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Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functors

Les foncteurs de type Gysin-$(Z/2Z)^d$

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ABSTRACT. Let $d \geq 1$ be an integer and $\mathcal{K}_d$ be a contravariant functor from the category of subgroups of $(\mathbb{Z}/2\mathbb{Z})^d$ to the category of graded and finite $\mathbb{F}_2$-algebras. In this paper, we generalize the conjecture of G. Carlsson [C3], concerning free actions of $(\mathbb{Z}/2\mathbb{Z})^d$ on finite CW-complexes, by suggesting, that if $\mathcal{K}_d$ is a Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functor (that is to say, the functor $\mathcal{K}_d$ satisfies some properties, see 2.2), then we have:

$$(C_d) : \sum_{i\geq 0} dim_{\mathbb{F}_2}(\mathcal{K}_d(0))^i \geq 2^d$$

We prove this conjecture for $1 \leq d \leq 3$ and we show that, in certain cases, we get an independent proof of the following results (for $d = 3$ see [C4]):

If the group $(\mathbb{Z}/2\mathbb{Z})^d$, $1 \leq d \leq 3$, acts freely and cellularly on a finite CW-complex X, then $\sum_{i\geq 0} dim_{\mathbb{F}_2} H^i(X;\, \mathbb{F}_2) \geq 2^d$.

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1. Introduction

Since the work of Paul A. Smith arround 1938 [Sm] (see also [MB]) known as “Smith theory" the following problem has been posed.

$(\mathcal{P}_{d,k})$: *Which group $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely and cellularly on a product of k spheres ?*

The case $k = 1$, which is easy, was proved since 1935 ([Sm], [M] and [MTW]); the result is that, if $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely on the sphere S^n then $d \leq 1$.

The case $k = 2$ has been proved by A.Heller [He] in 1959 using a combinatory method which, apparently, doesn't extend to the case of three spheres (see [DV]). The result is that, if $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely and cellularly on the product of two spheres $S^{n_1} \times S^{n_2}$, then $d \leq 2$.

Some works concerning the problem $(\mathcal{P}_{d,k})$ “(such as [Co])" allow to have a generalization. The following statement is a classical conjecture:

$(C_{d,S})$: *The group $(\mathbb{Z}/2\mathbb{Z})^{d+1}$ doesn't act freely and cellularly on a product of d spheres, $d \geq 1$.*

The conjecture $(C_{3,S})$ was proved by G. Carlsson in 1987 [C4].

Among other works concerning the conjecture $(C_{d,S})$, we can cite [AB], [C2], [Han], [OY] and [R]. Carlsson's work [C2] concerns the case where the spheres have the same dimension and the action of the group $(\mathbb{Z}/2\mathbb{Z})^d$ on homology is trivial. The work of Adem-Browder [AB] concerns the case of the group $(\mathbb{Z}/p\mathbb{Z})^d$, p an odd prime.

In the middle of $1980s$ the conjecture ($C_{d,S}$) was generalized by G. Carlsson [C3] (and S. Halperin [Hal] for Torus) who suggest the following “Halperin-Carlsson" Conjecture (it is also called “toral rank conjecture" in some literature).

($C_{d,X}$): Let X be a finite CW-complex on which the group $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely and cellularly then, $$\sum_{i\geq 0} dim_{\mathbb{F}_2} H^i(X;\ \mathbb{F}_2) \geq 2^d.$$

In this paper we generalize the conjecture $C_{d,X}$ in the following sense which will be more precise in section 2.2. Let's call a Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functor a contravariant functor $\mathcal{K}_{(\mathbb{Z}/2\mathbb{Z})^d}$, or $\mathcal{K}_d$ for simplicity, from the category of subgroups of $(\mathbb{Z}/2\mathbb{Z})^d$ to the category of graded, finite and unitary $\mathbb{F}_2$-algebras such that:

- For every subgroup W of $(\mathbb{Z}/2\mathbb{Z})^d$, the graded, finite and unitary $\mathbb{F}_2$-algebra $\mathcal{K}_d(W)$ is an $H^*(W;\ \mathbb{F}_2)$-algebra,
- For every subgroup W of $(\mathbb{Z}/2\mathbb{Z})^d$ and for every U a subgroup of W of codimension one, there exist an exact sequence of $H^*(W;\ \mathbb{F}_2)$-modules of the form:

$$\ldots \longrightarrow \mathcal{K}_d(W)^{*-1} \xrightarrow{t.} \mathcal{K}_d(W)^* \xrightarrow{\mathcal{K}_d(i)} \mathcal{K}_d(U)^* \xrightarrow{\psi} \mathcal{K}_d(W)^* \xrightarrow{t.} \ldots$$

where

- $i : U \hookrightarrow W$ is the inclusion,
- $\mathcal{K}_d(U)$ is an $H^*(W;\ \mathbb{F}_2)$-algebra via $i^* : H^*(W;\ \mathbb{F}_2) \to H^*(U;\ \mathbb{F}_2)$,
- $\mathcal{K}_d(W)$ is an $H^*(W/U;\ \mathbb{F}_2)$-algebra via $\pi^* : H^*(W/U;\ \mathbb{F}_2) \to H^*(W;\ \mathbb{F}_2)$, $\pi : W \to W/U$ is the natural projection,
- $H^*(W/U;\ \mathbb{F}_2) \simeq \mathbb{F}_2[t]$.

We propose the following conjecture:

(C_d): Let $\mathcal{K}_d$ be a Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functor, then: $$\sum_{i\geq 0} dim_{\mathbb{F}_2}\big(\mathcal{K}_d(0)\big)^i \geq 2^d.$$

The conjecture C_d implies the conjecture $C_{d,X}$ because if X is a finite CW-complex on which the group $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely and cellularly, then the functor $\mathcal{K}_d$ defined by $\mathcal{K}_d(W) = H^*_W(X;\ \mathbb{F}_2)$ is a Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functor whose 0^{th}-term is $H^*(X;\ \mathbb{F}_2)$.

The aim of this paper is to prove, in certain cases, the conjecture C_d for $1 \leq d \leq 3$.

The paper is structured as follows. In the second section we fix the notations and we give some properties of Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functors. The third section will concern the proof of the main result of this paper.

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2. On Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functors

In this section we fix some notations, introduce the Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functors and give some of their properties.

2.1. *Notations*

Let V be an elementary abelian 2-group that is, a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$, $d \geq 1$; the integer d is called the rank of V and will be denoted by $d = rk(V)$. The mod. 2 cohomology of V will be simply denoted H^*V. Let's recall that H^*V is a polynomial algebra over $\mathbb{F}_2$ on d generators t_i, $1 \leq i \leq d$, of degree one.

We denote by $(t)_0^k = \mathbb{F}_2[t]/ < t^{k+1} >$ where $< t^s >$, $s \in \mathbb{N}$, is the ideal of $\mathbb{F}_2[t]$ of elements of degree $\geq s$.

Let X be a CW-complex. Throughout this paper, the action of V on X will be considered cellulary (see [TD] [Chap. II, Sect. 1] for the notion of equivariant CW-complexes).

2.2. *Gysin-V-functors*

Let V be an elementary abelian 2-group of rank ≥ 1. The set $\mathcal{W}$ of subgroups of V is ordered by inclusion and then can be considered as a category. Let $\mathbb{K}_f$ be the category of graded, finite and unitary $\mathbb{F}_2$-algebras; we denote by H^*V-$\mathbb{K}_f$ the category of graded, finite and unitary H^*V-$\mathbb{F}_2$-algebras. An object of this category is a graded, finite and unitary $\mathbb{F}_2$-algebra K equipped with a map of graded unitary $\mathbb{F}_2$-algebras $H^*V \otimes K \to K$.

Definition 2.2.1. A Gysin-V-functor is a contravariant functor

$$\mathcal{K}_V : \mathcal{W} \rightsquigarrow \mathbb{K}_f,\ W \mapsto \mathcal{K}_V(W) = K_W$$

such that:

(i) For every subgroup W of V, the algebra K_W is an object of the category H^*W-$\mathbb{K}_f$.

(ii) For every subgroup W of V and for every subgroup U of W of codimension one, there exist an exact sequence of H^*W-modules of the form:

$$G(U,W): \quad \ldots \longrightarrow (K_W)^{*-1} \xrightarrow{t.} (K_W)^* \xrightarrow{\mathcal{K}_V(i)} (K_U)^* \xrightarrow{\psi} (K_W)^* \xrightarrow{t.} \ldots$$

where

- $i : U \hookrightarrow W$ is the inclusion,
- K_U is an H^*W-algebra via $i^* : H^*W \to H^*U$,
- $H^*(W/U) \simeq \mathbb{F}_2[t]$ and $t. : K_W \to K_W$ is the $H^*(W/U)$-structure of K_W via the morphism $\pi^* : H^*(W/U) \to H^*W$ induced by the projection $\pi : W \to W/U$.

Vocabularies and notations 2.2.2.
2.2.2.1. The exact sequence $G(U, W)$ will be called the Gysin sequence associated to the subgroups U and W of V ($U \subseteq W$ of codimension one).

2.2.2.2. When the structures and morphisms are fixed, a Gysin-V-functor will be simply denoted $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$.

Let E be an elementary abelian 2-group, let $E' \subseteq E$ be a subgroup and let M_E be an H^*E-module. The module M_E is an $H^*(E/E')$-module via $\pi^* : H^*(E/E') \to H^*E$ where $\pi : E \to E/E'$ is the projection. We denote $\widetilde{H}^*(E/E')$ the augmentation ideal of $H^*(E/E')$.

Definition 2.2.3. With the previous notations, we denote

$$\begin{aligned}\overline{M_E}^{E/E'} &= M_E/\widetilde{H}^*(E/E').M_E \\ &= \mathbb{F}_2 \underset{\mathrm{H}^*(E/E')}{\otimes} M_E = Tor_0^{\mathrm{H}^*(E/E')}(M_E,\ \mathbb{F}_2)\end{aligned}$$

The previous Gysin sequence $G(U, W)$ induces a short exact sequence of H^*U-modules:

$$\overline{G}(U, W) : \quad 0 \longrightarrow \overline{K_W}^{W/U} \xrightarrow{\mathcal{K}_V(i)} K_U \xrightarrow{\psi} \tau^{W/U}\big(K_W\big) \longrightarrow 0$$

where

$$\begin{aligned}\tau^{W/U}\big(K_W\big) &= \ker\big(t. : (K_W)^* \to (K_W)^{*+1}\big) \\ &= Tor_1^{\mathrm{H}^*(W/U)}\big(K_W,\ \mathbb{F}_2\big)\end{aligned}$$

One can construct various examples of Gysin-V-functors; some of them are purely algebraic examples and the other comes from topology.

2.2.4. Examples

Example 2.2.4.1. Let $K_0 = \langle \iota, x_1, x_2, x_4, y_4, x_5 \rangle$ be the graded, finite and unitary $\mathbb{F}_2$-algebra generated by six generators: ι of degree zero, x_i of degree i, $i = 1, 2, 4, 5$ and y_4 of degree 4. These generators satisfy the following relations:

$$\begin{cases} x_j^2 = y_4^2 = 0,\ j = 1, 2, 4, 5, \\ x_1 x_4 = x_1 y_4 = 0, \\ x_2 x_4 = x_2 x_5 = 0, \\ x_2 y_4 = x_1 x_5. \end{cases}$$

In K_0 the elements x_1x_2 and x_1x_5 are non trivial. As an $\mathbb{F}_2$-vector space $K_0 = \langle \iota, x_1, x_2, x_1x_2, x_4, y_4, x_5, x_1x_5\rangle$.

Let $H^*(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2[t]$. We consider the graded, finite and unitary $\mathbb{F}_2[t]$-$\mathbb{F}_2$-algebra $K_{\mathbb{Z}/2\mathbb{Z}} = \langle \mu, t, z_1, z_2\rangle$ generated by four generators: μ of degree zero, t and z_1 of degree one and z_2 of degree two. In $K_{\mathbb{Z}/2\mathbb{Z}}$ we have the relations:

$$\begin{cases} z_1^2 = z_2^2 = 0, \\ t^5\mu = t^4 z_1 = t^4 z_2 = 0. \end{cases}$$

In $K_{\mathbb{Z}/2\mathbb{Z}}$ the elements z_1z_2, $t^4\mu$, t^3z_1, t^3z_2 and $t^3z_1z_2$ are non trivial. We then have:

$$\begin{cases} \overline{K_{\mathbb{Z}/2\mathbb{Z}}}^{\mathbb{Z}/2\mathbb{Z}} = \langle \mu, z_1, z_2, z_1z_2\rangle \text{ as an } \mathbb{F}_2\text{-vector space}, \\ \tau^{\mathbb{Z}/2\mathbb{Z}}(K_{\mathbb{Z}/2\mathbb{Z}}) = \{t^4\mu, t^3z_1, t^3z_2, t^3z_1z_2\} \text{ as an } \mathbb{F}_2\text{-vector space}. \end{cases}$$

Consider the following sequence of $\mathbb{F}_2$-vector spaces

$$0 \longrightarrow \overline{K_{\mathbb{Z}/2\mathbb{Z}}}^{\mathbb{Z}/2\mathbb{Z}} = \langle \mu, z_1, z_2, z_1z_2\rangle \xrightarrow{\sigma} K_0 = \langle \iota, x_1, x_2, x_1x_2, x_4, y_4, x_5, x_1x_5\rangle \xrightarrow{\psi}$$
$$\tau^{\mathbb{Z}/2\mathbb{Z}}(K_{\mathbb{Z}/2\mathbb{Z}}) = \{t^4\mu, t^3z_1, t^3z_2, t^3z_1z_2\} \longrightarrow 0$$

where

- $\sigma(\mu) = \iota$, $\sigma(z_1) = x_1$, $\sigma(z_2) = x_2$ and $\sigma(z_1z_2) = x_1x_2$.
- $\psi(\iota) = \psi(x_1) = \psi(x_2) = \psi(x_1x_2) = 0$
- $\psi(x_4) = t^4\mu$, $\psi(y_4) = t^3z_1$, $\psi(x_5) = t^3z_2$ and $\psi(x_1x_5) = t^3z_1z_2$.

We verify that this sequence is exact and, by definition, that $\mathcal{K}_{\mathbb{Z}/2\mathbb{Z}} = \{K_0, K_{\mathbb{Z}/2\mathbb{Z}}\}$ is a Gysin-$\mathbb{Z}/2\mathbb{Z}$-functor with $\mathcal{K}_{\mathbb{Z}/2\mathbb{Z}}(i) = \sigma$, $i : \{0\} \hookrightarrow \mathbb{Z}/2\mathbb{Z}$ is the inclusion.

Example 2.2.4.2. Let V be an elementary abelian 2-group and let X be a finite CW-complex on which the group V acts freely. For every subgroup W of V, we denote by $X_{hW} = EW \times_W X$ the Borel construction which is the quotient of $EW \times X$ by the diagonal action of W. Here EW is a contractible space on which W acts freely; $BW = EW/W$ is a classifying space of W. The mod.2 cohomology of the space X_{hW}, $H^*(X_{hW}) = H^*_W X$, is called the mod.2 equivariant cohomology of X. We denote by $\pi_W : X_{hW} \to BW$ the map induced by $X \to \{*\}$. It is clear that $H^*_W X$ is a graded H^*W-module (resp. H^*V-module) via $\pi^*_W : H^*W \to H^*_W X$ (resp. via $i^* : H^*V \to H^*W$, where $i : W \hookrightarrow V$ is the natural inclusion). We verify that $H^*_W X$ is an object of the category H^*W-$\mathbb{K}_f$ that is a graded, finite and unitary $\mathbb{F}_2$-algebra equipped with a map of graded unitary $\mathbb{F}_2$-algebras $H^*W \otimes H^*_W X \to H^*_W X$.

The contravariant functor $\mathcal{K}_V : \mathcal{W} \rightsquigarrow \mathbb{K}_f$, $W \mapsto \mathcal{K}_V(W) = H^*_W X$ is a Gysin-V-functor because:

(i) For every subgroup W of V, $H^*_W X$ is an object of the category H^*W-$\mathbb{K}_f$. The H^*W-module $H^*_W X$ is finite because when X is a W-CW-complex and the action of W is free, then the Borel construction X_{hW} is homotopy equivalent to the orbit space X/W wich implies $H^*_W X \cong H^*(X/W)$.

(ii) Let $W \subseteq V$ be a subgroup and let $U \subset W$ be a subgroup of codimension 1. The inclusion $i : U \hookrightarrow W$ induces the following two sheets covering: $W/U \cong \mathbb{Z}/2\mathbb{Z} \to X_{hU} \xrightarrow{i} X_{hW}$ with

$B(\pi) \circ \pi_W : X_{hW} \to BW \to B(W/U)$ as a classifying map, $\pi : W \to W/U$ is the natural projection.

Let $H^*(W/U) \cong \mathbb{F}_2[t]$, we also denote by t the non trivial element $(B(\pi) \circ \pi_W)^*(t)$ of $H^1_W X$. The Gysin exact sequence associated to the previous covering is the following exact sequence of H^*W-modules:

$$\ldots \longrightarrow H^{*-1}_W X \xrightarrow{t.} H^*_W X \xrightarrow{i^*} H^*_U X \xrightarrow{tr} H^*_W X \xrightarrow{t.} \ldots$$

where tr is the the transfer ([Sp], [Z]) and for $x \in H^*_W X$, $t.x = (B(\pi) \circ \pi_W)^*(t) \smile x$.

This shows that $\mathcal{K}_V = \{K_W = H^*_W X,\ W \text{ a subgroup of } V\}$ is a Gysin-V-functor. This example comes from "topology" via the equivariant cohomology of a free action of V on a finite CW-complex X.

2.3. *Some properties of Gysin-V-functors*

Let's recall some definitions and fix some notations. Let E be a finite graded $\mathbb{F}_2$-vector space.

• We denote by $\| E \|$ the *norm* of E which is the maximum of the set $\{k \in \mathbb{N},\ E^k \neq 0\}$.

• Let V be an elementary abelian 2-group. If E is an H^*V-module and $x \in E$, we denote by $\langle x \rangle_V$ the sub-H^*V-module of E generated by the element x.

Definitions 2.3.1. (i) A finite graded $\mathbb{F}_2$-vector space E is called:

(i-a) *connected* if $E^0 \cong \mathbb{Z}/2\mathbb{Z}$.

(i-b) *bi-connected* if: $\begin{cases} E \text{ is connected}: \ E^0 \cong \mathbb{Z}/2\mathbb{Z}, \\ \text{and} \\ E^{\|E\|} \cong \mathbb{Z}/2\mathbb{Z}. \end{cases}$

(ii) A Gysin-V-functor $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ is *connected* (resp. *bi-connected*) if K_0 is *connected* (resp. K_0 is *bi-connected*).

(iii) A finite CW-complex X is *bi-connected* if H^*X is *bi-connected*.

We have the following property of Gysin-V-functors.

Lemma 2.3.2. Let V be an elementary abelian 2-group and let $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ be a bi-connected Gysin-V-functor. Then, for each subgroup W of V, the graded finite $\mathbb{F}_2$-algebra K_W is bi-connected and we have $\| K_W \| = \| K_0 \|$.

Proof. The proof is by induction on the rank of the subgroup of V. Let $\mathcal{K}_V$ be a bi-connected Gysin-V-functor and let $U \subseteq V$ be a subgroup of rank one. The Gysin exact sequence of graded $\mathbb{F}_2$-vector spaces:

$$\overline{G}(0,U): \quad 0 \longrightarrow \overline{K_U}^U \xrightarrow{\mathcal{K}_V(i)} K_0 \xrightarrow{\psi} \tau^U(K_U) \longrightarrow 0$$

shows that:

2.3.2.1. In degree zero, we have: $(\mathcal{P}_0):\ \mathbb{Z}/2\mathbb{Z} \cong \left(K_0\right)^0 \cong \left(\overline{K_U}^U\right)^0 \oplus \left(\tau^U(K_U)\right)^0.$

Since $\big(\tau^U(K_U)\big)^0 \subseteq \big(K_U\big)^0 = \big(\overline{K_U}^U\big)^0$, then $\big(\overline{K_U}^U\big)^0 = 0$ implies $\big(\tau^U(K_U)\big)^0 = 0$. This contradicts the equality $(\mathcal{P}_0)$. Then we deduce that: $\big(\tau^U(K_U)\big)^0 = 0$ and $\mathbb{Z}/2\mathbb{Z} \cong \big(\overline{K_U}^U\big)^0 \cong \big(K_U\big)^0$. This shows that K_U is connected.

2.3.2.2. In degree $\parallel K_0 \parallel$, we have: $(\mathcal{P}_{\|K_0\|})$: $\mathbb{Z}/2\mathbb{Z} \cong \big(K_0\big)^{\|K_0\|} \cong \big(\overline{K_U}^U\big)^{\|K_0\|} \oplus \big(\tau^U(K_U)\big)^{\|K_0\|}$. Since K_U is a graded finite H^*U-module, we have: $\parallel K_U \parallel = \parallel \tau^U(K_U) \parallel$. The following inequalities follow: $\parallel \overline{K_U}^U \parallel \leq \parallel K_U \parallel = \parallel \tau^U(K_U) \parallel \leq \parallel K_0 \parallel$. We deduce from $(\mathcal{P}_{\|K_0\|})$ that: $\big(\overline{K_U}^U\big)^{\|K_0\|} = 0$ and $\big(\tau^U(K_U)\big)^{\|K_0\|} \cong \mathbb{Z}/2\mathbb{Z}$. This shows that:

$\parallel K_U \parallel = \parallel \tau^U(K_U) \parallel \geq \parallel K_0 \parallel$. So, we have the equality: $\parallel K_U \parallel = \parallel \tau^U(K_U) \parallel = \parallel K_0 \parallel$.

We proved that K_U is bi-connected and $\parallel K_U \parallel = \parallel K_0 \parallel$.

The lemma holds by induction on the rank of subgroups of V using the same method. ∎

Let E be a graded finite $\mathbb{F}_2$-vector space. We denote by $d(E) = \sum_{i\geq 0} dim_{\mathbb{F}_2} E^i$ the (total) dimension of E. We have:

Proposition 2.3.3. Let V be an elementary abelian 2-group and $\mathcal{K}_V = \{K_W,\ W$ subgroup of $V\}$ be a Gysin-V-functor. Then, the dimension of K_0 is even: $d(K_0) \equiv 0\ (mod.\ 2)$.

Proof. Let $U \subset V$ be a subgroup of rank one, then the Gysin exact sequence

$$\overline{G}(0,U):\quad 0 \longrightarrow \overline{K_U}^U \xrightarrow{\mathcal{K}_V(i)} K_0 \xrightarrow{\psi} \tau^U(K_U) \longrightarrow 0$$

Shows that $d(K_0) = d\big(\overline{K_U}^U\big) + d\big(\tau^U(K_U)\big)$. The propostion 2.3.3 is a consequence of the following lemma. ∎

Lemma 2.3.4. Let U be an elementary abelian 2-group of rank one and let M be a graded finite H^*U-module. We have: $d\big(\overline{M}^U\big) = d\big(\tau^U(M)\big)$.

Proof. The proof is a consequence of the following exact sequence:

$$0 \longrightarrow \tau^U(M) \longrightarrow M \xrightarrow{.t} M \longrightarrow \overline{M}^U \longrightarrow 0\,.$$

Exactness implies that its Euler characteristics is zero, giving $d\big(\overline{M}^U\big) = d\big(\tau^U(M)\big)$. ∎

Remark 2.3.5. Here is an example of application of the previous lemma. Let U_i, $i = 1, 2$, be an elementary abelian 2-group of rank one, let $V = U_1 \oplus U_2$ and let $H^*U_i \cong \mathbb{F}_2[t_i],\ i = 1, 2$. Let M be a graded finite H^*V-module, x_1 and x_2 two elements of M such that the finite $\mathbb{F}_2[t_1]$-modules

$\overline{M}^{U_2}$ and $\tau^{U_2}(M)$ are monogenic: $\begin{cases} \overline{M}^{U_2} \cong \big(t_1\big)_0^{k_1} x_1, \\ \tau^{U_2}(M) \cong \big(t_1\big)_0^{k_2} x_2. \end{cases}$ (see notations 2.1).

In this case we have $d\big(\overline{M}^{U_2}\big) = k_1 + 1$ and $d\big(\tau^{U_2}(M)\big) = k_2 + 1$. The lemma 2.3.4, applied for the H^*U_2-module M, implies the equality: $k_1 = k_2$.

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2.4. *On the extension of Gysin-V-functors*

Let V be an elementary abelian 2-group, V' a subgroup of V and $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ be a Gysin-V-functor. Then, $\mathcal{K}_{V'} = \{K_W,\ W \text{ subgroup of } V'\}$ is a Gysin-V'-functor called a "sub-Gysin-functor" of $\mathcal{K}_V$. We say also that $\mathcal{K}_V$ is an extension of the Gysin-V'-functor $\mathcal{K}_{V'}$.

It is interesting to know when a Gysin-V-functor extends because this question is related to the extension of a free action of the group V on a finite CW-complex. We have.

Proposition 2.4.1. Let V be an elementary abelian 2-group and let $\mathcal{K}_V$ be a Gysin-V-functor, $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$, such that $\overline{K_V}^V \cong \mathbb{Z}/2\mathbb{Z}$, then $\mathcal{K}_V$ doesn't extend.

Proof. Suppose that the Gysin-V-functor $\mathcal{K}_V$ extends to $\mathcal{K}_H$, $H = V \oplus \mathbb{Z}/2\mathbb{Z}$ then, the Gysin exact sequence of graded finite H^*V-modules:

$$\overline{G}(V,H):\quad 0 \longrightarrow \overline{K_H}^{H/V} \xrightarrow{\ \mathcal{K}_H(i)\ } K_V \xrightarrow{\ \psi\ } \tau^{H/V}(K_H) \longrightarrow 0\,,$$

where $i : V \hookrightarrow H$ is the inclusion, shows that:

$$\begin{aligned}\left(K_V\right)^0 &\cong \left(\overline{K_H}^{H/V}\right)^0 \oplus \left(\tau^{H/V}(K_H)\right)^0\\ &\cong \left(K_H\right)^0 \oplus \left(\tau^{H/V}(K_H)\right)^0\end{aligned}$$

Since $\left(\tau^{H/V}(K_H)\right)^0 \subseteq \left(K_H\right)^0$ and $\left(K_V\right)^0 \cong \mathbb{Z}/2\mathbb{Z}$ because $\overline{K_V}^V \cong \mathbb{Z}/2\mathbb{Z}$, we deduce that the morphism $\overline{\left(\overline{K_H}^{H/V}\right)}^V \xrightarrow{\ \mathcal{K}_H(i)\ } \overline{K_V}^V \cong \mathbb{Z}/2\mathbb{Z}$ is an epimorphism and $\overline{\tau^{H/V}(K_H)}^V = 0$. This leads to a contradiction because the $H^*(H)$-module K_H is non-trivial and finite. ■

Proposition 2.4.2. Let V be an elementary abelian 2-group, $\mathcal{K}_V$ a bi-connected Gysin-V-functor and ι_V the unit of the $\mathbb{F}_2$-algebra K_V: $(K_V)^0 \cong \mathbb{Z}/2\mathbb{Z} =< \iota_V >$.

If the norm of K_V is equal to the norm of its sub-H^*V-module generated by ι_V:

$\parallel K_V \parallel = \parallel \langle \iota_V \rangle_V \parallel$, then $\mathcal{K}_V$ doesn't extend.

Proof. Let $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ be a bi-connected Gysin-V-functor and suppose that $\mathcal{K}_V$ extends to $\mathcal{K}_H$, where $H = V \oplus \mathbb{Z}/2\mathbb{Z}$. By the lemma 2.3.2, the Gysin-H-functor $\mathcal{K}_H$ is bi-connected and we have: $\parallel K_V \parallel = \parallel K_H \parallel$ and $(K_H)^0 \cong \mathbb{Z}/2\mathbb{Z} =< \iota_H >$. Since the map $\mathcal{K}_H(i) : K_H \to K_V$, induced by the inclusion of V in H, is a map of unitary-(connected)-$\mathbb{F}_2$-algebras then, $\mathcal{K}_H(i)(\iota_H) = \iota_V$. The H^*H-morphism $\mathcal{K}_H(i) : \langle \iota_H \rangle_H \to \langle \iota_V \rangle_V$ is surjective because the morphism $H^*H \to H^*V$ is surjective for every $V \subseteq H$.

Let's denote by $j : \langle \iota_H \rangle_H \hookrightarrow K_H$ the natural inclusion. We have the following commutative diagram whose second line is the short Gysin exact sequence $\overline{G}(V,H)$ of H^*V-modules.

$$\begin{array}{ccccccccc}
 & & Im(\overline{j}^{H/V}) & \xrightarrow{\ \mathcal{K}_H(i)_{|}\ } & \langle \iota_V \rangle_V & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \overline{K_H}^{H/V} & \xrightarrow{\ \mathcal{K}_H(i)\ } & K_V & \xrightarrow{\ \psi\ } & \tau^{H/V}(K_H) & \longrightarrow & 0
\end{array}$$

This shows that the sub-H^*V-module $Im(\overline{j}^{H/V})$ of $\overline{K_H}^{H/V}$ is isomorphic to sub-H^*V-module $\langle\iota_V\rangle_V$ of K_V $\big(\mathcal{K}_H(i)_{|} : Im(\overline{j}^{H/V}) \to \langle\iota_V\rangle_V$ is an isomorphism$\big)$. This implies the inequality between norms: $\| \overline{K_H}^{H/V} \| \geq \| Im(\overline{j}^{H/V}) \| = \| \langle\iota_V\rangle_V \| = \| K_V \|$.

Since $\overline{K_H}^{H/V}$ is a sub-H^*V-module of K_V, we have that $\| \overline{K_H}^{H/V} \| \leq \| K_V \|$. So we have the equality: $\| \overline{K_H}^{H/V} \| = \| K_V \|$.

The Gysin exact sequence $\overline{G}(V,H)$ of H^*V-modules:

$$0 \longrightarrow \overline{K_H}^{H/V} \xrightarrow{\mathcal{K}_H(i)} K_V \xrightarrow{\psi} \tau^{H/V}(K_H) \longrightarrow 0$$

implies the following isomorphism: $\big(K_V\big)^{\|K_V\|} \cong \big(\overline{K_H}^{H/V}\big)^{\|K_V\|} \oplus \big(\tau^{H/V}(K_H)\big)^{\|K_V\|}$.

Since the Gysin-V-functor $\mathcal{K}_V$ is bi-connected, $\big(K_V\big)^{\|K_V\|} \cong \mathbb{Z}/2\mathbb{Z}$, and $\| \overline{K_H}^{H/V} \| = \| K_V \|$, we deduce from the previous isomorphism:

$$\begin{cases} (i)\ \big(\overline{K_H}^{H/V}\big)^{\|K_V\|} \cong \big(K_V\big)^{\|K_V\|} \cong \mathbb{Z}/2\mathbb{Z}, \\ (ii)\ \big(\tau^{H/V}(K_H)\big)^{\|K_V\|} = 0. \end{cases}$$

By lemma 2.3.2, $\| K_V \| = \| K_H \|$; since K_H is a graded, finite and non-trivial $H^*(H/V)$-module, then $\big(\tau^{H/V}(K_H)\big)^{\|K_V\|} \neq 0$. This contradicts the point (ii). ■

3. The main result

Let V be an elementary abelian 2-group of rank d and $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ be a Gysin-V-functor. Let's denote $d(K_0) = \Sigma_{i\geq 0} dim_{\mathbb{F}_2}\big(K_0\big)^i$ the total dimension of the graded finite $\mathbb{F}_2$-vector space K_0.

The main result of this paper is to show, in certain cases, that $d(K_0)$ is related to the rank of the group V, as suggested by the conjecture (C_d), $d(K_0) \geq 2^{rk(V)}$.

More precisely, we have:

Theorem 3.1. *Let V be an elementary abelian 2-group and $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ be a Gysin-V-functor. Then,*

(i) For $rk(V) = 1$, $d(K_0) \geq 2$ so the conjecture (C_1) holds.

(ii) For $rk(V) = 2$, if the Gysin-V-functor $\mathcal{K}_V$ is connected, we have the inequality: $d(K_0) \geq 4$, so the conjecture (C_2) holds.

(ii) For $rk(V) = 3$, if the Gysin-V-functor $\mathcal{K}_V$ is bi-connected, we have the inequality: $d(K_0) \geq 8$, so the conjecture (C_3) holds.

As an application of this theorem we get an independent proof of the results concerning $(C_{d,X})$ for $d \leq 3$.

Proposition 3.2. *Let V be an elementary abelian 2-group and let X be a finite CW-complex on which the group V acts freely. Then,* *(i) For $rk(V) = 1$, we have: $d(H^*X) \geq 2$.*

(ii) For $rk(V) = 2$ and X connected, we have: $d(H^*X) \geq 4$.

(ii) For $rk(V) = 3$ and X bi-connected, we have: $d(H^*X) \geq 8$.

Proof. Let V be an elementary abelian 2-group and let X be a finite CW-complex on which the group V acts freely. By the example 2.2.4.2, the contravariant functor $\mathcal{K}_V : \mathcal{W} \rightsquigarrow \mathbb{K}_f,\ W \mapsto \mathcal{K}_V(W) = H^*_W X$ is a Gysin-V-functor whose 0^{th}-term $K_0 = \mathcal{K}_V(0) = H^*X$. ∎

Let S^n, $n \geq 1$, be the standard unit sphere in $\mathbb{R}^{n+1}$, then the product $S^{n_1} \times ... \times S^{n_k}$, $k \geq 1$, is a bi-connected CW-complex. By the proposition 3.2, if an elementary abelian 2-group V, $1 \leq rk(V) \leq 3$, acts freely on a product of k spheres then, $k \geq rk(V)$.

3.1. *proof of theorem 3.1*

To prove theorem 3.1 we consider the following three cases:

3.1.1. *The case $rk(V) = 1$.*

The proposition 2.3.3 shows that if $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ is a Gysin-V-functor, $rk(V) = 1$, then $d(K_0) \equiv 0\ (mod.\ 2)$. This implies that $d(K_0) \geq 2$ because the graded $\mathbb{F}_2$-vector space K_0 is not trivial.

3.1.2. *The case $rk(V) = 2$.*

Let $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ be a Gysin-V-functor, $rk(V) = 2$, and suppose that $d(K_0) < 4$. Since $d(K_0) \equiv 0\ (mod.\ 2)$ (see proposition 2.3.3) and K_0 non-trivial, we deduce that $d(K_0) = 2$.

Let $U \subseteq V$ be a subgroup of rank one and consider the short exact sequence of graded $\mathbb{F}_2$-vector spaces associated to the couple $(\{0\} \subset U)$ of subgroups of V

$$\overline{G}(\{0\}, U) : \quad 0 \longrightarrow \overline{K_U}^U \xrightarrow{\mathcal{K}_V(i)} K_0 \xrightarrow{\psi} \tau^U(K_U) \longrightarrow 0\ ,$$

$i : \{0\} \hookrightarrow U$ denotes the inclusion. This shows that: $d(K_0) = 2 = d(\overline{K_U}^U) + d(\tau^U(K_U))$. The lemma 2.3.4 implies that: $d(\overline{K_U}^U) = 1$. Since the Gysin-V-functor $\mathcal{K}_V$ is **connected**, we have: $\mathbb{Z}/2\mathbb{Z} \cong (K_U)^0 \cong (\overline{K_U}^U)^0 \cong \overline{K_U}^U$.

The proposition 2.4.1 shows that, in this case, the Gysin-U-functor $\mathcal{K}_U = \{K_W,\ W \text{ subgroup of } U\}$ can not extend to $\mathcal{K}_V$. This leads to a contradiction.

3.1.3. *The case $rk(V) = 3$.*

Let $\mathcal{K}_V = \{K_W,\ W \text{ subgroup of } V\}$ be a bi-connected Gysin-V-functor, $rk(V) = 3$, and suppose that $d(K_0) < 8$. Since $d(K_0) \equiv 0\ (mod.\ 2)$ (see proposition 2.3.3) and K_0 non-trivial, then we have three possibility: $d(K_0) = 2$, $d(K_0) = 4$ and $d(K_0) = 6$. We will show that the three cases $d(K_0) = 2$, $d(K_0) = 4$ and $d(K_0) = 6$ are impossible. Let $U_i, 1 \leq i \leq 3$, be a rank one subgroup of V such that: $V \cong U_1 \oplus U_2 \oplus U_3$.

3.1.3.1 The case $d(K_0) = 2$ is impossible by the previous case 3.1.2. We proved in 3.1.2 that if $d(K_0) = 2$, then K_0 can't be the 0^{th}-term of a Gysin-E-functor with E an elementary abelian 2-group of rank 2 and a fortiori of rank ≥ 2.

3.1.3.2 Suppose that $d(K_0) = 4$. The Gysin exact sequence of graded finite $\mathbb{F}_2$-vector spaces

$$\overline{G}(\{0\}, U_1): \quad 0 \longrightarrow \overline{K_{U_1}}^{U_1} \xrightarrow{\mathcal{K}_V(i_1)} K_0 \xrightarrow{\psi} \tau^{U_1}(K_{U_1}) \longrightarrow 0\,,$$

($i_1 : \{0\} \hookrightarrow U_1$ is the inclusion), shows that $d(K_0) = 4 = d\big(\overline{K_{U_1}}^{U_1}\big) + d\big(\tau^{U_1}(K_{U_1})\big)$. The lemma 2.3.4 implies that $d(\overline{K_{U_1}}^{U_1}) = 2$, that is: $\overline{K_{U_1}}^{U_1} \cong \langle \overline{\iota_1}, \overline{g_1} \rangle$ is the $\mathbb{F}_2$-vector space generated by two generators $\overline{\iota_1}$ and $\overline{g_1}$ where $\iota_1 \in \big(K_{U_1}\big)^0 \cong \mathbb{Z}/2\mathbb{Z}$ is the unit and $g_1 \in \big(K_{U_1}\big)^{k_1}$, $k_1 \geq 1$.

Since $\mathcal{K}_{U_1}$ is a sub-Gysin-functor of $\mathcal{K}_{U_1 \oplus U_2}$ whose 0^{th}-term K_0 is bi-connected, then by 2.4.2, the norm of K_{U_1} is bigger than the norm of the sub-H^*U_1-module generated by ι_1. We have:

$$\begin{aligned} \parallel K_{U_1} \parallel &= \parallel \langle g_1 \rangle_{U_1} \parallel \\ &> \parallel \langle \iota_1 \rangle_{U_1} \parallel \end{aligned}$$

This shows, in particular, that $\langle \iota_1 \rangle_{U_1} \cap \langle g_1 \rangle_{U_1} = 0$. We have then an isomorphism of H^*U_1-modules: $K_{U_1} \cong \langle \iota_1 \rangle_{U_1} \oplus \langle g_1 \rangle_{U_1}$.

Let $\iota_{1,2}$ be the unit of the graded $\mathbb{F}_2$-algebra $K_{U_1 \oplus U_2}$. Consider the Gysin exact sequence $\overline{G}(U_1, U_1 \oplus U_2)$ of H^*U_1-modules

$$0 \longrightarrow \overline{K_{U_1 \oplus U_2}}^{U_2} \xrightarrow{\mathcal{K}_{U_1 \oplus U_2}(i_1)} K_{U_1} \cong \langle \iota_1 \rangle_{U_1} \oplus \langle g_1 \rangle_{U_1} \xrightarrow{\psi} \tau^{U_2}(K_{U_1 \oplus U_2}) \longrightarrow 0\,,$$

$i_1 : U_1 \hookrightarrow U_1 \oplus U_2$ denotes the natural inclusion.

We have $\psi(\langle \iota_1 \rangle_{U_1}) = 0$ since $\mathcal{K}_{U_1 \oplus U_2}(i_1)(\iota_{1,2}) = \iota_1$. We have also the following equality:

$$\tau^{U_2}(K_{U_1 \oplus U_2}) = \langle \psi(g_1) \rangle_{U_1},$$

which is a consequence of the following points:

(i)
$$\begin{aligned} \parallel K_{U_1} \parallel &= \parallel K_{U_1 \oplus U_2} \parallel, \text{ by lemma 2.3.2} \\ &= \parallel \tau^{U_2}(K_{U_1 \oplus U_2}) \parallel, \text{ since } K_{U_1 \oplus U_2} \text{ is finite} \end{aligned}$$

(ii) $\big(\overline{K_{U_1 \oplus U_2}}^{U_2}\big)^{\parallel K_{U_1} \parallel} = 0$ since the Gysin-V-functor, $\mathcal{K}_V$, is bi-connected so:

$$\big(K_{U_1}\big)^{\parallel K_{U_1} \parallel} \cong \mathbb{Z}/2\mathbb{Z} \cong \big(\tau^{U_2}(K_{U_1 \oplus U_2})\big)^{\parallel K_{U_1} \parallel}.$$

(iii) The point (ii) and the H^*U_1-linearity of ψ implies the equality of norms:

$$\parallel \langle g_1 \rangle_{U_1} \parallel = \parallel \langle \psi(g_1) \rangle_{U_1} \parallel .$$

Note that $\psi(g_1) \neq 0$ because if not, since ψ is H^*U_1-linear, then $\psi\big(K_{U_1} \cong \langle \iota_1 \rangle_{U_1} \oplus \langle g_1 \rangle_{U_1}\big) = \tau^{U_2}(K_{U_1 \oplus U_2}) = 0$ which contradicts the fact that $K_{U_1 \oplus U_2}$ is finite and non-trivial.

In conclusion, we have: $\begin{cases} (i)\ \overline{K_{U_1 \oplus U_2}}^{U_2} \cong \langle \iota_1 \rangle_{U_1}, \\ (ii)\ \tau^{U_2}(K_{U_1 \oplus U_2}) \cong \langle \psi(g_1) \rangle_{U_1}. \end{cases}$

The point (i) implies that $\overline{K_{U_1\oplus U_2}}^{U_1\oplus U_2} \cong \mathbb{Z}/2\mathbb{Z}$ and the proposition 2.4.1 shows the contradiction since the Gysin-V-functor $\mathcal{K}_V$ extends $\mathcal{K}_{U_1\oplus U_2}$ ($V \cong U_1 \oplus U_2 \oplus U_3$).

3.1.3.3 Suppose that $d(K_0) = 6$. To show a contradiction, in this case, we will analyse the graded, finite and unitary H^*W-$\mathbb{F}_2$-algebras K_W for $W = U_1$ and $W = U_1 \oplus U_2$.

I1. Informations on K_{U_1}.

By the same previous method, using the Gysin exact sequence

$$\overline{G}(\{0\}, U_1): \quad 0 \longrightarrow \overline{K_{U_1}}^{U_1} \xrightarrow{\mathcal{K}_{U_1}(i_1)} K_0 \xrightarrow{\psi} \tau^{U_1}(K_{U_1}) \longrightarrow 0\,,$$

we show that $d(\overline{K_{U_1}}^{U_1}) = 3$ that is: $\overline{K_{U_1}}^{U_1} \cong \langle \overline{\iota_1}, \overline{g_1}, \overline{g_2}\rangle$ is the $\mathbb{F}_2$-vector space generated by three generators $\overline{\iota_1}$, $\overline{g_1}$ and $\overline{g_2}$: $\iota_1 \in \big(K_{U_1}\big)^0 \cong \mathbb{Z}/2\mathbb{Z}$ is the unit, $g_1 \in \big(K_{U_1}\big)^{k_1}$, $k_1 \geq 1$ and $g_2 \in \big(K_{U_1}\big)^{k_2}$, $k_2 \geq 1$.

Since the bi-connected Gysin-U_1-functor $\mathcal{K}_{U_1}$ extends, then by proposition 2.4.2, the norm of the graded finite $\mathbb{F}_2$-vector space K_{U_1} is reached as the norm of a sub-$\mathbb{F}_2$-vector space generated by a generator different of ι_1, for example g_1. We have: $\parallel \langle \iota_1\rangle_{U_1} \parallel < \parallel K_{U_1} \parallel = \parallel \langle g_1\rangle_{U_1} \parallel$.

We verify that $\langle \iota_1\rangle_{U_1} \cap \langle g_1\rangle_{U_1} = 0$ and we have a short exact sequence of H^*U_1-modules of the form: $\big(E(U_1)\big): \quad 0 \longrightarrow \langle \iota_1\rangle_{U_1} \oplus \langle g_1\rangle_{U_1} \longrightarrow K_{U_1} \longrightarrow C_{U_1} \longrightarrow 0$ where C_{U_1} is a graded finite monogenic H^*U_1-module generated by the element g_2.

In refers to 2.1, let $H^*U_i \cong \mathbb{F}_2[t_i]$, $i = 1, 2$, the polynomial algebra over $\mathbb{F}_2$ on one generator t_i of degree one, $< t^s >$, $s \in \mathbb{N}$, be the ideal of $\mathbb{F}_2[t]$ of elements of degree $\geq s$ and $\big(t\big)_0^k = \mathbb{F}_2[t]/ < t^{k+1} >$.

With these notations we have:

I1.1 $\langle \iota_1\rangle_{U_1} \cong \big(t_1\big)_0^{n_1}\iota_1$, $n_1 \geq 1$.

I1.2 $C_{U_1} \cong \big(t_1\big)_0^{l_1} g_2$ with $\boldsymbol{l_1 \leq n_1}$ because, in the graded finite unitary $\mathbb{F}_2$-algebra K_{U_1}, we have: $g_2 = \iota_1.g_2$. This implies that: $t_1^s g_2 = (t_1^s\iota_1).g_2$, $s \in \mathbb{N}$.

I1.3 Remark. In $I1.1$ the integer n_1 is ≥ 1 because if not $n_1 = 0$ which means that: $t_1.\iota_1 = 0$. This implies that $\iota_1 \in \big(\tau^{U_1}(K_{U_1})\big)^0$. Since $\big(K_0\big)^0 \cong \big(\overline{K_U}^U\big)^0 \oplus \big(\tau^U(K_U)\big)^0$, $\big(\overline{K_U}^U\big)^0 \neq 0$, $\big(\overline{K_U}^U\big)^0 \subseteq \big(K_0\big)^0$ and K_0 connected: $\big(K_0\big)^0 \cong \mathbb{Z}/2\mathbb{Z}$, we have: $\big(\tau^U(K_U)\big)^0 = 0$. So the contradiction.

I2. Informations on $K_{U_1\oplus U_2}$.

Let $\iota_{1,2}$ be the unit of the graded $\mathbb{F}_2$-algebra $K_{U_1\oplus U_2}$. Since the bi-connected Gysin-$U_1 \oplus U_2$-functor $\mathcal{K}_{U_1\oplus U_2}$ extends, then by proposition 2.4.2, the norm of the graded finite $\mathbb{F}_2$-vector space $K_{U_1\oplus U_2}$ is reached as the norm of a sub-$\mathbb{F}_2$-vector space generated by a generator ξ different of $\iota_{1,2}$. We have: $\parallel \langle \iota_{1,2}\rangle_{U_1\oplus U_2} \parallel < \parallel K_{U_1\oplus U_2} \parallel = \parallel \langle \xi\rangle_{U_1\oplus U_2} \parallel$.

Consider the Gysin exact sequence:

$$\overline{G}(U_1, U_1 \oplus U_2): \quad 0 \longrightarrow \overline{K_{U_1\oplus U_2}}^{U_2} \xrightarrow{\mathcal{K}_{U_1\oplus U_2}(i_1)} K_{U_1} \xrightarrow{\psi} \tau^{U_2}(K_{U_1\oplus U_2}) \longrightarrow 0\,,$$

where $i_1 : U_1 \hookrightarrow U_1 \oplus U_2$ is the inclusion. We have: $\mathcal{K}_{U_1\oplus U_2}(i_1)(\iota_{1,2}) = \iota_1$ and $\mathcal{K}_{U_1\oplus U_2}(i_1)(\xi) \neq 0$ in K_{U_1}.

The Gysin exact sequence, $\overline{G}(U_1, U_1 \oplus U_2)$, of H^*U_1-modules, shows that:

I2.1 The element $\psi(g_1)$ is a generator of the H^*U_1-module $\tau^{U_2}(K_{U_1\oplus U_2})$ and $\psi_| : \langle g_1\rangle_{U_1} \to \tau^{U_2}(K_{U_1\oplus U_2})$ is injective. As explained at the end of 3.1.3.2, this is because: $\| K_{U_1} \| = \| K_{U_1\oplus U_2} \| = \| \tau^{U_2}(K_{U_1\oplus U_2}) \|$, $\big(\overline{K_{U_1\oplus U_2}}^{U_2}\big)^{\|K_{U_1}\|} = 0$ and $\| \langle g_1\rangle_{U_1} \| = \| \langle\psi(g_1)\rangle_{U_1} \| = \| K_{U_1} \| = \| \tau^{U_2}(K_{U_1\oplus U_2}) \|$.

Note that if $\psi(g_1) = 0$, then there exist a generator x of $\overline{K_{U_1\oplus U_2}}^{U_2}$ such that: $\mathcal{K}_{U_1\oplus U_2}(i_1)(x) = g_1$. Since $\mathcal{K}_{U_1\oplus U_2}(i_1)$ is injective and H^*U_1-linear, then: $\| \langle x\rangle_{U_1} \| = \| \langle g_1\rangle_{U_1} \|$. This shows that:

$$\| \overline{K_{U_1\oplus U_2}}^{U_2} \| = \| \langle x\rangle_{U_1} \| = \| \langle g_1\rangle_{U_1} \| = \| K_{U_1} \| = \| K_{U_1\oplus U_2} \| = \| \tau^{U_2}(K_{U_1\oplus U_2}) \| .$$

Since the Gysin-$U_1 \oplus U_2$-functor $\mathcal{K}_{U_1\oplus U_2}$ is bi-connected, then:

$$\big(K_{U_1}\big)^{\|K_{U_1}\|} = \mathbb{Z}/2\mathbb{Z} = \big(\tau^{U_2}(K_{U_1\oplus U_2})\big)^{\|K_{U_1}\|},$$

which shows that: $\big(\overline{K_{U_1\oplus U_2}}^{U_2}\big)^{\|K_{U_1}\|} = 0$. This contradicts the equality: $\| \overline{K_{U_1\oplus U_2}}^{U_2} \| = \| K_{U_1} \|$.

I2.2 Note that $\psi(g_2) \neq 0$ if not $\tau^{U_2}(K_{U_1\oplus U_2}) = \langle\psi(g_1)\rangle_{U_1}$. Since $\langle\iota_{1,2}\rangle_{U_1\oplus U_2} \hookrightarrow K_{U_1\oplus U_2}$, then $\tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}) \hookrightarrow \tau^{U_2}(K_{U_1\oplus U_2}) = \langle\psi(g_1)\rangle_{U_1}$. This show that:

$$\| \langle\iota_{1,2}\rangle_{U_1\oplus U_2} \| = \| \tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}) \| = \| \tau^{U_2}(K_{U_1\oplus U_2}) \| = \| K_{U_1\oplus U_2} \| .$$

Since the bi-connected Gysin-$U_1 \oplus U_2$-functor $\mathcal{K}_{U_1\oplus U_2}$ extends, then by proposition 2.4.2, we have a contradiction.

Let $\langle g_2\rangle_{U_1}$ be the sub-H^*U_1-module of K_{U_1} generated by g_2. We have:

$$\| \langle g_2\rangle_{U_1} \| < \| K_{U_1} \| = \| \langle g_1\rangle_{U_1} \| .$$

If not, $\| \langle g_2\rangle_{U_1} \| = \| K_{U_1} \| = \| \tau^{U_2}(K_{U_1\oplus U_2}) \|$ which implies that $\psi_| : \langle g_2\rangle_{U_1} \to \tau^{U_2}(K_{U_1\oplus U_2})$ is injective (see I2.1). In this case $\mathcal{K}_{U_1\oplus U_2}(i_1)(\xi) = 0$, so the contradiction.

I2.3 Let $\langle\xi\rangle_{U_1}$ be the sub-H^*U_1-module of $\overline{K_{U_1\oplus U_2}}^{U_2}$ generated by the element ξ. Since $\mathcal{K}_{U_1\oplus U_2}(i_1)$ is H^*U_1-linear, injective and $\mathcal{K}_{U_1\oplus U_2}(i_1)(\xi) \neq 0$, we have: $\mathcal{K}_{U_1\oplus U_2}(i_1)(\xi)$ is a non-trivial element of $\langle g_2\rangle_{U_1}$, because:

a) $\mathcal{K}_{U_1\oplus U_2}(i_1)(\xi) \notin \langle\iota_1\rangle_{U_1}$ since $\mathcal{K}_{U_1\oplus U_2}(i_1)(\iota_{1,2}) = \iota_1$ and $\mathcal{K}_{U_1\oplus U_2}(i_1)$ is injective and H^*U_1-linear.

b) $\mathcal{K}_{U_1\oplus U_2}(i_1)(\xi) \notin \langle g_1\rangle_{U_1}$ since, if not, we will have the equality:

$$\| \langle\xi\rangle_{U_1} \| = \| \langle g_1\rangle_{U_1} \| = \| K_{U_1} \| .$$

This contradicts the fact that $\big(\overline{K_{U_1\oplus U_2}}^{U_2}\big)^{\|K_{U_1}\|} = 0$, (see I2.1), since the Gysin-$U_1 \oplus U_2$-functor $\mathcal{K}_{U_1\oplus U_2}$ is bi-connected.

We have then the following exact sequence of H^*U_1-modules:

$$0 \longrightarrow \langle\xi\rangle_{U_1} \xrightarrow{\mathcal{K}_{U_1\oplus U_2}(i_1)} \langle g_2\rangle_{U_1} \xrightarrow{\quad\psi\quad} \langle\psi(g_2)\rangle_{U_1} \longrightarrow 0 ,$$

which shows: $\| \langle\xi\rangle_{U_1} \| = \| \langle g_2\rangle_{U_1} \|$ and $\| \langle\psi(g_2)\rangle_{U_1} \| < \| \langle g_2\rangle_{U_1} \|$. This implies:

$$\| \langle\psi(g_2)\rangle_{U_1} \| < \| \langle g_2\rangle_{U_1} \| < \| \langle g_1\rangle_{U_1} \| = \| \langle\psi(g_1)\rangle_{U_1} \| .$$

As a consequence we have: $\langle\psi(g_1)\rangle_{U_1} \cap \langle\psi(g_2)\rangle_{U_1} = 0$, in $\tau^{U_2}(K_{U_1\oplus U_2})$, and the following H^*U_1-isomorphism: $\tau^{U_2}(K_{U_1\oplus U_2}) \cong \langle\psi(g_1)\rangle_{U_1} \oplus \langle\psi(g_2)\rangle_{U_1}$. This proves, in particular, that: $\tau^{U_2}\big(\langle\xi\rangle_{U_1\oplus U_2}\big) \subseteq \langle\psi(g_1)\rangle_{U_1}$ (same norm). It results that the H^*U_1-module $\tau^{U_2}\big(\langle\xi\rangle_{U_1\oplus U_2}\big)$ is monogenic isomorphic to $(t_1)_0^s.y$ for some generator y.

As a consequence of the previous results, we have:

Proposition 3.3. *There is an isomorphism of $H^*(U_1 \oplus U_2)$-modules:*

$$K_{U_1\oplus U_2} \cong \langle\iota_{1,2}\rangle_{U_1\oplus U_2} \oplus \langle\xi\rangle_{U_1\oplus U_2}.$$

Proof. We will prove that: $\langle\iota_{1,2}\rangle_{U_1\oplus U_2} \cap \langle\xi\rangle_{U_1\oplus U_2} = 0$. Since $\langle\iota_{1,2}\rangle_{U_1\oplus U_2} \cap \langle\xi\rangle_{U_1\oplus U_2}$ is a finite $H^*(U_1 \oplus U_2)$-module, it suffices to show that: $\tau^{U_2}\big(\langle\iota_{1,2}\rangle_{U_1\oplus U_2} \cap \langle\xi\rangle_{U_1\oplus U_2}\big) = 0$. ■

Lemma 3.4. *With the previous notations, we have:* $\tau^{U_2}\big(\langle\iota_{1,2}\rangle_{U_1\oplus U_2} \cap \langle\xi\rangle_{U_1\oplus U_2}\big) = 0.$

Proof. We have: $\tau^{U_2}\big(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}\cap\langle\xi\rangle_{U_1\oplus U_2}\big)=\tau^{U_2}\big(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}\big)\cap\tau^{U_2}\big(\langle\xi\rangle_{U_1\oplus U_2}\big)$. Since $\tau^{U_2}\big(\langle\xi\rangle_{U_1\oplus U_2}\big) \cong (t_1)_0^s.y$. This means that:

$$s+ \mid y \mid=\parallel \tau^{U_2}\big(\langle\xi\rangle_{U_1\oplus U_2}\big) \parallel=\parallel \tau^{U_2}(K_{U_1\oplus U_2}) \parallel=\parallel K_{U_1\oplus U_2}) \parallel>\parallel \langle\iota_{1,2}\rangle_{U_1\oplus U_2} \parallel=\parallel \tau^{U_2}\big(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}\big) \parallel,$$

$\mid y \mid$ is the degree of y.

Let $z \in \tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}) \cap (t_1)_0^s.y$, then $z = t_1^p.y,\ p \leq s$. This implies that there exist $k \in \mathbb{N}$ such that: $t_1^k z = t_1^s.y \neq 0$ "(element of maximal degree)". Since $t_1^k z \in \langle\iota_{1,2}\rangle_{U_1\oplus U_2}$ and $\parallel \langle\iota_{1,2}\rangle_{U_1\oplus U_2} \parallel< s+ \mid y \mid$, then: $t_1^k z = 0$, so the contradiction. ■

Since $K_{U_1\oplus U_2} \cong \langle\iota_{1,2}\rangle_{U_1\oplus U_2} \oplus \langle\xi\rangle_{U_1\oplus U_2}$, then: $\tau^{U_2}(K_{U_1\oplus U_2}) \cong \tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}) \oplus \tau^{U_2}\big(\langle\xi\rangle_{U_1\oplus U_2}\big)$. This proves, in particular that: $\tau^{U_2}\big(\langle\xi\rangle_{U_1\oplus U_2}\big) \cong \langle\psi(g_1)\rangle_{U_1}$.

We have the following commutative diagram, **(D)**, of H^*U_1-modules, whose second horizontal line is the Gysin exact sequence $\overline{G}(U_1, U_1 \oplus U_2)$ and its second vertical line is the exact sequence $E(U_1)$ (see I.1 of 3.1.3.3),

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\langle\iota_{1,2}\rangle_{U_1\oplus U_2}}^{U_2} & \longrightarrow & \langle\iota_1\rangle_{U_1} \oplus \langle g_1\rangle_{U_1} & \xrightarrow{\psi_|} & \langle\psi(g_1)\rangle_{U_1} \cong \tau^{U_2}\big(\langle\xi\rangle_{U_1\oplus U_2}\big) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{K_{U_1\oplus U_2}}^{U_2} & \xrightarrow{\mathcal{K}_{U_1\oplus U_2}(i_1)} & K_{U_1} & \xrightarrow{\psi} & \tau^{U_2}\big(K_{U_1\oplus U_2}\big) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\langle\xi\rangle_{U_1\oplus U_2}}^{U_2} & \xrightarrow{\overline{\mathcal{K}}_{U_1\oplus U_2}(i_1)} & C_{U_1} & \xrightarrow{\overline{\psi}} & \tau^{U_2}\big(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}\big) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $i_1 : U_1 \hookrightarrow U_1 \oplus U_2$ is the natural inclusion and where $\overline{\mathcal{K}}_{U_1\oplus U_2}(i_1)$ (resp. $\overline{\psi}$) is induced by $\mathcal{K}_{U_1\oplus U_2}(i_1)$ (resp. ψ).

By analysing the previous diagram **(D)**, we verify that:

I2.2.1 $\overline{\langle\iota_{1,2}\rangle_{U_1\oplus U_2}}^{U_2} \cong \langle\iota_1\rangle_{U_1} \cong (t_1)_0^{n_1}\iota_1$, $n_1 \geq 1$, (see $I1.1$).

I2.2.2 Since $C_{U_1} \cong (t_1)_0^{l_1} g_2$, $l_1 \leq n_1$, (see I1.2), then its quotient $\tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2})$ is isomorphic to $(t_1)_0^{m_1}\psi(g_2)$, $m_1 \leq l_1$.

I3. The contradiction.

The last line of the previous diagram **(D)**, which is an exact sequence of graded finite monogenic H^*U_1-modules, can now be written, using $I2.2.2$, as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\langle\xi\rangle_{U_1\oplus U_2}}^{U_2} & \xrightarrow{\overline{\mathcal{K}}_{U_1\oplus U_2}(i_1)} & C_{U_1} & \xrightarrow{\overline{\psi}} & \tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}) & \longrightarrow & 0\,. \\ & & & & \downarrow\cong & & \downarrow\cong & & \\ & & & & (t_1)_0^{l_1} g_2 & & (t_1)_0^{m_1}\psi(g_2) & & \end{array}$$

Since $\overline{\mathcal{K}}_{U_1\oplus U_2}(i_1)$ is injective, H^*U_1-linear and $\overline{\mathcal{K}}_{U_1\oplus U_2}(i_1)(\xi) \neq 0$, then we have:
$\| \overline{\langle\xi\rangle_{U_1\oplus U_2}}^{U_2} \| = \| C_{U_1} \| > \| \tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}) \|$. This is equivalent to:
$\| (t_1)_0^{l_1} g_2 \| = l_1 + k_2 > \| (t_1)_0^{m_1}\psi(g_2) \| = m_1 + k_2$, k_2 is the degree of g_2. We have then,
$\boldsymbol{l_1 > m_1}$ **.**

In conclusion, we have: $\begin{cases} \overline{\langle\iota_{1,2}\rangle_{U_1\oplus U_2}}^{U_2} \cong (t_1)_0^{n_1}\iota_1, \text{ see } I2.1, \\ \tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2}) \cong (t_1)_0^{m_1}\psi(g_2), \\ m_1 < n_1, \text{ because } m_1 < l_1 \leq n_1. \end{cases}$

The lemma 2.3.4 (see also the remark 2.3.5) shows the equality of dimensions:

$$d\big(\overline{\langle\iota_{1,2}\rangle_{U_1\oplus U_2}}^{U_2}\big) = n_1 + 1 = d\big(\tau^{U_2}(\langle\iota_{1,2}\rangle_{U_1\oplus U_2})\big) = m_1 + 1,$$

so the contradiction.

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