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Existence and multiplicity of solutions for $\alpha(x)$-Kirchhoff Equation with indefinite weights

Existence et multiplicité de solution pour l'équation $\alpha(x)$-Kirchhoff à poids indéfinis

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ABSTRACT. In this paper, we investigate the existence of at least three weak solutions for a class of nonlocal elliptic equations with Navier boundary value conditions. The proof of our result uses the basic theory and critical point theory of variable exponential Lebesgue Sobolev spaces. Moreover a generalization of Corollary 1.1 in [21] is obtained.
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1 Introduction

This article is concerned by the obtention of at least three weak solutions for the following problem:

$$(\mathbf{P}_\lambda)\begin{cases} M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big)\Delta^2_{\alpha(x)}v = \lambda(F(x)v^{\beta(x)-2}v + G(x)v^{\delta(x)-2}v), & \text{in } \Omega, \\ v = \Delta v = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ and λ is a positive parameter, F and G are two functions such that F in $L^{l_1(x)}(\Omega)$ and G in $L^{l_2(x)}(\Omega)$, where $L^{l_1(x)}(\Omega)$ and $L^{l_2(x)}(\Omega)$ are a generalized Sobolev spaces. The functions $\alpha, \beta, \delta, l_1$ and $l_2 \in C(\overline{\Omega})$ verify some inequalities which will be stated later. M is a continuous function satisfying some conditions.

The operator $\Delta^2_{\alpha(x)}$ is defined as $\Delta^2_{\alpha(x)}v = \Delta(|\Delta v|^{\alpha(x)-2}\Delta v)$ and called the $\alpha(x)$-biharmonic operator. This operator is a natural generalization of the α-biharmonic operator $\Delta^2_\alpha v = \Delta(|\Delta v|^{\alpha-2}\Delta v)$, where $\alpha > 1$ is a real constant.

In [32], Zhang and Miao studied the problem

$$\begin{cases} \Big(a + b\int\limits_\Omega \frac{|\Delta v|^{\alpha(x)}}{\alpha(x)}dx\Big)\Delta^2_{\alpha(x)}v = f(x,v), & \text{in } \Omega, \\ v = \Delta v = 0, & \text{on } \partial\Omega, \end{cases}$$

where f is a nonlinear term in $C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ satisfy Carathéodory condition. The constants a and b satisfy $a>0$ and $b\geq 0$. In [32] the authors proved an existence and multiplicity result for the above nonlocal elliptic equation by using the basic theory and critical point theory of variable exponential Lebesgue Sobolev spaces.

Moreover, recently, Allali and Taarabti in [11] investigated the problem

$$-M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\Delta^2_{p(x)}u = f(x,u) \quad \text{in } \Omega,$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}=\frac{\partial}{\partial \nu}\left(|\Delta u|^{p(x)-2}\Delta u\right)=0 \quad \text{on } \partial\Omega.$$

Under suitable conditions and using variational approach and Krasnoselskii's genus theory, the authors proved the existence and multiplicity of solutions to the above problem. Using the same tools as in [11], Rezvazni, Alimohammady and Agheli in [28], established the existence of infinitely solutions for the problem

$$\begin{cases} -\left(a+b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) div\left(|\nabla u|^{p(x)-2}\nabla u\right)=\lambda|u|^{q(x)-2}u-g(x)|u|^{p(x)-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $a,b>0$ are constants, $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, λ is a positive real parameter and g is a continuous function and $p(x),q(x)$ are real continuous functions on $\bar{\Omega}$ with $1<q(x)<p(x)<p^*(x)=\frac{Np(x)}{N-p(x)}$ and $p(x)<N$ for all $x\in\bar{\Omega}$.

The problems with negative nonlocal terms and weight function were considered in [16]

$$\begin{cases} \left(a-b\int_\Omega \frac{|\Delta v|^{\alpha(x)}}{\alpha(x)}dx\right)\Delta^2_{\alpha(x)}v=\lambda V(x)|v|^{\beta(x)-2}v, & \text{in } \Omega,\\ v=\Delta v=0, & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

for the problem (1.1) the authors established the existence of at least three solutions. Their arguments is mountain pass theorem and Ekeland's principle.

According to the critical theorem of Bonanno and Marano [6] and variational methods, the authors in [21] provided the existence of at least three weak solutions for the following problem

$$\begin{cases} \Delta\left(a(x,\Delta v)\right)=\lambda(V_1(x)|v|^{\beta(x)-2}v+V_2(x)|v|^{\delta(x)-2}v) & \text{in } \Omega,\\ v=\Delta v=0 & \text{on } \partial\Omega,\end{cases}$$

where λ is a positive parameter, V_1 and V_2 are two functions such that : V_1 in $L^{l_1(x)}(\Omega)$ and V_2 in $L^{l_2(x)}(\Omega)$, where $L^{l_1(x)}(\Omega)$ and $L^{l_2(x)}(\Omega)$ are a generalized Sobolev spaces. The functions $\beta,\ \delta,\ l_1,\ l_2\in C(\overline{\Omega})$ satisfy some inequalities.

Recently, much attention has been paid to the study of problems involving elliptic equations with non-standard growth conditions and corresponding non-local equations, both from a purely mathematical point of view and for specific application. Specifically, they arise spontaneously in many different contexts, such as and not limited to electro-rheological fluids, elastic mechanics, porous medium models and biological systems [27, 29, 33, 1, 7]. We recall that such problems are proposed by Kirchhoff in [22] as an existence of the classical D'Alembert wave equations for free vibration of elastic strings. Note that elliptic and singular elliptic problems with $\alpha(x)$-Laplace operator can be found in [1, 2, 3, 4, 5, 8, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25, 26].

The problem $(\mathbf{P}_\lambda)$ is a new variant of the Dirichlet problem of type Kirchhoff [22] that has never been handled before. Problem $(\mathbf{P}_\lambda)$ is called nonlocal problem, which implies that the equations in $(\mathbf{P}_\lambda)$ are no longer pointwise. This leads to some mathematical difficulties makes research on such issues particularly interesting.

In the sequel, let $l(x) := \sup\{l > 0 \mid b(x,l) \subseteq \Omega\}$ for all $x \in \Omega$, where b is the ball centered at x and of radius l. It's easy to see that there is $x_0 \in \Omega$ such that $b(x_0, \tilde{R}) \subseteq \Omega$, where $\tilde{R} = \sup\limits_{x\in\Omega} l(x)$.

To prove our aim result, we assume the following conditions

$(\mathbf{H1})$ Assume that $F \in L^{l_1(x)}(\Omega)$ such that

$$F(x) := \begin{cases} \leq 0, & \text{for } x \in \Omega \setminus b(x_0, \tilde{R}), \\ \geq f, & \text{for } x \in b(x_0, \frac{\tilde{R}}{2}), \\ > 0, & \text{for } x \in b(x_0, \tilde{R}) \setminus b(x_0, \frac{\tilde{R}}{2}), \end{cases}$$

and $G \in L^{l_2(x)}(\Omega)$ such that

$$G(x) := \begin{cases} \leq 0, & \text{for } x \in \Omega \setminus b(x_0, \tilde{R}), \\ \geq g, & \text{for } x \in b(x_0, \frac{\tilde{R}}{2}), \\ > 0, & \text{for } x \in b(x_0, \tilde{R}) \setminus b(x_0, \frac{\tilde{R}}{2}), \end{cases}$$

where f and g are positive constants.

$(\mathbf{H2})$ $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and increasing function satisfying there exists $m_2 \geq m_1 > 0$ and $q \geq p > 1$ such that

$$m_1 t^{p-1} \leq M(t) \leq m_2 t^{q-1},\ \forall t \in \mathbb{R}^+ \text{ with } \quad \max(\sup_{x\in\Omega} \beta(x), \sup_{x\in\Omega} \delta(x)) < p \inf_{x\in\Omega} \alpha(x),$$

and

$$\sup_{x\in\overline{\Omega}} p(x) < \frac{N}{2} < \min(l_1(x), l_2(x)).$$

Our main tool is the following theorem, which we reformulate into a more convenient form.

Theorem 1. (BONANNO-MARANO *[[6], Theorem 3.6]*) *Let X be a reflexive real Banach space and $\Phi : X \to \mathbb{R}$ a coercive, continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X. Let $\Psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable functional whose Gateaux derivative is compact such that*

$$(a_0) \quad \inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that:

$$(a_1) \quad \frac{\sup\limits_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})};$$

(a_2) *for each* $\lambda \in \Lambda_r := \left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\limits_{\Phi(x)\le r} \Psi(x)}\right)$, *the functional $\Phi - \lambda\Psi$ is coercive.*

Then for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

The remainder of this paper is organized as follows. In the next section, we introduce the mathematical description of the variable exponent Lebesgue and Sobolev spaces. In Section 3, we give our main result and proofs.

2 Mathematical backgrounds

In this part, we need definitions and elementary properties of Sobolev spaces with nonstandard growth conditions $L^{\alpha(x)}(\Omega)$, $W^{2,\alpha(x)}(\Omega)$ and $W_0^{1,\alpha(x)}(\Omega)$ (more details, see [13] and [24]).

Set

$$C_+(\overline{\Omega}) := \{k : k \in C(\overline{\Omega}), k(x) > 1, \forall \ x \in \overline{\Omega}\}.$$

For $\theta > 0, \ k \in C_+(\overline{\Omega})$, let

$$k^- := \min_{x\in\Omega} k(x), \qquad k^+ := \max_{x\in\Omega} k(x),$$

and

$$[\theta]^k := \max\{\theta^{k^-}, \theta^{k^+}\}, \qquad [\theta]_k := \min\{\theta^{k^-}, \theta^{k^+}\}.$$

Remark 2.1. *One can see easily* $[\theta]^{\frac{1}{k}} := \max\{\theta^{\frac{1}{k^+}}, \theta^{\frac{1}{k^-}}\}$ *and* $[\theta]_{\frac{1}{k}} := \min\{\theta^{\frac{1}{k^+}}, \theta^{\frac{1}{k^-}}\}$.

Let $\alpha \in C_+(\overline{\Omega})$, denote by

$$L^{\alpha(x)}(\Omega) = \{v : v \text{ is a measurable real-valued function with } \int_\Omega |v(x)|^{\alpha(x)}dx < \infty\}.$$

This space is equipped with the Luxemburg

$$|v|_{\alpha(x)} = inf\{\mu > 0 : \int_\Omega |\frac{v(x)}{\mu}|^{\alpha(x)}dx \le 1\}.$$

We recall that the above normed space is a Banach spaces, reflexive if and only if $1 < \alpha^- \leq \alpha^+ < \infty$. Besides, if α_1, α_2 satisfies $\alpha_1(x) \leq \alpha_2(x)$ a.e. $x \in \Omega$ then the embedding $L^{\alpha_2(x)}(\Omega) \hookrightarrow L^{\alpha_1(x)}(\Omega)$ is continous. For every $v \in L^{\alpha(x)}(\Omega)$ and $w \in L^{\alpha'(x)}(\Omega)$ with $\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1$, the Hölder inequality holds true (See [13] and [23])

$$|\int_\Omega vwdx| \leq (\frac{1}{\alpha^-} + \frac{1}{(\alpha')^-})|v|_{\alpha(x)}|w|_{\alpha'(x)}. \tag{2.1}$$

On this space, we define the modular on $L^{\alpha(x)}(\Omega)$ which is a map $\rho_{\alpha(x)} : L^{\alpha(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{\alpha(x)}(v) := \int_\Omega |v|^{\alpha(x)}dx$$

and satisfy some useful properties cited below.

Lemma 2.1. (See [23]) *For all $v \in L^{\alpha(x)}(\Omega)$, we have*

1. $|v|_{\alpha(x)} < 1 \ (resp. = 1, > 1) \Leftrightarrow \rho_{\alpha(x)}(v) < 1 \ (resp. = 1, > 1)$.
2. $\left[|v|_{\alpha(x)}\right]_\alpha \leq \rho_{\alpha(x)}(v) \leq \left[|v|_{\alpha(x)}\right]^\alpha$.
3. $\rho_{\alpha(x)}(v_n - v) \to 0 \Leftrightarrow |v_n - v|_{\alpha(x)} \to 0$.

Proposition 2.2. (See [10]) *Let α and β be two measurable functions with $\alpha \in L^\infty(\Omega)$ and $1 \leq \alpha(x)\beta(x) \leq \infty$ for a.e.$x \in \Omega$. If $v \in L^{\beta(x)}(\Omega)$ with $v \neq 0$, then we have*

$$\left[|v|_{\alpha(x)\beta(x)}\right]_\alpha \leq ||v|^{\alpha(x)}|_{\beta(x)} \leq \left[|v|_{\alpha(x)\beta(x)}\right]^\alpha.$$

For any positive integer m, the Lebesgue-sobolev space $W^{m,\alpha(x)}(\Omega)$ is defined as follows

$$W^{m,\alpha(x)}(\Omega) := \left\{v \in L^{\alpha(x)}(\Omega) \mid D^p v \in L^{\alpha(x)}(\Omega), |p| \leq m\right\},$$

where $p = (p_1, p_2, ..., p_N)$ is a multi-index and

$$|p| = \sum_{i=1}^N p_i, \ D^p v = \frac{\partial^{|p|}v}{\partial^{p_1}x_1....\partial^{p_N}x_N},$$

We equip this space with

$$\|v\|_{m,\alpha(x)} := \sum_{|p|\leq m} |D^p v|_{\alpha(x)}.$$

We define also $W_0^{m,\alpha(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,\alpha(x)}(\Omega)$ with respect to the norm $\|.\|_{m,\alpha(x)}$. We recall the following

Proposition 2.3. (See [23])

1. *$W^{m,\alpha(x)}(\Omega)$ is separable reflexive Banach space.*

2. *If $\beta \in C_+(\overline{\Omega})$ is such that $\beta(x) < \alpha^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{m,\alpha(x)}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega)$ is compact and continuous.*

So, $W^{2,\alpha(x)}(\Omega)$ and $W_0^{1,\alpha(x)}(\Omega)$ are separable and reflexive Banach spaces. And

$$E := W^{2,\alpha(x)}(\Omega) \cap W_0^{1,\alpha(x)}(\Omega),$$

is also a separable and reflexive Banach space, when equipped with the norm $\|v\|_E = \|v\|_{W^{2,\alpha(x)}(\Omega)} + \|v\|_{W_0^{1,\alpha(x)}(\Omega)}$, thus $\|v\| = |v|_{\alpha(x)} + |\nabla v|_{\alpha(x)} + \sum_{|p|=2} |D^p v|_{\alpha(x)}$. In Zang and Fu [30], the equivalence of the norms was proved, and it was even proved that the norm $|\triangle v|_{\alpha(x)}$ is equivalent to the norm $\|v\|$ (see [30], Theorem 4.4). Note that $(E, \|.\|)$ is a separable and reflexive Banach space.

In the sequel define $\rho_{\alpha(x)} : E \to \mathbb{R}$ by

$$\rho_{\alpha(x)}(v) := \int_{\Omega} |\Delta v|^{\alpha(x)} dx,$$

one has

Lemma 2.2. *For all $v \in L^{\alpha(x)}(\Omega)$, we have*

1. $\|v\| < 1$ $(resp. = 1, > 1) \Leftrightarrow \rho_{\alpha(x)}(v) < 1$ $(resp. = 1, > 1)$.
2. $[\|v\|]_{\alpha} \leq \rho_{\alpha(x)}(v) \leq [\|v\|]^{\alpha}$.
3. $\rho_{\alpha(x)}(v_n - v) \to 0 \Leftrightarrow \|v_n - v\| \to 0$.

Now, we recall a proposition which will be needed later

Proposition 2.4. (See [14]) *If E is a reflexive Banach space, Y is a Banach space, $Z \subset E$ is nonempty, closed and convex and $J : Z \to Y$ is completely continuous, then J is compact.*

3 The main result and proof

In the sequel, let

$$S := T(\tilde{R}^N - (\frac{\tilde{R}}{2})^N), \; T := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})},$$

here Γ is the Euler function. Let $C_1, C_2 > 0$ are the best constants for which the inequality (3.6) below holds.

Our main result is as follows.

Theorem 3.1. *Under the assumptions* $(\mathbf{H1})$ *and* $(\mathbf{H2})$ *, there exist $R > 0$ and $d > 0$ such that*

$$R < \frac{m_1}{p(\alpha^+)^p} \left[\frac{2d(N-1)}{\tilde{R}^2}\right]_{\alpha p} S^p, \tag{3.1}$$

and

$$\overline{T}_R := \frac{1}{R}\Big\{\frac{(\alpha^+)^{\frac{\beta^+}{\alpha^-}}}{\beta^-}[C_1]^\beta|F|_{l_1(x)}\Big[[R]^{\frac{1}{\alpha}}\Big]^\beta + \frac{(\alpha^+)^{\frac{\delta^+}{\alpha^-}}}{\delta^-}[C_2]^\delta|G|_{l_2(x)}\Big[[R]^{\frac{1}{\alpha}}\Big]^\delta\Big\} \tag{3.2}$$

$$< \quad \gamma_d := \frac{\big(\frac{1}{\beta^+}f[d]_\beta + \frac{1}{\delta^+}g[d]_\delta\big)T(\frac{\tilde{R}}{2})^N}{\frac{m_2}{q(\alpha^-)^q}\Big[\frac{4d(N-1)}{\tilde{R}^2}\Big]^{\alpha q}S^q}.$$

Then for every $\lambda \in \overline{\Lambda}_R := \Big(\frac{1}{\gamma_d}, \frac{1}{\overline{T}_R}\Big)$, *problem* $(\boldsymbol{P}_\lambda)$ *have at least three weak solutions.*

Corollary 3.2. *Assume that* $(\mathbf{H1})$ *and* $(\mathbf{H2})$ *are fulfilled, then there exists* $R, d > 0$ *with*

$$R < \frac{1}{\alpha^+}\Big[\frac{2d(N-1)}{\tilde{R}^2}\Big]_\alpha S, \tag{3.3}$$

and

$$\overline{T}_R := \frac{1}{R}\Big\{\frac{(\alpha^+)^{\frac{\beta^+}{\alpha^-}}}{\beta^-}[C_1]^\beta|F|_{l_1(x)}\Big[[R]^{\frac{1}{\alpha}}\Big]^\beta + \frac{(\alpha^+)^{\frac{\delta^+}{\alpha^-}}}{\delta^-}[C_2]^\delta|G|_{l_2(x)}\Big[[R]^{\frac{1}{\alpha}}\Big]^\delta\Big\} \tag{3.4}$$

$$\tag{3.5}$$

$$< \quad \gamma_d := \frac{\big(\frac{1}{\beta^+}f[d]_\beta + \frac{1}{\delta^+}g[d]_\delta\big)T(\frac{\tilde{R}}{2})^N}{\frac{1}{\alpha^-}\Big[\frac{4d(N-1)}{\tilde{R}^2}\Big]^{\alpha}S}.$$

Then for every $\lambda \in \overline{\Lambda}_R := \Big(\frac{1}{\gamma_d}, \frac{1}{\overline{T}_R}\Big)$, *problem*

$$\begin{cases} \Delta^2_{\alpha(x)}v = \lambda(F(x)v^{\beta(x)-2}v + G(x)v^{\delta(x)-2}v), & \text{in } \Omega, \\ v = \Delta v = 0, & \text{on } \partial\Omega, \end{cases}$$

have at least three weak solutions.

Remark 3.3. *Note that corollary 3.2 is identical to corollary 1.1 obtained by Kefi et al in* [21].

Remark 3.4. (see [17]) *Let* $l_1'(x), l_2'(x)$, *the conjugate exponents of the functions* $l_1(x), l_2(x)$ *respectively and let* $\eta_1(x) := \frac{l_1(x)\beta(x)}{l_1(x)-\beta(x)}$, $\eta_2(x) := \frac{l_2(x)\delta(x)}{l_2(x)-\delta(x)}$. *Then there exist compact and continuous embedding* $E \hookrightarrow L^{l_1'(x)\beta(x)}(\Omega)$, $E \hookrightarrow L^{l_2'(x)\delta(x)}(\Omega)$, $E \hookrightarrow L^{\eta_1(x)}(\Omega)$ *and* $E \hookrightarrow L^{\eta_2(x)}(\Omega)$ *and the best constants* $C_1,\ C_2 > 0$ *with*

$$|v|_{l_1'(x)\beta(x)} \le C_1\|v\| \ \text{ and } \ |v|_{l_2'(x)\delta(x)} \le C_2\|v\|. \tag{3.6}$$

Note that a weak solution of problem $(\mathbf{P}_\lambda)$ satisfy the following definition

Definition 3.5. *We say that $v \in E$ is weak solution of $(\boldsymbol{P}_\lambda)$ if*

$$M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big)\int\limits_\Omega(|\Delta v|^{\alpha(x)-2}\Delta v\Delta w)dx = \lambda\int\limits_\Omega(F(x)|v|^{\beta(x)-2}+G(x)|v|^{\delta(x)-2})vwdx$$

for any $w \in E$.

The energy functional corresponding to problem $(\mathbf{P}_\lambda)$ is defined on E as:

$$J_\lambda(v) = I(v) - \lambda\psi(v),$$

where

$$I(v) = \widehat{M}\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big) \quad \text{with} \quad \widehat{M}(t) = \int\limits_0^t M(s)ds$$

and

$$\psi(v) = \int\limits_\Omega \Big(\frac{F(x)}{\beta(x)}|v|^{\beta(x)} + \frac{G(x)}{\delta(x)}|v|^{\delta(x)}\Big)dx.$$

It is obviously that the functional I is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $v \in E$ is the functional $I'(v) \in E^*$, given by

$$\langle I'(v), w\rangle = M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big)\int\limits_\Omega(|\Delta v|^{\alpha(x)-2}\Delta v\Delta w)dx.$$

By Proposition 2.2 and inequality (3.6), we have

$$\begin{aligned}|\psi(v)| &\leq \frac{1}{\beta^-}|F(x)|_{l_1(x)}[|v|_{l_1'(x)\beta(x)}]^\beta + \frac{1}{\delta^-}|G(x)|_{l_2(x)}[|v|_{l_2'(x)\delta(x)}]^\delta \\ &\leq \frac{1}{\beta^-}|F(x)|_{l_1(x)}[C_1\|v\|]^\beta + \frac{1}{\delta^-}|G(x)|_{l_2(x)}[C_2\|v\|]^\delta.\end{aligned}$$

therefore ψ is indeed well-defined. Finally, using $(\mathbf{H2})$, J_λ is well-defined.

In the sequel, we shall need the following lemma.

Lemma 3.1. (i) *The functional I is coercive on E and $I' : E \to E^*$ is a strictly monotone homeomorphism.*
(ii) *$\psi' : E \to E^*$ is completely continuous.*

Proof.

(i) It is clear from Lemma 2.2 and hypothesis $(\mathbf{H2})$ that for every $v \in E$ such that $\|v\| > 1$, one has

$$I(v) = \widehat{M}\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big) \geq \frac{m_1}{p(\alpha^+)^{p-1}}\|v\|^{p\alpha^-}, \tag{3.7}$$

and thus I is coercive.

For the rest of the proof of Lemma 3.1, we can use the same idea given by Dai in [9]. So the functional I' is continuous because M is continuous. Without loss of generality and for any $v,\ w \in E$ with $v \neq w$, we assume that $\int_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx \geq \int_\Omega \frac{1}{\alpha(x)}|\Delta w|^{\alpha(x)}dx$. Since $M(t)$ is a monotone function, we have

$$M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big) \geq M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta w|^{\alpha(x)}dx\Big). \tag{3.8}$$

By the Cauchy's inequality, we have

$$\Delta v \Delta w \leq |\Delta v||\Delta w| \leq \frac{|\Delta v|^2 + |\Delta w|^2}{2}. \tag{3.9}$$

Using (3.9) and Young's inequality, we can easily obtain (more details, see [9])

$$\int\limits_\Omega |\Delta v|^{\alpha(x)-2}|\Delta v|^2 dx + \int\limits_\Omega |\Delta w|^{\alpha(x)-2}|\Delta w|^2 dx \leq \int\limits_\Omega \Big(|\Delta v|^{\alpha(x)} + |\Delta w|^{\alpha(x)}\Big)dx. \tag{3.10}$$

Therefore, using (3.8) and (3.10), we obtain

$$\begin{aligned}
\langle I'(v) - I'(w), v - w\rangle &= \langle I'(v), v\rangle - \langle I'(v), w\rangle - \langle I'(w), v\rangle + \langle I'(w), w\rangle \\
&= M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big)\int\limits_\Omega |\Delta v|^{\alpha(x)}dx \\
&\quad -M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big)\int\limits_\Omega (|\Delta v|^{\alpha(x)-2}\Delta v \Delta w)dx \\
&\quad -M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta w|^{\alpha(x)}dx\Big)\int\limits_\Omega (|\Delta w|^{\alpha(x)-2}\Delta v \Delta w)dx \\
&\quad +M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta w|^{\alpha(x)}dx\Big)\int\limits_\Omega |\Delta w|^{\alpha(x)}dx \\
&\geq M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta w|^{\alpha(x)}dx\Big)\Big(\int\limits_\Omega \frac{1}{2}(|\Delta v|^{\alpha(x)-2} - |\Delta w|^{\alpha(x)-2})(|\Delta v|^2 - |\Delta w|^2)dx\Big) \\
&\geq 0,
\end{aligned}$$

which implies, I' is monotone. It remains to show that I' is strictly monotone. Indeed, if $\langle I'(v) - I'(w), v - w\rangle = 0$, then we have

$$\int\limits_\Omega \frac{1}{2}(|\Delta v|^{\alpha(x)-2} - |\Delta w|^{\alpha(x)-2})(|\Delta v|^2 - |\Delta w|^2)dx = 0,$$

so $|\Delta v| = |\Delta w|$. Thus, we obtain

$$\begin{aligned}
\langle I'(v) - I'(w), v - w\rangle &= \langle I'(v), v - w\rangle - \langle I'(w), v - w\rangle \\
&= M\Big(\int\limits_\Omega \frac{1}{\alpha(x)}|\Delta v|^{\alpha(x)}dx\Big)\Big(\int\limits_\Omega |\Delta v|^{\alpha(x)-2}(\Delta v - \Delta w)^2 dx\Big) \\
&= 0,
\end{aligned}$$

i.e. $v-w$ is a constant. In view of $v=w=0$ on $\partial\Omega$, we have $v\equiv w$, which is contrary with $u\neq w$. Therefore, $\langle I'(v)-I'(w), v-w\rangle>0$. Which implies that I' is a strictly monotone operator in E. From the strict monotonicity of I', note that I' is an injection. By the condition (**H2**) it is clear that for any $v\in X$ with $\|v\|>1$, one has

$$\frac{\langle I'(v),v\rangle}{\|v\|}\geq\frac{\left(\frac{m_1\|v\|^{p\alpha^-}}{p(\alpha^+)^{p-1}}\right)\int_\Omega|\Delta v|^{\alpha(x)}dx}{\|v\|}=\left(\frac{m_1}{p(\alpha^+)^{p-1}}\right)\|v\|^{(p+1)\alpha^--1},$$

So I' is coercive, thus I' is a sujection in view of Minty-Browder theorem [31]. Therefore I' has an inverse mapping $(I')^{-1}:E^*\longrightarrow E$.

Hence the continuity of $(I')^{-1}$ is sufficient to ensure I' to be homeomorphism.

(ii) We prove that ψ' is compact. Let $v_n\rightharpoonup v$ in E then v_n converge uniformly to v on Ω (see [31]). Furthermore

$$\begin{aligned}|\langle\psi'(v_n)-\psi'(v),w\rangle| &\leq \int_\Omega|F(x)||v_n|^{\beta(x)-1}-|v|^{\beta(x)-1}||w|dx+\int_\Omega|G(x)||v_n|^{\delta(x)-1}-|v|^{\delta(x)-1}||w|dx\\ &\leq |F(x)|_{l_1(x)}||v_n|^{\beta(x)-1}-|v|^{\beta(x)-1}|_{\frac{\beta(x)}{\beta(x)-1}}|w|_{\eta_1(x)}\\ &+ |G(x)|_{l_2(x)}||v_n|^{\delta(x)-1}-|v|^{\delta(x)-1}|_{\frac{\delta(x)}{\delta(x)-1}}|w|_{\eta_2(x)}.\end{aligned}$$

Using Lemma 1 in [18] and Remark 3.4, one has

$$|\langle\psi'(v_n)-\psi'(v),w\rangle|\to 0,\qquad n\to+\infty.$$

Which implies ψ' is completely continuous. By Proposition 2.4, ψ' is compact.

□

3.1 Proof of Theorem 3.1

Since the functionals I and ψ satisfy the assumptions mentioned in Theorem 1. We can start our proof. In fact, let the function $V_d\in E$ defined by

$$V_d:=\begin{cases}0, & \text{if } x\in\Omega\setminus b(x_0,\tilde R),\\ \frac{2d}{\tilde R}(\tilde R-|x-x_0|), & \text{if } x\in b(x_0,\tilde R)\setminus b(x_0,\frac{\tilde R}{2}),\\ d, & \text{if } x\in b(x_0,\frac{\tilde R}{2}),\end{cases}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^N$. We can see that

$$\Delta V_d=\begin{cases}0, & \text{if } x\in\Omega\setminus b(x_0,\tilde R)\cup b(x_0,\frac{\tilde R}{2}),\\ \frac{-2d(N-1)}{\tilde R|x-x_0|}, & \text{if } x\in b(x_0,\tilde R)\setminus b(x_0,\frac{\tilde R}{2}).\end{cases}$$

Using Lemma 2.1 and the condition $(\mathbf{H2})$, deduce that

$$\frac{m_1}{p(\alpha^+)^p}\Big[\frac{2d(N-1)}{\tilde{R}^2})\Big]_{\alpha p}S^p \le I(V_d) \le \frac{m_2}{q(\alpha^-)^q}\Big[\frac{4d(N-1)}{\tilde{R}^2}\Big]^{\alpha q}S^q, \tag{3.11}$$

$$\begin{aligned}\psi(V_d) &\ge \int\limits_{b(x_0,\frac{\tilde{R}}{2})}\frac{F(x)}{\beta(x)}|V_d|^{\beta(x)}dx + \int\limits_{b(x_0,\frac{\tilde{R}}{2})}\frac{G(x)}{\delta(x)}|V_d|^{\delta(x)}dx \\ &\ge \Big(\frac{1}{\beta^+}f[d]_\beta + \frac{1}{\delta^+}g[d]_\delta\Big)T\Big(\frac{\tilde{R}}{2}\Big)^N.\end{aligned}$$

and hence

$$\frac{\psi(V_d)}{I(V_d)} \ge \frac{\Big(\frac{1}{\beta^+}f[d]_\beta + \frac{1}{\delta^+}g[d]_\delta\Big)T\Big(\frac{\tilde{R}}{2}\Big)^N}{\frac{m_2}{q(\alpha^-)^q}\Big[\frac{4d(N-1)}{\tilde{R}^2}\Big]^{\alpha q}S^q} = \gamma_d.$$

Next, from $R < \frac{m_1}{p(\alpha^+)^p}\Big[\frac{2d(N-1)}{\tilde{R}^2})\Big]_{\alpha p}S^p$, we get $R < I(V_d)$. Now, for each $v \in I^{-1}(]-\infty, R])$, due to condition $(\mathbf{H_2})$, one has that

$$\frac{1}{\alpha^+}\big[\|v\|\big]_\alpha \le R. \tag{3.12}$$

By proposition 2.2, inequalities (3.12) and (3.6) we have

$$\begin{aligned}\psi(v) &\le \frac{1}{\beta^-}|F|_{l_1(x)}\big[C_1\|v\|\big]^\beta + \frac{1}{\delta^-}|G|_{l_2(x)}\big[C_2\|v\|\big]^\delta \\ &\le \frac{1}{\beta^-}|F|_{l_1(x)}[C_1]^\beta\big[(\alpha^+)^{\frac{1}{\alpha^-}}[R]^{\frac{1}{\alpha}}\big]^\beta + \frac{1}{\delta^-}|G|_{l_2(x)}[C_2]^\delta\big[(\alpha^+)^{\frac{1}{\alpha^-}}[R]^{\frac{1}{\alpha}}\big]^\delta \\ &\le \frac{(\alpha^+)^{\frac{\beta^+}{\alpha^-}}}{\beta^-}[C_1]^\beta|F|_{l_1(x)}\big[[R]^{\frac{1}{\alpha}}\big]^\beta + \frac{(\alpha^+)^{\frac{\delta^+}{\alpha^-}}}{\delta^-}[C_2]^\delta|G|_{l_2(x)}\big[[R]^{\frac{1}{\alpha}}\big]^\delta.\end{aligned} \tag{3.13}$$

Therefore

$$\frac{1}{R}\sup_{I(v)\le R}\psi(v) \le \overline{T}_R.$$

After that, we will prove that for any $\lambda > 0$, $I - \lambda\psi$ is coercive. In fact by using Remark 3.4, one has

$$\psi(v) \le \frac{1}{\beta^-}|F|_{l_1(x)}[C_1\|v\|]^\beta + \frac{1}{\delta^-}|G|_{l_2(x)}[C_2\|v\|]^\delta. \tag{3.14}$$

For $\|v\| > 1$ and using the relations (3.7) and (3.14), we get

$$I(v) - \lambda\psi(v) \ge \frac{m_1}{p(\alpha^+)^p}\|v\|^{p\alpha^-} - \lambda\Big(\frac{1}{\beta^-}|F|_{l_1(x)}[C_1\|v\|]^\beta + \frac{1}{\delta^-}|G|_{l_2(x)}[C_2\|v\|]^\delta\Big).$$

Since $\max(\beta^+,\ \delta^+) < p\alpha^-$, so $I - \lambda\psi$ is coercive and due to

$$\overline{\Lambda} := \Big(\frac{1}{\gamma_d}, \frac{1}{\overline{T}_R}\Big) \subseteq \Big(\frac{I(V_d)}{\psi(V_d)}, \frac{R}{\sup\limits_{I(v)\le R}\psi(v)}\Big),$$

Theorem 1 asserts that for any $\lambda \in \overline{\Lambda}_R$, $I - \lambda\psi$ have at least three critical points in E which are weak solutions for problem $(\mathbf{P}_\lambda)$ and this completes the proof. □

3.2 Proof of Corollary 3.2

Put $m_1 = m_2 = 1$ and $t := \int\limits_{\Omega} \frac{1}{\alpha(x)} |\Delta v|^{\alpha(x)} dx = 1$, so due to assertion $(\mathbf{H2})$, one has $M(t) = 1$ and consequently $\widehat{M}(t) = t$. Inequality (3.7) becomes

$$I(v) = \widehat{M}\Big(\int\limits_{\Omega} \frac{1}{\alpha(x)} |\Delta v|^{\alpha(x)} dx \Big) \geq \frac{1}{\alpha^+} \|v\|^{\alpha^-},$$

with $max(\beta^+, \delta^+) < \alpha^-$ which guarantees the coercivity of the functional I and $I - \lambda\psi$. Moreover inequality (3.11) becomes

$$\frac{1}{\alpha^+} \Big[\frac{2d(N-1)}{\tilde{R}^2}) \Big]_{\alpha} S \leq I(V_d) \leq \frac{1}{\alpha^-} \Big[\frac{4d(N-1)}{\tilde{R}^2} \Big]^{\alpha} S.$$

Finally a simple calculation achieve the proof of Corollary 3.2.

3.3 Final comments

(i) The problem $(\mathbf{P}_\lambda)$ corresponds to the *subcritical* setting described in Remarks 3.4. We believe that worthy research directions fit either into the *critical* or *supercritical* framework (in the Sobolev variable exponent sense). Even in the case of *almost-critical* with *lack of compactness* the consequences are unknown. More precisely, in the same notation as Remark 3.4, it is a very interesting open problem to study the qualitative analysis of the solution of $(\mathbf{P}_\lambda)$ provided that there exists $z_1,\ z_2 \in \Omega$ such that

$$\max(r_1(z_1), s_1'(z_1) q(z_1)) = p^*(z_1) \quad \text{and} \quad \max(r_2(z_2) s_2'(z_2) \alpha(z_2)) = p^*(z_2)$$

but

$$\max(r_1(x), s_1'(x) q(x)) < p^*(x) \quad \text{for all } z \in \Omega \setminus \{z_1\}$$

and

$$\max(r_2(x) s_2'(x) \alpha(x)) < p^*(x) \quad \text{for all } z \in \Omega \setminus \{z_2\}.$$

(ii) Another very interesting research direction is to extend the approach developed in this paper to the operators who describe the capillary phenomenon. More precisely it's intriguing to treat operators of type

$$-\text{div}\Big((1 + \frac{|\nabla v|^{p(x)}}{\sqrt{1 + |\nabla v|^{2p(x)}}}) |\nabla v|^{p(x)-2} w \Big) + b(x) |v|^{p(x)-2} w,$$

provided that $p^+ < N$ and a source term which contain indefinite weights especially since until now there was no work that has been completed in this direction.

Declaration

Competing interests

Not applicable

Data Availability Statement

Not applicable

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