Advances in Pure and Applied Mathematics
2023, vol. 14, n° special, 31-47 pages, DOI: 10.21494/ISTE.OP.2023.0936 ISTE OpenScience

Topological invariants for the scalar curvature problem on manifolds

Invariants topologiques pour le problème de la courbure scalaire sur les variétés

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ABSTRACT. In [7], A.Bahri introduced two topological invariants μ and τ to study the prescribed scalar curvature problem on standard spheres of high dimensions. In this paper we first extend μ and τ to the problem on general riemannian manifolds. Second we analyze, as suggested in [7], the relation between these two quantities and we prove under topological conditions that $\mu = \tau$.
MSC: 58E05, 35J60.
KEYWORDS. Scalar curvature, Yamabe flow, Bahri's topological invariants.

1. Scalar curvature problem

Let $(M^n, g_0),\ n \geq 3$, be a closed riemannian manifold endowed with its initial metric g_0. The prescribed scalar curvature problem on (M^n, g_0) is the problem of finding suitable conditions on a given function K on M^n, so that it can be realized as the scalar curvature of a new metric g conformally equivalent to g_0. This is equivalent to solving the nonlinear elliptic equation

$$\begin{cases} -4\dfrac{n-1}{n-2}\Delta_{g_0} u + R_{g_0} u = K u^{\frac{n+2}{n-2}} \\ \qquad u > 0 \quad \text{on } M^n, \end{cases} \tag{1.1}$$

where Δ_{g_0} is the Laplace-Beltrami operator and R_{g_0} is the scalar curvature of the metric g_0.

Problem (1.1) has an underlying variational structure. The associated variational function being

$$J(u) \;=\; \frac{\int_{M^n} -L_{g_0} u u \mathrm{d}v_{g_0}}{\left(\int_{M^n} K u^{\frac{2n}{n-2}} \mathrm{d}v_{g_0}\right)^{\frac{n-2}{n}}},\ u \in H^1(M^n)\backslash\{0\}.$$

Here $-L_{g_0} = -4\frac{n-1}{n-2}\Delta_{g_0} u + R_{g_0} u$ is the conformal Laplacian of (M^n, g_0). Let

$$\Sigma = \left\{ u \in H^1(M^n),\ \|u\|^2 = \int_{M^n} -L_{g_0} u u \mathrm{d}v_{g_0} = 1 \right\} \text{ and } \Sigma^+ = \left\{ u \in \Sigma,\ u \geq 0 \right\}.$$

A direct computation shows that solutions of problem (1.1) correspond to the critical points of $J(u)$ subjected to the constraint $u \in \Sigma^+$. However, J fails to satisfy the Palais-Smale condition, since the exponent $\frac{2n}{n-2}$ is critical and embedding $H^1(M^n) \hookrightarrow L^{\frac{2n}{n-2}}(M^n)$ is not compact. This leads to the failure of the classical existence mechanisms.

By a direct integration, it is easy to see that a necessary condition to solve (1.1) is that $\max_{M^n} K > 0$. Moreover, in the case of $M^n = S^n$, there are well known obstructions found by Kazdan-Warner [25] and later by Bourguignon-Ezin [12]. Hence it is not expectable to solve (1.1) for all functions K. Particulary, a strictly monotone function on S^n which is rotationally symmetric cannot be the scalar curvature of a metric conformally equivalent to the standard one.

The prescribed scalar curvature problem has always been one of major subject in differential geometry and non linear analysis. Numerous studies were dedicated to this topic and various sufficient conditions were found with various methods. Among them let us cite Bahri [7] and Bahri-coron [7] (topological methods), Ambrosetti-Garcia-Peral [3] and Chang-Yang [14] (perturbation methods), W. Chen-C.Li [16], A. Malchiodi [30] and Y. Li [28] (sub-critical approximations).

Most of the results on problem (1.1) are concerned with the lower dimensional case, see for example [1, 2, 10, 15, 18, 20, 23, 26, 27, 32]. Much less is known for the higher dimensional case, see for example [5, 9, 19, 22, 24, 29, 33].

In a seminal paper [7], A. Bahri studies problem (1.1) on the standard dimensional spheres. He introduced two topological invariants μ and τ to study the problem in high dimensions $(n \geq 7)$ and derived the following existence result:

If $\tau = 1$ or $\mu = 0$ (in $\mathbb{Z}/_{2\mathbb{Z}}$), then (1.1) has a solution.

Our first aim in this paper is to extend these topological invariants to the problem on general riemannian manifolds and therefore extend the existence result of Bahri to any n-dimensional closed riemannian manifold, $n \geq 7$. Second we shall analyze, as suggested in ([7], P 374), the relation between the quantities μ and τ and discuss some topological conditions to ensure an equality between these two topological invariants.

The rest of this paper is organized as follows. We first introduce the assumptions, notations and definitions that we will use. Second, we prove technical estimates useful to extend the existence result of Bahri to problem (1.1) on general closed riemannian manifolds. Finally, we analyze the relation between the topological invariants μ and τ and we prove under some topological conditions that μ equals τ.

2. Variational calculus

Fix a positive constant ρ_0 less than the injective radius of (M^n, g_0). Denote $\{x_i\}$ the geodesic coordinates of g_0 in $B(a, 2\rho_0)$, $a \in M^n$. We introduce a smooth positive function h_a, $a \in M^n$, defined by

$$h_a(x) = \frac{1}{n-2}\big(2(R_{g_0})_{ij}(a) - \frac{R_{g_0}(a)(g_0)_{ij}(a)}{n-1}\big)x_i x_j,\ x \in B(a, 2\rho_0)$$

where $(R_{g_0})_{ij}$ is the Ricci tensor of (M^n, g_0). Let

$$\widetilde{g} = e^{h_a} g_0.$$

From [4] we know that $(R_{\widetilde{g}})_{ij}(a) = 0$.

Denote again $\{x_i\}$ the geodesic coordinates of $\widetilde{g}$ in $B(a, 2\rho_0)$. We define

$$\delta_{(a,\lambda)}(x) = \big(\frac{\lambda}{1 + \lambda^2 \mathrm{d}_{\widetilde{g}}(x, a)^2}\big)^{\frac{n-2}{2}},\ x \in B(a, 2\rho_0),\ \lambda \gg 1.$$

Let $\psi_a(x) = e^{\frac{n-2}{4}h_a(x)}$ and let $\omega_a(x)$ be a cut-off function on M^n defined by $\omega_a = 1$ on $B(a, \rho_0)$ and $\omega_a = 0$ on $B(a, 2\rho_0)^c$. We define

$$\widehat{\delta}_{(a,\lambda)}(x) = \omega_a(x)\psi_a(x)\delta_{(a,\lambda)}(x) + (1 - \omega_a(x))\frac{1}{\lambda^{\frac{n-2}{2}}}.$$

The failure of the Palais-Smale condition is characterized as follows.

Let ω be a solution of (1.1) or zero and let $p \in \mathbb{N}$ and $\varepsilon > 0$ small enough. We define

$$\begin{aligned} V(p,\varepsilon,\omega) \;=\Big\{ \;& u \in \Sigma,\ \exists a_1,\cdots,a_p \in M^n, \exists \lambda_1,\cdots,\lambda_p > \varepsilon^{-1},\ \exists \alpha_0,\alpha_1,\cdots,\alpha_p > 0,\ s.t., \\ & \Big\|u - \alpha_0\omega - \sum_{i=1}^{p}\alpha_i\widehat{\delta}_{(a_i,\lambda_i)}\Big\| < \varepsilon, \ \text{ with } \Big|\alpha_i^{\frac{4}{n-2}}K(a_i)J(u)^{\frac{n}{n-2}} - 1\Big| < \varepsilon, \\ & \forall i = 1,\cdots,p,\ \Big|\alpha_0^{\frac{4}{n-2}}J(u)^{\frac{n}{n-2}} - 1\Big| < \varepsilon,\ \forall i = 1,\cdots,p \text{ and } \varepsilon_{ij} < \varepsilon,\ \forall i \neq j\Big\}, \end{aligned}$$

where $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j \mathrm{d}^2_{g_0}(a_i,a_j)\right)^{-\frac{n-2}{2}}$.

Proposition 2.1. *[6] Let (u_k) be a non precompact Sequence of Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ tends to zero. Then there exists $p \in \mathbb{N}$, $(\varepsilon_k) \searrow 0$ and a subsequence of (u_k) denoted again (u_k) such that $u_k \in V(p, \varepsilon_k, \omega)$, $\forall k \in \mathbb{N}$.*

Let ω be a non degenerate critical point of J in Σ^+. It is known (see [6] and [8]) that for any $u \in V(p,\varepsilon,\omega)$ there exists unique $\alpha_0, \alpha_1, \cdots, \alpha_p$, $a_1, \cdots, a_p$, $\lambda_1, \cdots, \lambda_p$ and $h \in T_\omega W_u(\omega)$ such that $\alpha_0(\omega + h) + \sum_{i=1}^{p}\alpha_i\widehat{\delta}_{(a_i,\lambda_i)} \in V(p,\varepsilon,\omega)$ and

$$u = \sum_{i=1}^{p}\alpha_i\widehat{\delta}_{(a_i,\lambda_i)} + \alpha_0(\omega + h) + v,$$

where $v \in H^1(M^n)$ satisfying

$$(V_0):\ \langle v,\varphi\rangle = 0,\ \forall \varphi \in \left\{\omega,\ h,\ \widehat{\delta}_{(a_i,\lambda_i)},\ \frac{\partial\widehat{\delta}_{(a_i,\lambda_i)}}{\partial\lambda_i},\ \frac{\partial\widehat{\delta}_{(a_i,\lambda_i)}}{\partial a_i},\ i = 1,\cdots,p\right\}.$$

We point out here that in a generic situation we can assume that all the critical points of J are non degenerate, see [34].

For $u = \alpha_0(\omega + h) + \sum_{i=1}^{p}\alpha_i\widehat{\delta}_{(a_i,\lambda_i)} \in V(p,\varepsilon,\omega)$, we introduce the minimization problem

$$\min\left\{J(u+v),\ v \text{ satisfies } (V_0)\right\}.$$

According to [6] and [7], the above minimization problem has a unique solution $\bar{v} = \bar{v}(\alpha_i, a_i, \lambda_i, h)$. Moreover there exists a change of variables $V = v - \bar{v}$ such that

$$J(\alpha_0(\omega + h) + \sum_{i=1}^{p}\alpha_i\widehat{\delta}_{(a_i,\lambda_i)} + v) = J(\alpha_0(\omega + h) + \sum_{i=1}^{p}\alpha_i\widehat{\delta}_{(a_i,\lambda_i)} + \bar{v}) + \|v\|^2.$$

On the V variable, we decrease $\|V\|$ according to the differential equation $\dot{V} = -\beta V$, $\beta \gg 1$. Therefore, as $t = 1$, $V(t) = e^{-\beta t}V(0)$ will have a very small norm as β very large. Therefore in order to define our deformation, we can work as if $V = 0$.

We now define a critical point at infinity of J.

Definition 2.2. *[6], [7] Let ω be a solution of (1.1) or zero. Let $y_1, \cdots, y_p \in M^n$, $p \in \mathbb{N}$. We say that $(\omega, y_1, \cdots, y_p)$ is a critical point at infinity of J, if there exists a decreasing pseudogradient $\bigvee$ of J and a flow line $u(t)$ of $\bigvee$ such that for any t large $u(t) = \alpha_0(t)(\omega + h(t)) + \sum_{i=1}^{p}\alpha_i(t)\widehat{\delta}_{(a_i(t),\lambda_i(t))} + \overline{v}(t) \in V(p, \varepsilon(s), \omega)$, with $\varepsilon(t) \searrow 0$ and $a_i(t) \to y_i$ as $t \to \infty$, $\forall i = 1, \cdots, p$. Moreover, J has a Morse reduction around $(\omega, y_1, \cdots, y_p)$.*

The next Proposition describes the critical points at infinity of J in dimension $n \geq 7$ under the following non degeneracy-condition:

(nd) $K\colon M^n \to \mathbb{R}$ is a positive Morse function such that $\Delta_{g_0}K(y) \neq 0$ if $\nabla_{g_0}K(y) = 0$.

Proposition 2.3. *Let $n \geq 7$. There exists a family $\mathcal{P}_1$ of bounded pseudogradients of the functional J such that for any $V \in \mathcal{P}_1$ the following holds:*

Let ω be a (non degenerate) critical point of J or zero. For any $p \in \mathbb{N}$ and $\varepsilon > 0$ small enough there exists a positive constant c such that for any $u = \alpha_0(\omega + h) + \sum_{i=1}^{p}\alpha_i\widehat{\delta}_{(a_i,\lambda_i)} \in V(p, \varepsilon, \omega)$ we have.

(i) $\langle \partial J(u), V(u)\rangle \leq -c\Bigg(\|h\|^2 \sum_{i=1}^{p}\big(\frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i}\big) + \sum_{i\neq j}\varepsilon_{ij}\Bigg)$,

(ii) $\Big\langle \partial J(u + \overline{v}), V(u) + \frac{\partial \overline{v}}{\partial(\alpha a,\lambda,h)}\Big\rangle \leq -c\Bigg(\|h\|^2 \sum_{i=1}^{p}\big(\frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i}\big) + \sum_{i\neq j}\varepsilon_{ij}\Bigg)$,

(iii) $\max_{1\leq i\leq p}\lambda_i(t)$ *remains bounded along any flow line* $u(t) = \alpha_0(\omega + h(t)) + \sum_{i=1}^{p}\alpha_i(t)\widehat{\delta}_{(a_i(t),\lambda_i(t))}$ *of V only if for any $i = 1, \cdots, p$ $a_i(t) \to y_i$, with $\nabla K(y_i) = 0$, $-\Delta K(y_p) > 0$ and $y_i \neq y_j$, $\forall 1 \leq i \neq j \leq p$. In this case, $\lambda_i(t) \to \infty$, $\forall i = 1, \cdots, p$.*

(iv) *$J(u(t))$ has an expansion of Morse type when t tends to ∞.*

Proof. See the construction of ([11], Section 3) in the case of $\omega = 0$. If $\omega \neq 0$, we have the following expansion of J in $V(p, \varepsilon, \omega)$. See [21] for a Similar Statement.

$$J(\sum_{i=1}^{p}\alpha_i\widehat{\delta}_{(a_i,\lambda_i)} + \alpha_0(\omega + h)) = \frac{S_n \sum_{i=1}^{p}\alpha_i^2 + \alpha_0\|\omega\|^2}{\Big(S_n\sum_{i=1}^{p}\alpha_i^{\frac{2n}{n-2}}K(a_i) + \alpha_0^{\frac{2n}{n-2}}\|\omega\|^2\Big)^{\frac{n-2}{n}}} \Bigg[1 - C_1\sum_{i=1}^{p}\alpha_i^{\frac{2n}{n-2}}\frac{\Delta K(a_i)}{\lambda_i^2} - C_2\sum_{i=1}^{p}\frac{\alpha_i\alpha_0\omega(a_i)}{\lambda_i^{\frac{n-2}{2}}} - C_3\sum_{i\neq j}C_{ij}\alpha_i\alpha_j\varepsilon_{ij}\Bigg].$$

Observe that for $n \geq 7$, the term $\frac{\omega(a_i)}{\lambda_i^{n-2}}$ is dominated by $\frac{\Delta K(a_i)}{\lambda_i^2}$, (if $\Delta K(a_i) \neq 0$), for λ large enough. Therefore, the construction of $\mathcal{P}_1$ proceeds as if $\omega = 0$ in [11]. □

Next, we denote ω_∞ a critical point or a critical point at infinity of J. We also denote $i(\omega)_\infty$ its Morse index, $W_u(\omega)_\infty$ its unstable manifold and $W_s(\omega)_\infty$ its stable manifold of flow lines relatively to a vector field $V \in \mathcal{P}_1$.

$W_u(\omega)_\infty$ is a $i(\omega)_\infty$-dimensional submanifold of M^n. after truangulation, $W_u(\omega)_\infty$can be partitioned into $i(\omega)_\infty$-simplices. Therefore $W_u(\omega)_\infty$ defines a $\mathbb{Z}_{/2\mathbb{Z}}$-chain of dimension $i(\omega)_\infty$ in M^n.

Let $(\omega)_\infty$ and $(\omega')_\infty$ be tow critical or critical points at infinity of J. We say that $W_u(\omega)_\infty \cap W_s(\omega')_\infty$ is transverse, if $W_u(\omega)_\infty \cap J^{-1}(\{c\})$ and $W_s(\omega')_\infty \cap J^{-1}(\{c\})$ have transverse intersections with the hypersurface $J^{-1}(\{c\})$. Here, c is a regular value of J. In this case by a dimension argument, one deduce that $i(\omega)_\infty \geq i(\omega')_\infty + 1$.

In order to define the topological invariants μ and τ, we introduce the so-called the intersection numbers of two submanifolds of M^n. Let Γ_1 and Γ_2 be two submanifolds of M^n such that $\Gamma_1 \cap \Gamma_2$ is transverse and $\dim \Gamma_1 = codim \Gamma_2$. By a dimension argument $\Gamma_1 \cap \Gamma_2$ is decomposed by isolated points. In the case of $\Gamma_1 \cap \Gamma_2$ is compact, we denote

$$\Gamma_1 \cdot \Gamma_2$$

the intersection number of $\Gamma_1 \cap \Gamma_2$. Particularly, if $i(\omega)_\infty = k+1$ and $i(\omega')_\infty = k$, we define

$$i(\omega_\infty, \omega'_\infty) = \left[W_u(\omega)_\infty \cap J^{-1}(\{c\})\right] \cdot \left[W_s(\omega')_\infty \cap J^{-1}(\{c\})\right],$$

where c is a regular value of J between $J(\omega_\infty)$ and $J(\omega'_\infty)$.

Notice that $i(\omega_\infty, \omega'_\infty)$ counts the number of the flow lines of V descending from $(\omega)_\infty$ to $(\omega')_\infty$.

3. Topological results

Let $X \subset M^n$, be a fixed closed stratified set of dimension $k \geq 1$. For $p \geq 2$ we define

$$B_p(X) = \left\{ \sum_{i=1}^{p} \alpha_i \delta_{a_i}, \;\; \alpha_i \in [0,1], \; a_i \in X, \; \forall i = 1, \cdots, p \text{ and } \sum_{i=1}^{p} \alpha_i = 1 \right\},$$

where δ_{a_i} denotes the Dirac mass at a_i.

Let λ_0 be fixed positive constant large enough. We set

$$\begin{aligned} f_{\lambda_0} \colon B_p(X) &\rightarrow \Sigma^+ \\ \sum_{i=1}^{p} \alpha_i \delta_{a_i} &\mapsto \frac{\sum_{i=1}^{p} \alpha_i \widehat{\delta}_{(a_0,\lambda_0)}}{\left\| \sum_{i=1}^{p} \alpha_i \widehat{\delta}_{(a_0,\lambda_0)} \right\|}. \end{aligned}$$

$B_p(X)$ and $f_{\lambda_0}(B_p(X))$ are two compact manifolds of dimension $p(k+1)-1$ with no boundary of dimension $p(k+1)-2$.

Let $\nu^+(X)$ be a tabular neighborhood of X in M^n. Let $z_i, \cdots, z_p$ be a fixed p-points in X.

For any $z_i,\ i=1,\cdots,p$, we denote $\nu^+(z_i)$ the fiber of $\nu^+(X)$ at z_i.

Let ε_1 be a fixed positive constant small enough. We define

$$\begin{aligned}\Gamma_{\varepsilon_1}(z_1,\cdots,z_p) &= \Big\{ \sum_{i=1}^{p} \frac{1}{K(z_i+h_i)^{\frac{n-2}{4}}}\widehat{\delta}_{(z_i+h_i,\lambda_i)} + v,\ v \text{ satisfies } (V_0),\ \|v\|<\varepsilon_1,\ \lambda_i>\varepsilon_1^{-1}, \\ &\qquad h_i\in\nu^+(z_i),\ \forall i=1,\cdots,p \ \text{ and } \sum_{i=1}^{p}|h_i|^2<\varepsilon_1\Big\}.\end{aligned}$$

$\Gamma_{\varepsilon_1}(z_1,\cdots,z_p)$ is a submanifold of Σ of codimension $p(k+1)-1$.

Let δ_0 be a fixed positive constat small enough. We define

$$c_{\delta_0}(z_1,\cdots,z_p) = \Gamma_{\varepsilon_1}(z_,\cdots,z_p)\cap J^{-1}\big(\{c_\infty(z_1,\cdots,z_p)+\delta_0\}\big),$$

where $c_\infty(z_1,\cdots,z_p) = \big(S_n\sum_{i=1}^{p}\frac{1}{K(z_i)^{\frac{n-2}{2}}}\big)^{\frac{2}{n}}$. $c_{\delta_0}(z_1,\cdots,z_p)$ is a Fredholm submanifold of Σ of codimension $p(k+1)$.

For $V\in\mathcal{P}_1$, we denote $\Phi(t,\cdot)$ the 1_parameter group generated by V. Define

$$W_u(f_{\lambda_0}(B_p(X))) = \Big\{\Phi(t,u),\ t\geq 0 \text{ and } u\in f_{\lambda_0}(B_p(X))\Big\}.$$

IT is easy to see that

$$\dim W_u(f_{\lambda_0}(B_p(X))) = \operatorname{codim} c_{\delta_0}(z_1,\cdots,z_p) = p(k+1).$$

We therefore define:

Definition 3.1.

$$\tau(z_1,\cdots,z_p) = \tau = W_u(f_{\lambda_0}(B_p(X)))\cdot c_{\delta_0}(z_1,\cdots,z_p).$$

We point out that in [7], X is defined by a stable manifold of a critical point of K of coindex k. Here X represents any closed stratified set of dimension k. We shall prove the following Theorem.

Theorem 3.2. *If*

$$\tau = 1 \ (\text{ in } \mathbb{Z}_{/2\mathbb{Z}})$$

then K is the scalar curvature of a metric g conformally equivalent to g_0.

Proof. Denote y_0 an absolute maximum of K on M^n, denote y_{i_0} an element of X such that

$$K(y_{i_0}) = \min_{x\in X} K(x),$$

and for $* := p(k+1)-1$, we denote

A_* the set of all the critical or critical points at infinity of J not containing y_0 in its description.

Let $K_\varepsilon = 1 + \varepsilon K$ and J_ε be the associated variational functional. For $t \in [0,1]$ we set

$$K_t = tK + (1-t)K_\varepsilon.$$

Observe that K_t, $t \in [0,1]$ have the same critical points on M^n with the same sign of Laplacian. Therefore the above Proposition 2.3 holds for all $K_t,\ t \in [0,1]$.

For any $V^t \in \mathcal{P}_1,\ t \in [0,1]$, we consider the $(*+1)$-dimensional chain of Σ^+ introduced first by A.Bahri in [7].

$$C(V^t) = \overline{W_u^t\Big(f_{\lambda_0}(B_p(X))\Big)} + \sum_{\omega_* \in A_*} \Big(f_{\lambda_0}(B_p(X)) \cdot W_s(\omega_\infty)\Big)\overline{W_u^t(y_0, \omega_\infty)}.$$

Remark. Let ω be a critical point of J. the unstable manifold $W_u(\omega)$ of ω with respect to the gradient flow $\eta(\cdot,\cdot)$ of $(-\partial J)$ is as usual defined by:

$$W_u(\omega) = \{u \in \Sigma,\ \ \eta(t,u) \longrightarrow \omega,\ \text{as}\ t \longrightarrow -\infty\}.$$

$W_u(\omega)$ is a manifold with boundary of finite dimension. If ω is nondegenerate, the dimension of $W_u(\omega)$ equals to the Morse index of J at ω given by the dimension of the negative space in the expansion of J (up to order 2) around ω. If ω is degenerate, the dimension of $W_u(\omega)$ cannot be computed by the expansion of J (up to order 2) around ω, since a null space of dimension ≥ 1 appears in the expansion. However, the behavior of flow lines around degenerate critical points (or mixed critical points at infinity) is well studied in ([7], P.335 and P.460-466).

Now using the computation of [7], we have.

$$\partial(C(V^t)) = f_{\lambda_0}(B_p(X)),\ \ \forall t \in [0,1].$$

Recall that $f_{\lambda_0}(B_p(X))$ is independent of $K_t,\ t \in [0,1]$. Therefore $C(V^0) + C(V^1)$ defines a cycle with $\mathbb{Z}_{/2\mathbb{Z}}$ coefficients of dimension $(*+1)$ in Σ^+. Using the fact that Σ^+ is a contractible space, the cycle $C(V^0) + C(V^1)$ defines a boundary of a $(*+2)$-dimensional chain θ of Σ^+. Assume that $\theta \cap c_{\delta_0}(z_1, \cdots, z_p)$ is compact and intersects transversally, then $\theta \cap c_{\delta_0}(z_1, \cdots, z_p)$ defines a manifold of dimension 1 and

$$\sharp\partial\big(\theta \cap c_{\delta_0}(z_1, \cdots, z_p)\big) = C(V^0) \cdot c_{\delta_0}(z_1, \cdots, z_p) + C(V^1) \cdot c_{\delta_0}(z_1, \cdots, z_p) = 0 \text{ in } \mathbb{Z}_{/2\mathbb{Z}}.$$

Thus

$$C(V^1) \cdot c_{\delta_0}(z_1, \cdots, z_p) = C(V^0) \cdot c_{\delta_0}(z_1, \cdots, z_p), \quad \text{in } \mathbb{Z}_{/2\mathbb{Z}}. \tag{3.1}$$

At $t = 0$, we can suppose that ε is small that all the critical points at infinity of J of p-masses, p is large, can not dominated by $f_{\lambda_0}(B_p(X))$. Moreover $J_\varepsilon\big(f_{\lambda_0}(B_p(X))\big)$ lies under the level $C_2(z_1, \cdots, z_p) + \frac{\delta_0}{2}$. Therefore

$$W_u^0\big(f_{\lambda_0}(B_p(X))\big) \cap c_{\delta_0}(z_1, \cdots, z_p) = \emptyset \ \text{ and } C(V^0) \cdot c_{\delta_0}(z_1, \cdots, z_p) = 0.$$

This yields from (3.1) that

$$C(V^1) \cdot c_{\delta_0}(z_1, \cdots, z_p) = 0. \tag{3.2}$$

Let $\overline{A}_*$ be the subset of the critical points at infinity in A_* not containing a (true) critical point of J in Σ^+. We decompose $C(V)$, $(V = V^1)$, as the following

$$\begin{aligned} C(V) &= \overline{W_u\big(f_{\lambda_0}(B_p(X))\big)} + \sum_{\omega_\infty \in \overline{A}_*} \big(f_\lambda(B_p(X)) \cdot W_s(\omega)_\infty\big) W_u(y_0, \omega_\infty) \\ &+ \sum_{\omega_\infty \in A_* \setminus \overline{A}_*} \big(f_\lambda(B_p(X)) \cdot W_s(\omega)_\infty\big) W_u(y_0, \omega_\infty). \end{aligned}$$

From (3.2), we have

$$\begin{aligned} \tau(z_1, \cdots, z_p) &+ \sum_{\omega_\infty \in \overline{A}_*} \big(f_\lambda(B_p(X)) \cdot W_s(\omega)_\infty\big)\big(W_u(y_0, \omega_\infty) \cdot c_{\delta_0}(y_0, \omega_\infty)\big) \\ &+ \sum_{\omega_\infty \in A_* \setminus \overline{A}_*} \big(f_\lambda(B_p(X)) \cdot W_s(\omega)_\infty\big)\big(W_u(y_0, \omega_\infty) \cdot c_{\delta_0}(y_0, \omega_\infty)\big) = 0 \text{ in } \mathbb{Z}_{/2\mathbb{Z}}. \end{aligned} \tag{3.3}$$

We shall prove that

$$\sum_{\omega_\infty \in \overline{A}_*} \big(f_\lambda(B_p(X)) \cdot W_s(\omega)_\infty\big)\big(W_u(y_0, \omega_\infty) \cdot c_{\delta_0}(y_0, \omega_\infty)\big) = 0. \tag{3.4}$$

Let $\omega_\infty \in \overline{A}_*$. According to the topological computation of [7], it remains only to focus the case of $\omega_\infty = (y_1, \cdots, y_p)$ with

$$ind(K, y_i) = n - k, \ \forall i = 1, \cdots, p.$$

We distinguish 3-cases.

- Case 1. $c_\infty(\omega_\infty) \geq \frac{(PS_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2}{n}}}$.

  In this case ω_∞ is above $f_{\lambda_0}(B_p(X))$ and hence $\big(f_{\lambda_0}(B_p(X))\big) \cap W_s(\omega_\infty) = \emptyset$.

- Case 2. $C_\infty(\omega_\infty) < \frac{(PS_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2}{n}}}$ and $C_\infty(y_0, \omega_\infty) < \frac{(PS_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2}{n}}}$.

  We know from [7] that starting from $c_{\delta_0}(z_1, \cdots, z_p)$, the flow lines of V increase (with respect to J) until arriving to the level $\frac{(PS_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2}{n}}}$ without meeting any critical points at infinity of J. Therefore

$$W_u(y_0, \omega_\infty) \cdot c_{\delta_0}(z_1, \cdots, z_p) = 0.$$

- Case 3. $C_\infty(\omega_\infty) < \frac{(PS_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2}{n}}}$ and $C_\infty(y_0, \omega_\infty) \geq \frac{(PS_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2}{n}}}$.

  Let in this case $\varepsilon_0 > 0$ such that

$$\varepsilon_0 \frac{S_n^{\frac{2}{n}}}{K(y_0)^{\frac{n-2}{n}}} + C_\infty(\omega)_\infty < \frac{(PS_n)^{\frac{2}{n}}}{K(y_0)^{\frac{n-2}{n}}}.$$

We define a functional K_{ε_0} in M^n by:

$$K_{\varepsilon_0} = K \text{ in } M^n \setminus B(y_0, \rho) \text{ and } K_{\varepsilon_0} = \varepsilon_0 K \text{ in } B(y_0, \frac{\rho}{2}),$$

where ρ is a small positive constant. It is easy to see that K_{ε_0} satisfies the assumptions of the above Case 2. Thus

$$W_u^{\varepsilon_0}(y_0, \omega_\infty) \cdot c_{\delta_0}(z_1, \cdots, z_p) = 0 \tag{3.5}$$

For $t \in [0,1]$, define $K_t = tK + (1-t)K_{\varepsilon_0}$. Let

$$W = \bigcup_{t\in[0,1]} W_u^t(y_0, \omega_\infty).$$

W is a manifold of dimension $* + 2$. Assume that it dominates critical points at infinity ω_{*+1}^t of index $* + 1$ at a infinite number of times $t_1, \cdots, t_r$ with intersection numbers $\ell_1, \cdots, \ell_r$. It follows that,

$$\begin{aligned} \partial W &= W_u^{\varepsilon_0}(y_0, \omega_\infty) + W_u(y_0, \omega_\infty) + \bigcup_{t\in[0,1]} W_u^t(\omega_\infty) \\ &+ \sum_{\omega_{*+1}} (\omega_\infty \cdot \omega_{*+1}) \bigcup_{t\in[0,1]} W_u^t(y_0, \omega_{*+1}) + \sum_{i=1}^{r} \ell_i W_u(\omega_{*+1}^{t_i}). \end{aligned} \tag{3.6}$$

As $C_{\delta_0}(z_1, \cdots, z_p)$ is of codimension $* + 1$ and without boundary, we get

$$\sharp\partial(W \cap C_{\delta_0}(z_1, \cdots, z_p) = \sharp(\partial W \cap C_{\delta_0}(z_1, \cdots, z_p)) = 0. \tag{3.7}$$

Claim 1.

$$\bigcup_{t\in[0,1]} W_u^t(\omega_\infty) \cdot C_{\delta_0}(z_1, \cdots, z_p) = 0.$$

Indeed, we devide $W_u^t(\omega_\infty)$ into 2 subsets.

$$\begin{aligned} S_1^t &= \left\{ u \in W_u^t(\omega_\infty),\ s.t.,\ u = \sum_{i=1}^{p} \alpha_i \widehat{\delta}_{(x_i,\lambda_i)},\ \text{with } x_i \in B(y_0,\rho)^c,\ \forall i \right\}. \\ S_2^t &= \left\{ u \in W_u^t(\omega_\infty),\ s.t.,\ u = \sum_{i=1}^{p} \alpha_i \widehat{\delta}_{(x_i,\lambda_i)},\ \text{with } x_i \in B(y_0,\rho) \text{ for some index } i \right\}. \end{aligned}$$

In $B(y_0, \rho)^c$, $K_t = K$, $\forall t \in [0,1]$. Therefore $\bigcup_{t\in[0,1]} S_1^t = S_1^1$, which is of dimension $*$. So by a dimension argumentS_1^1 cannot dominate $C_{\delta_0}(z_1, \cdots, z_p)$.

Now, if $x_i \in B(y_0, \rho)$ for at least an index i, in this case the distance $\mathrm{d}(x_i, y_0)$ can only decrease along V. Thus, x_i can not dominate any z_i, $i = 1, \cdots, p$ and therefore

$$\bigcup_{t\in[0,1]} S_2^t \cdot C_{\delta_0}(z_1, \cdots, z_p) = 0.$$

Hence Claim 1 is valid.

Claim 2.

$$\bigcup_{t\in[0,1]} W_u^t(y_0, \omega_{*-1}) \cdot C_{\delta_0}(z_1, \cdots, z_p) = \sum_{i=1}^{r} \ell_i \big(W_u^{t_i}(\omega_{*+1}) \cdot C_{\delta_0}(z_1, \cdots, z_p) \big) = 0.$$

Indeed, for $t \in [0,1]$, $W_u^t(y_0, \omega_{*-1})$ can not dominate $C_{\delta_0}(z_1, \cdots, z_p)$ through direct connections, see [7]. Also, $W_u(\omega_{*+1})$ can not exists in the boundary of W only through indirect connections. Using dimension arguments of [7], Claim 2 follows.

Collecting Claims 1 and 2, (3.5), (3.6) and (3.7), we get

$$W_u(y_0, \omega_\infty) \cdot C_{\delta_0}(z_1, \cdots, z_p) = W_u^{\varepsilon_0}(y_0, \omega_\infty) \cdot C_{\delta_0}(z_1, \cdots, z_p) = 0,$$

and therefore (3.4) follows.

Consequently, if $\tau = 1$, we deduce from (3.3) and (3.4) that $A_* \backslash \overline{A}_*$ is not empty and therefore, the existence of a true critical point of J in Σ^+. □

We now introduce an other topological invariant μ defined by an intersection number of two submanifolds in Σ^+.

Let $k \in \mathbb{N}$ and let $y_1, \cdots, y_p$, p-critical points of K of index k such that $-\Delta K(y_i) > 0,\ i = 1, \cdots, p$. We denote

$$X = \bigcup_{i=1}^{p} \overline{W_s(y_i)}.$$

X is a stratified set of dimension k in M^n. It can be chosen for an appropriate k without boundary of dimension $k-1$. Let $\mathscr{C}_{y_0}(X)$ be a $(k+1)$-dimensional submanifold of $B_2(X)$ defined by

$$\mathscr{C}_{y_0}(X) = \big\{\alpha\delta_x + (1-\alpha)\delta_{y_0},\ x \in X \text{ and } \alpha \in [0,1]\big\}.$$

$f_{\lambda_0}(\mathscr{C}_{y_0}(X))$ defines a $(k+1)$-dimensional contractible set in Σ^+. We may perturb if necessary the flow lines of V, so that the intersection of $f_{\lambda_0}(\mathscr{C}_{y_0}(X))$ and $W_s(y_0, y_i)_\infty$ be transverse, for any $i = 1, \cdots, p$. Therefore the following intersection numbers is well defined.

Definition 3.3. *For any $i = 1, \cdots, p$, we define*

$$\mu(y_i) = f_{\lambda_0}(\mathscr{C}_{y_0}(X)) \cdot W_s(y_0, y_i), \ \textit{ in } \mathbb{Z}_{/2\mathbb{Z}}.$$

We then have

Theorem 3.4. $\mu(y_i) = 0,\ i = 1, \cdots, p \Rightarrow$ (1.1) *has a solution.*

Proof. Assume that (1.1) has no solution. We deform $f_{\lambda_0}(\mathscr{C}_{y_0}(X))$ using the flow lines of V. $f_{\lambda_0}(\mathscr{C}_{y_0}(X))$ retracts by deformation on a set diffeomorph to

$$X \cup \bigcup_{i=1}^{p} W_u(y_0, y_i)_\infty \cup \sigma,$$

where $\dim\ \sigma \leq k$. Using the assumption of the Theorem we get

$$f_{\lambda_0}(\mathscr{C}_{y_0}(X)) \simeq X \cup \sigma.$$

Using the sequence of the homology group of pairs $H_*(X \cup \sigma, X)$, we get after recalling that $f_{\lambda_0}(\mathscr{C}_{y_0}(X))$ is a contractible space, that

$$H_k(X) = 0.$$

This is impossible since X is k-dimensional stratified set without boundary. □

Remark. The proof of Theorem 3.4 is a simplified version of the one of ([7], Theorem B). So there is no differences in the proof of theorem 3.4 on general manifolds and Theorem B of [7] on spheres. However, the proof of Theorem 3.2 contain new justifications with respect to the one of ([7], Theorem A) . Namely, the discussion following the cases 1, 2 and 3 of the proof. These discussions are necessary in our statement, since, as mentioned before, We have not imposed any condition on the stratified set X.

4. Relation between the invariants μ and τ

In this section we study the relation between the quantities μ and τ introduced in the previous chapter. We discuss some topological conditions to ensure an equality between them. This leads of course to get more existence results of problem (1.1).

Let $k \in \mathbb{N}$ and y_{i_0} be a critical point of K of coindex k such that $-\Delta K(y_{i_0}) > 0$ and

$$K(y_{i_0}) = \max\big\{K(y),\ \nabla K(y) = 0,\ \Delta K(y) > 0,\ n - ind(K, y) = k\big\}.$$

Define

$$X = \overline{W_s(y_{i_0})},$$

$$\begin{aligned} f_{\lambda_0}(B_2(X)) &= \big\{\alpha\widehat{\delta}_{(x_1,\lambda)} + (1-\alpha)\widehat{\delta}_{(x_2,\lambda)},\ \alpha \in [0,1],\ x_i \in X\big\}, \\ f_{\lambda}(\mathscr{C}_{y_0}(X)) &= \big\{\alpha\widehat{\delta}_{(x,\lambda)} + (1-\alpha)\widehat{\delta}_{(y_0,\lambda)},\ \alpha \in [0,1],\ x \in X\big\}. \end{aligned}$$

As in the previous chapter we define for $z \in X$,

$$\tau(z) = W_u(f_{\lambda_0}(B_2(X))) \cdot C_{\delta_0}(y_{i_0}, z), \tag{4.1}$$

and

$$\mu(z) = f_{\lambda_0}(\mathscr{C}_z(X)) \cdot W_s(y_0, y_{i_0})_\infty. \tag{4.2}$$

Let $V(2,\varepsilon)(X)$ be the subset of $V(2,\varepsilon,\omega)$, such that $\omega = 0$ and the concentration points of any u in $V(2,\varepsilon,\omega)$ lie in X. Assume that

$$(H_1): \quad W_u(f_{\lambda_0}(B_2(X))) \cap W_s\Big(\bigcup_{z\in B_{y_0}} C_{\delta_0}(y_{i_0}, z)\Big) \subset V(2,\varepsilon,\omega)(X).$$

Here B_{y_0} is a neighborhood of y_0 in X such that for any $z \in B_{y_0}$, $C_{\delta_0}(y_{i_0}, z)$ is a Fredholm manifold of codimension $(k+2)$ without boundary. We then have

Proposition 4.1. *Under (H_1)-assumption $\mu(z)$ is independent of z, $\forall z \in X$*

Proof. Let z_0 and z_1 be two points in X. We define a continuous path $z(t)$, $t \in [0,1]$ in X connecting z_0 and z_1. We set

$$W = \bigcup_{t\in[0,1]} f_{\lambda_0}(\mathscr{C}_{z(t)}(X)).$$

W defines a compact manifold in dimension $k+2$. Using the fact that $W_s(y_0, y_{i_0})_\infty = k+1$, then $W \cap W_s(y_0, y_{i_0})_\infty$ is a compact manifold of dimension 1. Thus,

$$\begin{aligned} 0 &= \sharp\partial(W \cap W_s(y_0, y_{i_0})_\infty), \text{ in } \mathbb{Z}/_{2\mathbb{Z}}, \\ &= \mu(z_0) + \mu(z_1) + \sum_{\omega_{k+2}} \big[W_u(\omega_{k+2}) \cdot W_s(y_0, y_{i_0})\big] \cdot \big[W_s(\omega_{k+2}) \cdot W\big]. \end{aligned}$$

We claim that

$$\big[W_u(\omega_{k+2}) \cdot W_s(y_0, y_{i_0})\big] \cdot \big[W_s(\omega_{k+2}) \cdot W\big] = 0. \tag{4.3}$$

Indeed, if it is not zero, the critical point at infinity ω_{k+2} of Morse index $k+2$ dominates $(y_0, y_{i_0})_\infty$. Hence ω_{k+2} have to be of at least two masses. Moreover W dominates ω_{k+2}, then ω_{k+2} have to be of at most two masses. Thus ω_{k+2} contains exactly two masses which are critical points of K in X. Say y_i and y_j. Using the fact that

$$ind(y_i, y_j)_\infty = k+2 = ind(y_0, y_{i_0}) + 1,$$

then under (H_1) assumption, $y_i = y_0$ and $ind(y_j)_\infty = k+1$ or $y_j = y_{i_0}$ and $ind(y_i)_\infty = 1$. By dimension arguments these two cases are impossible. (4.3) is then valid and Proposition 1 follows. ☐

Proposition 4.2. *Under (H_1)-assumption, $\tau(z)$ is independent of z, $\forall z \in B_{y_0}$.*

Proof. Let z_0 and z_1 be two points in B_{y_0} and let $z(t),\ t \in [0,1]$ be a continuous path in B_{y_0} between z_0 and z_1. We define

$$W = \bigcup_{t \in [0,1]} C_{\delta_0}(z(t), y_{i_0}).$$

W is a manifold in Σ^+ of codimension $2k+1$. We can assume that the intersection between $W_u(f_{\lambda_0}(B_2(X)))$ and W is transverse. Then $W \cap W_u(f_{\lambda_0}(B_2(X)))$ defines a compact manifold of dimension 1. and we therefore have

$$\begin{aligned} 0 &= \sharp\partial(W \cap W_u(f_{\lambda_0}(B_2(X)))), \text{ in } \mathbb{Z}_{/2\mathbb{Z}}, \\ &= \tau(z_0) + \tau(z_1) + f_{\lambda_0}(B_2(X)) \cdot W + \sum_{\omega_{2k+1}} \big(f_{\lambda_0}(B_2(X)) \cdot W_s(\omega_{2k+1})\big) \cdot \big(W_u(\omega_{2k+1}) \cdot W\big). \end{aligned}$$

Observe that $f_{\lambda_0}(B_2(X)) \cap W = \varnothing$, since we can chose the concentration parameters of W large enough, and by the same argument of Proposition 4.1, we prove that

$$\big(f_{\lambda_0}(B_2(X)) \cdot W_s(\omega_{2k+1})\big) \cdot \big(W_u(\omega_{2k+1}) \cdot W\big) = 0.$$

The result of Proposition 4.2 follows. ☐

We now prove the following result.

Theorem 4.3. *Assume that J has no critical point in Σ^+ and assume (H_1). Then*

$$\mu = \tau.$$

Proof. Define

$$M_k = \bigcup_{z\in B_{y_0}} \big[W_u\big(\bigcup_{z'\in X} f_{\lambda_0}(\mathscr{C}_{z'}(X))\big)\cdot C_{\delta_0}(y_{i_0},z)\big].$$

We may assume that the intersection is transverse so that M_k defines a k-dimensional submanifold of Σ^+. We claim that

$$\partial M_k = \bigcup_{z\in \partial B_{y_0}} \big[W_u\big(\bigcup_{z'\in X} f_{\lambda_0}(\mathscr{C}_{z'}(X))\big)\cdot C_{\delta_0}(y_{i_0},z)\big] := \Gamma.$$

Indeed,

$$\partial M_k \;=\; W_u\big(f_{\lambda_0}(B_2(X))\big)\cap \partial\big(\bigcup_{z\in B_{y_0}} C_{\delta_0}(y_{i_0},z)\big) + \partial W_u\big(f_{\lambda_0}(B_2(X))\big)\cap \bigcup_{z\in B_{y_0}} C_{\delta_0}(y_{i_0},z).$$

Using the fact that there is no boundary for $C_{\delta_0}(y_{i_0},z)$, $z\in B_{y_0}$, we get

$$\partial M_k = \Gamma + f_{\lambda_0}(B_2(X))\cap\big(\bigcup_{z\in B_{y_0}} C_{\delta_0}(y_{i_0},z)\big) + \sum_{\omega_{2k+1}} \big(f_{\lambda_0}(B_2(X))\cdot\omega_{2k+1}\big) W_u(\omega_{2k+1})\cap\big(\bigcup_{z\in B_{y_0}} C_{\delta_0}(y_{i_0},z)\big).$$

Observe that for $\lambda_0 < \varepsilon_1^{-1}$, we then have

$$f_{\lambda_0}(B_2(X))\cap \bigcup_{z\in B_{y_0}} C_{\delta_0}(y_{i_0},z) = \varnothing,$$

and by (H_1) assumption, ω_{2k+1} is constructed by only two critical points in X. Therefore $\omega_{2k+1} = (y,y')_\infty$ and $i(y,y')_\infty \le 1+k+(k-1) = 2k$, which is impossible since $i(\omega_{2k+1}) = 2k+1$. It follows that

$$\sum_{\omega_{2k+1}} \big(f_{\lambda_0}(B_2(X))\cdot\omega_{2k+1}\big) W_u(\omega_{2k+1})\cap\big(\bigcup_{z\in B_{y_0}} C_{\delta_0}(y_{i_0},z)\big) = \varnothing,$$

and therefore $\partial M_k = \Gamma$.

Let $\Phi(.,u)$ be the 1-parameter group associated to $V\in\mathcal{P}_1$ and such that (H_1)-condition holds. For any $u\in M_k$ there exists a unique time $t_u>0$ such that $\Phi(-t_u,u)\in f_{\lambda_0}(B_2(X))$. The existence of t_u comes from the definition of M_k and the unicity comes from transferability and dimensions arguments.

We introduce the following two mappings of pairs:

$$\begin{aligned} g\colon (M_k,\partial M_k) &\longrightarrow \Big(X^2/_{\sigma_2}, \{y_{i_0}\}\underset{\sigma_2}{\times} X\backslash B_{y_0}\Big)\\ u=\alpha\widehat{\delta}_{(y_{i_0}+h_{i_0},\lambda_1)} + (1-\alpha)\widehat{\delta}_{(z+h,\lambda_2)} + v &\longmapsto \big(y_{i_0}, x(t_u, z+h)\big), \end{aligned}$$

where $x(.,a)$ is the restriction of $\Phi(.,u)$ on its concentration points.

Define also,

$$\begin{aligned} \varphi\colon (M_k,\partial M_k) &\longrightarrow \Big(X/_{\sigma_2}, \{y_{i_0}\}\underset{\sigma_2}{\times} \partial B_{y_0}\Big)\\ u=\alpha\widehat{\delta}_{(y_{i_0}+h_{i_0},\lambda_1)} + (1-\alpha)\widehat{\delta}_{(z+h,\lambda_2)} + v &\longmapsto (y_{i_0}, z). \end{aligned}$$

Here σ_2 is the permutations group of order 2.

We now consider the following two diagrams:

$$\begin{array}{ccc} & X^2 & \underset{\longrightarrow}{p_1} & X \\ & \downarrow p & & \\ (M_k, \partial M_k) & \underset{\longrightarrow}{\varphi} & \left(X^2/_{\sigma_2}, \{y_{i_0}\} \underset{\sigma_2}{\times} \partial B_{y_0} \right) & \end{array}$$

and

$$\begin{array}{ccc} & X^2 & \underset{\longrightarrow}{p_1} & X \\ & \downarrow p & & \\ (M_k, \partial M_k) & \underset{\longrightarrow}{g} & \left(X^2/_{\sigma_2}, \{y_{i_0}\} \underset{\sigma_2}{\times} X \backslash B_{y_0} \right). & \end{array}$$

Note that $(M_k, \partial M_k)$ defines a cycle of dimension k in its self. It cannot be realized as the boundary of chain of dimension $(k+1)$ in $(M_k, \partial M_k)$. Therefore, $[M_k, \partial M_k]_*$ is an orientation class of the pair $(M_k, \partial M_k)$ for $* = k$. We shall prove:

$$\tau = \big(\varphi^* \circ t_r^* \circ P_1^*\big)[X]^*\big([M_k, \partial M_k]_*\big),$$

and

$$\mu = \big(g^* \circ t_r^* \circ p_1^*\big)[X]^*\big([M_k, \partial M_k]_*\big).$$

Here tr^* is the transfer homomorphism, see [13].

Indeed, in $\mathbb{Z}/_{2\mathbb{Z}}$ we have

$$\begin{aligned} \big(\varphi^* \circ t_r^* \circ P_1^*\big)[X]^*\big([M_k, \partial M_k]_*\big) &= \big([X]^* \circ P_{1*} \circ t_r^* \circ \varphi_*\big)\big([M_k, \partial M_k]_*\big) \\ &= \big([X]^* \circ P_{1*} \circ t_r^*\big)\big([\varphi(M_k), \varphi(\partial M_k)]_*\big). \end{aligned} \tag{4.4}$$

Observe that for any $(y_{i_0}, z) \in \varphi(M_k)$ we have

$$\varphi^{-1}(\{(y_{i_0}, z)\}) = \left\{u \in C_{\delta_0}(y_{i_0}, z),\ s.t.,\ u \text{ is dominated by } f_{\lambda_0}(B_2(X))\right\}.$$

Therefore,

$$\sharp\varphi^{-1}(\{(y_{i_0}, z)\}) = \tau(y_{i_0}, z) = \tau,$$

since it is constant. Thus,

$$[\varphi(M_k), \varphi(\partial M_k)]_* = \tau\big[\{y_{i_0}\} \underset{\sigma_2}{\times} B_{y_0}, \{y_{i_0}\} \underset{\sigma_2}{\times} \partial B_{y_0}\big]_*. \tag{4.5}$$

By excision of $\overset{\circ}{B}_{y_0}$ in the pair $(X, \overline{B_{y_0}})$, we obtain that

$$\big[\{y_{i_0}\} \underset{\sigma_2}{\times} B_{y_0}, \{y_{i_0}\} \underset{\sigma_2}{\times} \partial B_{y_0}\big]_* = \big[\{y_{i_0}\} \underset{\sigma_2}{\times} X, \{y_{i_0}\} \underset{\sigma_2}{\times} \{y_0\}\big]_*.$$

It follows from (4.4) and (4.5) that

$$\begin{aligned}(\varphi^* \circ tr^* \circ p_1^*)[X]^*([M_k,\partial M_k]_*) &= \tau\big([X]^* \circ p_{1*} \circ tr_*\big)\big(\big[\{y_{i_0}\}_{\sigma_2} \times X, \{y_{i_0}\}_{\sigma_2} \times \{y_0\}\big]_*\big) \quad (4.6)\\ &= \tau\big([X]^* \circ p_{1*}\big)\big(\{y_{i_0}\} \times X \cup X \times \{y_{i_0}\}\big) = \tau[X]^*([X]_*)\\ &= \tau.\end{aligned}$$

From an other part,

$$\begin{aligned}(g^* \circ tr^* \circ p_1^*)[X]^*([M_k,\partial M_k]_*) &= \big([X]^* \circ p_{1*} \circ tr_* \circ g_*\big)\big(\big[M_k,\partial M_k\big]_*\big)\\ &= \big([X]^* \circ p_{1*} \circ tr_*\big)\big(\big[g(M_k), g(\partial M_k)\big]_*\big).\end{aligned}$$

For any $(y_{i_0}, z') \in g(M_k)$ we have

$$g^{-1}(\{y_{i_0}, z'\}) = \left\{ u \in \bigcup_{z \in B_{y_0}} C_{\delta_0}(y_{i_0}, z),\ s.t.,\ u \text{ is dominated by } f_{\lambda_0}(\mathcal{C}_{z'}(X)) \right\}.$$

Therefore,

$$\begin{aligned}\sharp g^{-1}(\{y_{i_0}, z'\}) &= \sharp W_u(f_{\lambda_0}(\mathcal{C}_{z'}(X))) \cdot \bigcup_{z \in B_{y_0}} C_{\delta_0}(y_{i_0}, z)\\ &= \sharp f_{\lambda_0}(\mathcal{C}_{z'}(X)) \cdot W_s(\bigcup_{z \in B_{y_0}} C_{\delta_0}(y_{i_0}, z))\\ &= \sharp f_{\lambda_0}(\mathcal{C}_{z'}(X)) \cdot W_s(y_0, y_{i_0})_\infty\\ &= \mu(z') = \mu,\end{aligned}$$

since $\mu(z')$ is constant for any $z' \in X$. It follows that

$$g(M_k) = \mu\big(\{y_{i_0}\} \times X \backslash B_{y_{i_0}}\big)$$

and

$$g(\partial M_k) = \mu\big(\{y_{i_0}\} \times \partial B_{y_{i_0}}\big).$$

Therefore,

$$\begin{aligned}(g^* \circ tr^* \circ p_1^*)[X]^*([M_k,\partial M_k]_*) &= \mu\big([X]^* \circ p_{1*} \circ tr_*\big)\big(\big[\{y_{i_0}\}_{\sigma_2} \times X\big]_*\big)\\ &= \mu\big([X]^* \circ p_{1*}\big)\big(\big[\{y_{i_0}\} \times X\big]_* + \big[X \times \{y_{i_0}\}\big]_*\big)\\ &= \mu[X]^*\big(\big[\{y_{i_0}\} \times X\big]_*\big) = \mu[X]^*([X]_*) = \mu. \quad (4.7)\end{aligned}$$

Lastly, we note that under (H_1)-condition, any flow line between $f_{\lambda_0}(B_2(X))$ and $\bigcup_{z \in B_{y_0}} C_{\delta_0}(y_{i_0}, z)$ lies in $V(2,\varepsilon)(X)$. Thus φ and g are two homotypes functions and therefore, $\varphi^* = g^*$, for any $* \in \mathbb{N}$. This with (4.6) and (4.7) yield that $\mu = \tau$. □

Remark. In a forth coming paper, we will discuss several possible alternative situations where condition (H_1) holds. For instance we think that if K is close to a positive constant on M^n, then (H_1) is satisfied.

This will give another proof of the recent result of [31] on Einstein manifolds and extend it to any closed-riemannian manifolds.

Bibliography

[1] M. Ahmedou and H. Chtioui, *Conformal metrics of prescribed scalar curvature on 4-manifolds: the degree zero case.* Arab. J. Math. (Springer) 6 (2017), no. 3, 127–136.

[2] A. Alghanemi and H.Chtioui, *Prescribing scalar curvaturs on n-dimensional manifolds*, $4 \leq n \leq 6$. C. R. Acad. Bulg. Sci., 73, No 2, (2020), 163 –169.

[3] A.Ambrosetti, A.Garcia and I.Peral, *Perturbation of* $\Delta u + u^{\frac{N+2}{N-2}=0}$*, the scalar curvature problem in* $\mathbb{R}^n$*, and related topics*. J. Funct. Anal. 165 (1999), no. 1, 117–149.

[4] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9), 55 (1976), 269-296.

[5] T. Aubin and A. Bahri, *Une hypothèse topologique pour le problème de la courbure scalaire prescrite*, (French) [A topological hypothesis for the problem of prescribed scalar curvature], J. Math. Pures Appl. (9), 76 (1997), 843-850.

[6] A.Bahri, *Critical point at infinity in some variational problems, Pitman Res. Notes Math*, Ser 182, Longman Sci. Tech. Harlow 1989.

[7] A. Bahri, *An invariant for yamabe-type ows with applications to scalar curvature problems in high dimensions*, A celebration of J. F. Nash Jr., Duke Math. J. 81 (1996), 323-466. NotesMath, Ser 182, Longman Sci. Tech. Harlow 1989.

[8] A. Bahri and J. M. Coron, *The scalar curvature problem on the standard three dimensional spheres*, J. Funct. Anal. 95 (1991), 106-172.

[9] M. Ben Ayed, Mohamed, H. Chtioui and M. Hammami, *The scalar-curvature problem on higher-dimensional spheres*. Duke Math. J. 93 (1998), no. 2, 379–424.

[10] R. Ben Mahmoud and H. Chtioui, *Existence results for the prescribed Scalar curvature on* S^3, Annales de l'Institut Fourier 61 (2011), 971-986.

[11] R. Ben Mahmoud and H. Chtioui, *Prescribing the Scalar Curvature Problem on Higher-Dimensional Manifolds*,Discrete and Continuous Dynamical Systems - Series A, 32, no. 5 (Mai 2012), 1857-1879.

[12] J. P. Bourguignon and J. P. Ezin, *Scalar curvature functions in conformal class of metric and conformal transformations*, Trans. AMS 301 (1987), 723-736.

[13] G. Bredon, *Introduction to compact transformation groups*, Academic Press , New York, 1972.

[14] S.-Y. Chang and P. Yang, *A perturbation result in prescribing scalar curvature on* S_n, Duke Math. J., 64 (1991), 27-69.

[15] S. A. Chang, M. J. Gursky and P. C. Yang, *The scalar curvature equation on 2 and 3 spheres*, Calc. Var. 1 (1993), 205-229.

[16] W. Chen and C. Li, *Prescribing scalar curvature on* S^n, Pacific J. of Math 199(2001), 61-78.

[17] W. Chen, C.Li, *A note on the Kazdan-Warner type conditions*, J. Differential Geom., 41, 2, (1995), 259-268.

[18] W. Chen, C.Li, emphGaussian curvature on singular surfaces, J. Geom. Anal., 3, 4, (1993), 315-334.

[19] C.C. Chen and C.S. Lin, *Prescribing the scalar curvature on* S^n, I. Apriori estimates J. Differential Geom. 57, (2001), 67-171.

[20] H. Chtioui, *Prescribing the Scalar Curvature Problem on Three and Four Manifolds, Advanced Nonlinear Studies*, 3, (2003), 457-470.

[21] H. Chtioui and A. Rigane, *On the prescribed Q-curvature problem*, Journal of Functional Analysis, 261, 10, (2011), 2999-3043

[22] H. Chtioui, R. Ben Mahmoud and D. A. Abuzaid, *Conformal transformation of metrics on the n-sphere*, Nonlinear Analysis: TMA, 82, (2013), 66-81.

[23] H. Chtioui, H. Hajaiej and M. Soula, *The scalar curvature problem on fourdimensional manifolds*, Commun. Pure Appl. Anal. , (2020), 19, (2): 723-746.

[24] H.Chtioui, *On the Chen-Lin conjecture for the prescribed scalar curvature problem*, arXiv:2009.06262 [math.DG]

[25] J. Kazdan and F. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures*, Ann. of Math (2) 101 (1975), 317-331.

[26] D. Koutroufiotis, *Gaussian curvature and conformal mapping*, J. Differential Geom., 7, (1972), 479-488.

[27] Y.Y. Li, *On Prescribing scalar curvature on S^3 and S^4*, C. R. Acad. Sci. Paris I 314 (1992), 55-59.

[28] Y.Y. Li, *Prescribing scalar curvature on S^n and related topics*, Part I, Journal of Differential Equations, 120 (1995), 319-410.

[29] Y.Y. Li, *Prescribing scalar curvature on Sn and related problems*. II. Existence and compactness. Comm. Pure Appl. Math. 49 (1996), no. 6, 541–597.

[30] A.Malchiodi, *The scalar curvature problem on S^n: an approach via Morse theory. Calc. Var. Partial Differential Equations*, 14 (2002), no. 4, 429–445.

[31] A. Malchiodi and M.Mayer: *Prescribing Morse scalar curvatures: pinching and Morse theory*, Comm. Pure Appl. Math., to appear.

[32] R. Schoen and D. Zhang, *Prescribed scalar curvature on the n-sphere, Calculus of Variations and Partial Differential Equations*, 4 (1996), 1-25.

[33] K. Sharaf, *On the prescribed scalar curvature problem on S^n*, Part1, Differential Geometry and its Application, 49, (2016), 423-446.

[34] S.Smale, *An infinite dimensional version of Sard s theorem*, Am. J. Math. 87, (1965), 861-866.

