})}+\|(1-\psi_R)\mathcal{N}[u]\,\|_{L^{q'}(J_2,L^{r'})}\\ &:= \|(I)\|_{L^{q'}(J_2)}+\|(II)\|_{L^{q'}(J_2)}.\end{aligned}\tag{3.23}$$

Now, by Hölder estimate via (3.23) and (3.22), one writes for certain $0<\theta\leq 1$,

$$\begin{aligned}(I) &\leq \|\psi_R u\|_f\|u\|_f^{2(p-1)}\Big(\||x|^{-\varrho}\|_{L^{a_1}(B(R))}\||x|^{-\varrho}\|_{L^c(B(R))}+\||x|^{-\varrho}\|_{L^{a_2}(B(R))}\||x|^{-\varrho}\|_{L^d(B^c(R))}\Big)\\ &\leq c\|\psi_R u\|^\theta\|u\|_{H^1}^{2(p-1)+1-\theta}\\ &\leq c\varepsilon^\theta.\end{aligned}\tag{3.24}$$

Here,

$$\begin{cases}1-\frac{1}{r}+\frac{\alpha}{2}=\frac{1}{a_1}+\frac{2p-1}{f}+\frac{1}{c}=\frac{1}{a_2}+\frac{2p-1}{f}+\frac{1}{d};\\ \frac{2}{d}<\varrho<\min\{\frac{2}{a_i},\frac{2}{c}\};\\ 0<\frac{1}{f}\leq\frac{1}{2}.\end{cases}\tag{3.25}$$

This gives

$$\begin{cases}2+\alpha-\frac{2}{r}-\frac{2(2p-1)}{f}=\frac{2}{c}+\frac{2}{a_1}>2\varrho;\\ 0<\frac{1}{f}\leq\frac{1}{2}.\end{cases}\tag{3.26}$$

So, the admissibility condition (2.8) implies that

$$\frac{1-s_c}{2}<\frac{1}{r}+\frac{2p-1}{f}<\frac{1}{2}\Big(2-2\varrho+\alpha\Big).$$

This is possible because $p\geq 2$. Moreover, by Hölder estimate via (3.23) and the properties of ψ, one writes

$$\begin{aligned}(II) &\leq c\|u\|_e^{2p-1}\Big(\||x|^{-\varrho}\|_{L^{g_1}(B^c(R))}\||x|^{-\varrho}\|_{L^h(B^c(R))}+\||x|^{-\varrho}\|_{L^{g_2}(B^c(R))}\||x|^{-\varrho}\|_{L^k(B(R))}\Big)\\ &\leq cR^{-(\varrho-\frac{2}{g})}\|u\|_{H^1}^{2p-1}\\ &\leq cR^{-(\varrho-\frac{2}{g})}.\end{aligned}\tag{3.27}$$

Here, $g:=\min\{g_1,g_2\}$ and

$$\begin{cases}1-\frac{1}{r}+\frac{\alpha}{2}=\frac{1}{g_1}+\frac{2p-1}{e}+\frac{1}{h}=\frac{1}{g_2}+\frac{2p-1}{e}+\frac{1}{k};\\ \max\{\frac{2}{g_i},\frac{2}{h}\}<\varrho<\frac{2}{k};\\ 0<\frac{1}{e}\leq\frac{1}{2}.\end{cases}\tag{3.28}$$

This reads

$$2+\alpha-2\varrho<\frac{2(2p-1)}{e}+\frac{2}{r}<2p-1+1-s_c=2p-1+\frac{2+\alpha-2\varrho}{2(p-1)}.$$

So, one needs $(2+\alpha-2\varrho)(1+\frac{1}{2(p-1)})<2p-1$ and so $2+\alpha-2\varrho<2(p-1)$. This is possible because $p>p_m$.

Now, by (3.23), (3.24) and (3.27), one gets for $0 < 2\beta < q'\theta$ and $R^{-(\varrho-\frac{2}{g})} < \varepsilon^\theta$,

$$\begin{aligned}\|F_2\|_{S^{s_c}(T,\infty)} &\leq \|(I)\|_{L^{q'}(J_2)} + \|(II)\|_{L^{q'}(J_2)} \\ &\leq c|J_2|^{\frac{1}{q'}}\Big(R^{-(\varrho-\frac{2}{g})} + \varepsilon^\theta\Big) \\ &\leq c\varepsilon^{-\frac{\beta}{q'}}\Big(R^{-(\varrho-\frac{2}{g})} + \varepsilon^\theta\Big) \\ &\leq c\varepsilon^\nu, \end{aligned} \tag{3.29}$$

where $\nu > 0$. The proof is closed via (3.10), (3.21) and (3.29). □

3.3 Virial/Morawetz estimate

The next radial identities will be useful.

$$\nabla = \frac{x}{r}\partial_r, \quad \Delta = \partial_r^2 + \frac{1}{r}\partial_r\,; \tag{3.30}$$

$$\frac{\partial^2}{\partial x_l \partial x_k} := \partial_l\partial_k = \Big(\frac{\delta_{lk}}{r} - \frac{x_l x_k}{r^3}\Big)\partial_r + \frac{x_l x_k}{r^2}\partial_r^2\,;. \tag{3.31}$$

In the rest of this note, one takes a smooth radial function $\zeta(x) := \zeta(|x|)$ such that $x \in \mathbb{R}^2$ and

$$\zeta : r \to \begin{cases} r^2, & \text{if} \quad 0 \leq r \leq 1; \\ r, & \text{if} \quad r > 2. \end{cases}$$

Now, for $R > 0$, take via (2.10) and (2.11),

$$\zeta_R := R^2\zeta(\frac{|\cdot|}{R}), \quad M_R := M_{\zeta_R} \quad \text{and} \quad V_R := V_{\zeta_R}.$$

Moreover, one assumes that in the centered annulus $C(R, 2R)$,

$$\partial_r\zeta > 0, \quad \partial_r^2\zeta \geq 0 \quad \text{and} \quad |\partial^\beta\zeta_R| \leq C_\alpha R|\cdot|^{1-\beta}, \quad \text{for all} \quad |\beta| \geq 1. \tag{3.32}$$

Note that on the centered ball of radius R, one has

$$\partial_{jk}\zeta_R = 2\delta_{jk}, \quad \Delta\zeta_R = 4 \quad \text{and} \quad \Delta^2\zeta_R = 0. \tag{3.33}$$

Moreover, on $B^c(2R)$, yield

$$\partial_{jk}\zeta_R = \frac{R}{|x|}\Big(\delta_{jk} - \frac{x_j x_k}{|x|^2}\Big), \quad \Delta\zeta_R = \frac{R}{|x|} \quad \text{and} \quad \Delta^2\zeta_R = \frac{R}{|x|^3}. \tag{3.34}$$

Now, one states a Morawetz type estimate.

Proposition 3.5. *There is $0 < \varepsilon << 1$ and $t_n, R_n \to \infty$ such that*

$$\int_0^T \Big(\int_{B(R)} |u(s,x)|^{2p}\,dx\Big)^{\frac{1}{p}} ds \lesssim T^{\frac{1}{1+\varepsilon}}; \tag{3.35}$$

$$\lim_{n\to\infty}\int_{B(R_n)} |u(t_n,x)|^{2p}\,dx = 0. \tag{3.36}$$

Proof. Taking account of Proposition 2.7, one writes

$$\begin{aligned}
V_R''[u] &= 4\int_{\mathbb{R}^2} \partial_l\partial_k\zeta_R\Re(\partial_k u\partial_l\bar{u})\,dx - \int_{\mathbb{R}^2}\Delta^2\zeta_R|u|^2\,dx \\
&+ 2(\frac{2}{p}-1)\int_{\mathbb{R}^2}\Delta\zeta_R\bar{u}\mathcal{N}[u]\,dx + \frac{4}{p}\int_{\mathbb{R}^2}\nabla\zeta_R\cdot\nabla(|x|^{-\varrho})|u|^p(\mathcal{J}_\alpha*|\cdot|^{-\varrho}|u|^p)\,dx \\
&+ \frac{4}{p}(\alpha-2)\int_{\mathbb{R}^2}|x|^{-\varrho}|u|^p\nabla\zeta_R(\frac{\cdot}{|\cdot|^2}\mathcal{J}_\alpha*|\cdot|^{-\varrho}|u|^p)\,dx \\
&:= (I)+(I)+(III),
\end{aligned} \tag{3.37}$$

where one decomposes the above integrals as $\Big(\int_{B(R)}+\int_{C(R,2R)}+\int_{B^c(2R)}\Big)$. By (3.33) via [19, Section 5], one has

$$(I) = 8\Big(\int_{B(R)}|\nabla u|^2\,dx - \frac{\gamma}{2p}\int_{B(R)}\bar{u}\mathcal{N}[u]\,dx\Big) + O\Big(\int_{B^c(R)}\bar{u}\mathcal{N}[u]\,dx\Big). \tag{3.38}$$

Moreover, taking $\not\nabla := \nabla - \frac{x\cdot\nabla}{|x|^2}x$ the angular gradient, by (3.34), it follows that

$$\begin{aligned}
(III) &= 4\int_{B^c(2R)}\frac{R}{|x|}\Big(\delta_{jk}-\frac{x_jx_k}{|x|^2}\Big)\Re(\partial_k u\partial_l\bar{u})\,dx - R\int_{B^c(2R)}\frac{|u|^2}{|x|^3}\,dx \\
&- 2(1-\frac{2}{p})\int_{B^c(2R)}\frac{R}{|x|}\bar{u}\mathcal{N}[u]\,dx - \frac{4\varrho}{p}\int_{B^c(2R)}\frac{R}{|x|}\bar{u}\mathcal{N}[u]\,dx + O\Big(\int_{B^c(R)}\bar{u}\mathcal{N}[u]\,dx\Big) \\
&= 4\int_{B^c(2R)}\frac{R}{|x|}|\not\nabla u|^2\,dx - R\int_{B^c(2R)}\frac{|u|^2}{|x|^3}\,dx \\
&- 2(1-\frac{2}{p})\int_{B^c(2R)}\frac{2R}{|x|}\bar{u}\mathcal{N}[u]\,dx - \frac{4\varrho}{p}\int_{B^c(2R)}\frac{R}{|x|}\bar{u}\mathcal{N}[u]\,dx + O\Big(\int_{B^c(R)}\bar{u}\mathcal{N}[u]\,dx\Big) \\
&\gtrsim -R^{-2}\int_{\mathbb{R}^2}|u|^2\,dx - \int_{B^c(R)}\bar{u}\mathcal{N}[u]\,dx.
\end{aligned} \tag{3.39}$$

Furthermore, by (3.32), one has

$$\begin{aligned}
(II) &:= 4\int_{C(R,2R)}\partial_l\partial_k\zeta_R\Re(\partial_k u\partial_l\bar{u})\,dx - \int_{C(R,2R)}\Delta^2\zeta_R|u|^2\,dx \\
&+ 2(\frac{2}{p}-1)\int_{C(R,2R)}\Delta\zeta_R\bar{u}\mathcal{N}[u]\,dx + \frac{4}{p}\int_{C(R,2R)}\nabla\zeta_R\cdot\nabla(|x|^{-\varrho})|u|^p(\mathcal{J}_\alpha*|\cdot|^{-\varrho}|u|^p)\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{4}{p}(\alpha-2) \int_{C(R,2R)} |x|^{-\varrho}|u|^p \nabla\zeta_R(\frac{\cdot}{|\cdot|^2}\mathcal{J}_\alpha * |\cdot|^{-\varrho}|u|^p)\, dx \\
&\gtrsim \quad -R^{-3}\int_{\mathbb{R}^2} |u|^2\, dx - \int_{B^c(R)} \bar{u}\mathcal{N}[u]\, dx. \qquad (3.40)
\end{aligned}
$$

Now, by (3.37), (3.38), (3.39) and (3.40), it follows that for certain $0 < \varepsilon << 1 << R$,

$$
\begin{aligned}
&V_R''[u] \\
\gtrsim & \int_{B(R)} |\nabla u|^2\, dx - \frac{\gamma}{2p}\int_{B(R)} \bar{u}\mathcal{N}[u]\, dx - R^{-2}\int_{\mathbb{R}^2} |u|^2\, dx - \int_{B^c(R)} \bar{u}\mathcal{N}[u]\, dx \\
\gtrsim & \int_{B(R)} |\nabla u|^2\, dx - \frac{\gamma}{2p}\int_{B(R)} \bar{u}\mathcal{N}[u]\, dx - R^{-2}\int_{\mathbb{R}^2} |u|^2\, dx - R^{-\varepsilon}\|u\|_{H^1}^{2p} \\
\gtrsim & \int_{B(R)} |\nabla u|^2\, dx - \frac{\gamma}{2p}\int_{B(R)} \bar{u}\mathcal{N}[u]\, dx - cR^{-2} - cR^{-\varepsilon} \qquad (3.41)
\end{aligned}
$$

Indeed, by Lemma 2.8 and Sobolev embeddings, one writes

$$
\begin{aligned}
\int_{B^c(R)} \bar{u}\mathcal{N}[u]\, dx &\lesssim \||x|^{-\varrho}\|_{L^{b_2}(B^c(R))}\Big(\|u\|_{a_1}^{2p}\||x|^{-\varrho}\|_{L^{b_1}(B(1))} + \|u\|_{a_2}^{2p}\||x|^{-\varrho}\|_{L^{b_2}(B^c(1))}\Big) \\
&\lesssim R^{\frac{2}{b_2}-\varrho}\|u\|_{H^1}^{2p}. \qquad (3.42)
\end{aligned}
$$

Here,

$$
\begin{cases}
1+\frac{\alpha}{2} = \frac{1}{b_2} + \frac{2p}{a_1} + \frac{1}{b_1} = \frac{2p}{a_2} + \frac{2}{b_2}; \\
\frac{2}{b_1} > \varrho > \frac{2}{b_2}; \\
0 < \frac{1}{a_i} \leq \frac{1}{2}, \quad 1 \leq i \leq 2.
\end{cases} \qquad (3.43)
$$

This gives

$$
\begin{cases}
2+\alpha-2p < 2+\alpha-\frac{4p}{a_2} = \frac{4}{b_2} < 2\varrho; \\
0 < \frac{1}{a_i} \leq \frac{1}{2}, \quad 1 \leq i \leq 2.
\end{cases} \qquad (3.44)
$$

Such a choice is possible because $p > p_m$. Moreover, by the identities

$$
\int_{\mathbb{R}^2} \bar{u}\mathcal{N}[u]\, dx = \int_{\mathbb{R}^2} (\psi_R\bar{u})\mathcal{N}[\psi_R u]\, dx + O\Big(\int_{B^c(R)} \bar{u}\mathcal{N}[u]\, dx\Big);
$$

$$
\int_{\mathbb{R}^2} \psi_R^2|\nabla u|^2\, dx = \|\nabla(\psi_R u)\|^2 + \int_{\mathbb{R}^2} \psi_R\Delta\psi_R|u|^2\, dx,
$$

via (3.41), (3.2) and Sobolev embedding, one writes

$$
\begin{aligned}
V_R''[u] + cR^{-2} + cR^{-\varepsilon} &\gtrsim \mathcal{I}(\psi_R u) \\
&\gtrsim \|\nabla(\psi_R u)\|^2
\end{aligned}
$$

$$\begin{aligned} &\gtrsim \|\psi_R u\|_{2p}^2 \\ &\gtrsim \Big(\int_{B(R)} |u|^{2p}\,dx\Big)^{\frac{1}{p}}. \end{aligned} \tag{3.45}$$

Integrating in time the estimate (3.45) and taking $0 < \varepsilon << 1$, it follows that

$$\begin{aligned} \int_0^T \Big(\int_{B(R)} |u(s,x)|^{2p}\,dx\Big)^{\frac{1}{p}} ds &\lesssim V_R'[u(T)] - V_R'[u_0] + cTR^{-2} + cTR^{-\varepsilon} \\ &\lesssim R + cTR^{-\varepsilon}. \end{aligned} \tag{3.46}$$

So, (3.35) follows by taking $R = T^{\frac{1}{1+\varepsilon}}$. Moreover, (3.35) gives

$$\frac{2}{T}\int_{\frac{T}{2}}^T \Big(\int_{B(R)} |u(s,x)|^{2p}\,dx\Big)^{\frac{1}{p}} ds \lesssim T^{-\frac{\varepsilon}{1+\varepsilon}}.$$

We conclude the proof of (3.36) by using the mean value Theorem. □

3.4 Proof of the scattering

Take $R, \varepsilon > 0$ given by Proposition 3.3 and $t_n, R_n \to \infty$ given by Proposition 3.5. Letting $n >> 1$ such that $R_n > R$, one gets by Hölder's inequality

$$\begin{aligned} \|u(t_n)\|_{L^2(B(R))}^2 &\leq |B(R)|^{\frac{p-1}{p}} \|u(t_n)\|_{L^{2p}(B(R))}^2 \\ &\leq R^{\frac{2(p-1)}{p}} \|u(t_n)\|_{L^{2p}(B(R_n))}^2 \\ &\lesssim \varepsilon^2. \end{aligned}$$

Hence, the scattering of energy global solutions to the focusing problem (1.1) follows from Proposition 3.3.

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