

Hereditary classes of ordered binary structures

Classes héréditaires des structures binaires ordonnées

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ABSTRACT. Balogh, Bollobás and Morris (2006) have described a threshold phenomenon in the behavior of the profile of hereditary classes of ordered graphs. In this paper, we give an other look at their result based on the notion of monomorphic decomposition of a relational structure introduced in [32]. We prove that the class \mathfrak{G} of ordered binary structures which do not have a finite monomorphic decomposition has a finite basis (a subset \mathfrak{A} such that every member of \mathfrak{G} embeds some member of \mathfrak{A}). In the case of ordered reflexive directed graphs, the basis has 1242 members and the profile of their ages grows at least as the Fibonacci function. From this result, we deduce that the following dichotomy property holds for every hereditary class \mathfrak{C} of finite ordered binary structures of a given finite type. Either there is an integer ℓ such that every member of \mathfrak{C} has a monomorphic decomposition into at most ℓ blocks and in this case the profile of \mathfrak{C} is bounded by a polynomial of degree $\ell - 1$ (and in fact is a polynomial), or \mathfrak{C} contains the age of a structure which does not have a finite monomorphic decomposition, in which case the profile of \mathfrak{C} is bounded below by the Fibonacci function.

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1. Introduction and presentation of the results

The framework of this paper is the theory of relations. It is about a counting function, the profile. The *profile* of a class \mathfrak{C} of finite relational structures (also called the *speed* by other authors) is the integer function $\varphi_{\mathfrak{C}}$ which counts for each non negative integer n the number $\varphi_{\mathfrak{C}}(n)$ of members of \mathfrak{C} defined on n elements, isomorphic structures being identified. For the last fifteen years, the behavior of this function has been discussed in many papers, particularly when \mathfrak{C} is *hereditary* (that is contains every substructure of any member of \mathfrak{C}) and is made of graphs (directed or not), of tournaments, of ordered sets, of ordered graphs and of ordered hypergraphs. As observed by P. J. Cameron [7], numerous results obtained about classes of permutations obtained during the same period fall under the frame of the profile of hereditary classes of relational structures, namely *bichains*, that is structures made of two linear orders on the same set. The results show that the profile cannot be arbitrary: there are *jumps* in its possible growth rates. Typically, its growth is polynomial or faster than every polynomial ([28] for ages, see [30] for a survey) and for several classes of structures, it is either at least exponential (e.g. for tournaments [3, 5], ordered graphs and hypergraphs [1, 2, 17] and permutations [16]) or at least with the growth of the partition function (e.g. for graphs [4]). For more, see the survey of Klazar [18] and for permutations the survey of Vatter [34].

This paper is motivated by Balogh, Bollobás and Morris results about the profile of hereditary classes of ordered graphs. They show in [2] that if \mathfrak{C} is a hereditary class of finite ordered graphs then its profile $\varphi_{\mathfrak{C}}$ is either polynomial or is ranked by the Fibonacci functions.

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Their theorem states:

Theorem 1.1. *If \mathcal{C} is a hereditary class of finite ordered graphs, then one of the following assertions holds.*

- (a) $\varphi_{\mathcal{C}}(n)$ is bounded above and there exist $M, N \in \mathbb{N}$ such that $\varphi_{\mathcal{C}}(n) = M$ for every $n \geq N$.
- (b) $\varphi_{\mathcal{C}}(n)$ is a polynomial in n . There exist $k \in \mathbb{N}$ and integers a_0, \dots, a_k such that, $\varphi_{\mathcal{C}}(n) = \sum_{i=0}^k a_i \binom{n}{i}$ for all sufficiently large n , and $\varphi_{\mathcal{C}}(n) \geq n$ for every $n \in \mathbb{N}$.
- (c) $F_{n,k} \leq \varphi_{\mathcal{C}}(n) \leq p(n)F_{n,k}$ for every $n \in \mathbb{N}$, for some $2 \leq k \in \mathbb{N}$ and some polynomial p , so in particular $\varphi_{\mathcal{C}}(n)$ is exponential.
- (d) $\varphi_{\mathcal{C}}(n) \geq 2^{n-1}$ for every $n \in \mathbb{N}$.

Here, $F_{n,k}$ denote the n^{th} generalized Fibonacci number of order k , defined by $F_{n,k} = 0$ if $n < 0$, $F_{0,k} = 1$ and $F_{n,k} = F_{n-1,k} + F_{n-2,k} + \dots + F_{n-k,k}$ for every $n \geq 1$. For $k = 2$, $F_{n,2}$ is the Fibonacci number F_n .

Their result extends Kaiser and Klazar result for classes of permutations (see [16]).

In this paper, we give an other look at Balogh, Bollobás and Morris result using notions of the theory of relations and notably the notion of monomorphic decomposition of a relational structure (described in Pouzet and Thiéry 2013 [32]).

Our technique allows to characterize the hereditary classes of ordered directed graphs and more generally ordered binary structures which have a polynomially bounded profile. It gives the jump of the profile between polynomials and the ordinary Fibonacci function $F_{-,2}$ but does not gives the hierarchy given in (c) and (d) of Theorem 1.1.

We recall that a *monomorphic decomposition* of a relational structure \mathcal{R} defined on a set V is a partition $(V_i)_{i \in I}$ of V such that the induced structures $\mathcal{R}_{\upharpoonright A}$ and $\mathcal{R}_{\upharpoonright A'}$ on two finite subsets A, A' of V are isomorphic provided that the sets $A \cap V_i$ and $A' \cap V_i$ have the same cardinality for each $i \in I$. We also recall that the *age* of a relational structure \mathcal{R} is the set $\text{Age}(\mathcal{R})$ of finite induced substructures of \mathcal{R} considered up to isomorphism. We will call *profile of \mathcal{R}* , denoted by $\varphi_{\mathcal{R}}$, the profile of its age $\text{Age}(\mathcal{R})$.

We prove first a dichotomy result about this notion:

Theorem 1.2. *Let \mathcal{C} be a hereditary class of finite relational structures with a fixed finite signature. Then*

1. *Either \mathcal{C} is a finite union of ages of relational structures, each admitting a finite monomorphic decomposition.*
2. *Or \mathcal{C} contains the age \mathcal{D} of a structure \mathcal{R} that does not have a finite monomorphic decomposition, this age being minimal with this property.*

From this follows easily a characterization of classes satisfying the first item of Theorem 1.2.

Corollary 1.3. *A class \mathcal{C} is a finite union of ages of relational structures that have a finite monomorphic decomposition if and only if there is an integer ℓ such that every member of \mathcal{C} has a monomorphic decomposition into at most ℓ blocks.*

Clearly, the profile $\varphi_{\mathcal{R}}$ of a structure \mathcal{R} admitting a monomorphic decomposition is bounded above by a polynomial (in fact, $\varphi_{\mathcal{R}}(n) \leq \binom{n+\ell-1}{\ell-1}$, where ℓ is the number of blocks of the monomorphic decomposition). Hence, if \mathfrak{C} satisfies (1) of Theorem 1.2, the profile of \mathfrak{C} is bounded by a polynomial. It was shown in [32] that $\varphi_{\mathcal{R}}$, hence $\varphi_{\mathfrak{C}}$, is a quasi-polynomial (that is a polynomial $a_{\ell-1}(n)n^{\ell-1} + \dots + a_1(n)n + a_0(n)$ whose coefficients $a_{\ell-1}(n), \dots, a_0(n)$ are periodic functions). In the case of ordered structures, this profile is in fact polynomial (see [25]).

The profile of a class verifying (2) of Theorem 1.2 is bounded below by the profile of the age \mathfrak{D} . For arbitrary relational structures, this profile can be bounded above by a polynomial. But, in the case of ordered structures, it is necessarily at least exponential [25] and, as we will see in Proposition 1.7, in the case of ordered binary structures, the profile of \mathfrak{D} is bounded below by the Fibonacci function.

Consequently:

Theorem 1.4. *If \mathfrak{C} is a hereditary class of ordered binary structures then*

1. *Either the profile is bounded above by a polynomial, and in this case there is an integer ℓ such that every member of \mathfrak{C} has a monomorphic decomposition in at most ℓ blocks.*
2. *Or the profile is bounded below by the Fibonacci function.*

Ordered structures that have a finite monomorphic decomposition have a particularly simple form. If $\mathcal{R} := (V, \leq, \rho_1, \dots, \rho_k)$ is such a structure, the chain (V, \leq) decomposes into finitely many intervals V_i such that the union of any local isomorphisms f_i of $(V_i, \leq_{|V_i})$ is a local isomorphism of \mathcal{R} (see Theorem 3.3 below). If \mathcal{R} is made of binary relations, these relations are quite close to the given order. For example, if \mathcal{R} is a *bichain*, that is $\mathcal{R} := (V, \leq, \leq')$, where \leq' is a linear order, then \leq' coincides with \leq or its opposite on each V_i . Note that any ordered binary structure can be viewed as superposition of graphs (symmetric and irreflexive) and unary relations on the same ordered set and this superposition has the same local isomorphisms as \mathcal{R} hence the same profile. Indeed, if $\mathcal{R} := (V, \leq, \rho_1, \dots, \rho_k)$ is an ordered binary structure, replace each ρ_i by the graphs ρ_i^+, ρ_i^- and the unary relation $u(\rho_i)$ defined by setting $\rho_i^+ := \{\{x, y\} \in [V]^2 : x < y \text{ and } (x, y) \in \rho_i\}$, $\rho_i^- := \{\{x, y\} \in [V]^2 : x < y \text{ and } (y, x) \in \rho_i\}$ and $u(\rho_i) := \{x \in V : (x, x) \in \rho_i\}$. For example, if \mathcal{R} has a finite monomorphic decomposition $(V_i)_{i=1, \dots, \ell}$ then the graphs are full or empty on each V_i .

The following result indicates that for ordered binary structures, the study of their profile reduces (roughly) to the case of single ordered binary relations, and in fact to unary ordered structures or to ordered graphs.

Proposition 1.5. *Let $k \in \mathbb{N}$. An ordered binary structure $\mathcal{R} := (V, \leq, \rho_1, \dots, \rho_k)$ has a finite monomorphic decomposition if and only if every structure $\mathcal{R}_i := (V, \leq, \rho_i)$, ($1 \leq i \leq k$), has such a decomposition.*

The case of an ordered unary relation is handled by a result of P. Jullien [14]: the profile is either polynomial or bounded below by the exponential function $n \mapsto 2^n$. The case of ordered graphs was handled by Balogh, Bollobás and Morris [2].

Finite ordered structures, of the form $\mathcal{R} := (V, \leq, u_1, \dots, u_k)$, consisting of a linear order \leq and unary relations u_i , can be represented by words over a finite alphabet A (namely the set $\{0, 1\}^k$). Embedding

between structures correspond to the subword ordering. A famous result of Higman [13] asserts that the set A^* of words over a finite alphabet A is well-quasi-ordered for this ordering. Hence, hereditary classes of words are finite unions of ideals. According to [14] (see Chapter 6, page 103), each ideal decomposes into a finite product of *elementary ideals*, sets of the form $\{\square, a\}$, where \square is the empty word and $a \in A$ and of *starred ages*, sets of the form B^* where B^* is the set of words over $B \subseteq A$ (cf. [15] for an extension to an ordered alphabet). The profile of B^* satisfies $\varphi_{B^*}(n) = |B|^n$. Thus the profile of a hereditary class of words is either polynomial or at least exponential.

In this paper, we give more information about the binary ordered structures yielding an exponential profile. The case of ordered structures not necessarily binary is handled in a forthcoming paper [25].

We say that ordered structures of the form $\mathcal{R} := (V, \leq, \rho_1, \dots, \rho_k)$ have *type* k . As a consequence of Ramsey's theorem, we obtain:

Theorem 1.6. *The collection \mathfrak{S}_k of ordered binary structures of type k that do not have a finite monomorphic decomposition has a finite basis, that is contains a finite set \mathfrak{A}_k such that every member of \mathfrak{S}_k embeds some member of \mathfrak{A}_k . If $k = 1$, the subset $\check{\mathfrak{A}}_1$ of \mathfrak{A}_1 made of ordered reflexive graphs has one thousand two hundred and forty two members and none embeds into an other one.*

Members \mathcal{R} of \mathfrak{A}_k have the following common features. First, their domain V is either $\mathbb{N} \times \{0, 1\}$, or $(\mathbb{N} \times \{0, 1\}) \cup \{a\}$ for some fixed element a . Next, if \mathcal{R} is such a structure and u is any one-to-one order-preserving map on (\mathbb{N}, \leq) , then the map (u, Id) on $\mathbb{N} \times \{0, 1\}$ defined by $(u, Id)(x, i) = (u(x), i)$ for $i \in \{0, 1\}$ and that fixes a if $a \in V$ preserves \mathcal{R} . They are *almost multichainable* in the sense of [30].

Ordered reflexive directed graphs \mathcal{G} belonging to $\check{\mathfrak{A}}_1$ depend on three parameters p, l, k . There are integers such that $1 \leq p \leq 10$, $1 \leq l \leq 6$ and the value of k depends upon p, l (see Section 6).

We show that the profile of every element of $\check{\mathfrak{A}}_1$ is at least exponential:

Proposition 1.7. *The profile of a member of $\check{\mathfrak{A}}_1$ is either given by one of the five following functions: $\varphi_1(n) := 2^n - 1$, $\varphi_2(n) := 2^n - n$, $\varphi_3(n) := 2^{n-1}$, $\varphi_4(n) := 2^{n-1} + 1$ and the Fibonacci sequence, or is bounded below by one of them.*

This paper is organized as follows. Section 2 contains the definitions and basic notions. Section 3 contains a presentation of the notion of monomorphic decomposition. We give the proof of Theorem 1.2 and Corollary 1.3 in Subsection 3.2 and the proof of Proposition 1.5 in Subsection 3.3. In section 5 we give the detailed proof of the first part of Theorem 1.6 with a description of members of \mathfrak{A}_k . The proof of the second part is given in Section 7, with a study of profiles of members of $\check{\mathfrak{A}}_1$ from which Proposition 1.7 follows.

The results presented in this paper are included in Chapter 8 of our doctoral thesis [22]. They are mentioned in an abstract posted on ArXiv [24]. Our results have been presented at the 9th International Colloquium on Graph Theory and combinatorics (ICGT 2014) held in Grenoble (France), June 30-July 4, 2014, and at the conference-school on Discrete Mathematics and Computer Science (DIMACOS'2015) held in Sidi Bel Abbès (Algeria), November 15-19, 2015. We are pleased to thank the organizers of these conferences for their help.

2. Basic concepts

Our terminology follows Fraïssé [10]. For properties of profiles we refer to the survey of Pouzet [30].

2.1. Relational structures, embeddability, hereditary classes and ages

Let m be a non-negative integer. A m -ary relation with domain V is a subset ρ of V^m . When needed, we identify ρ with its characteristic function χ_ρ sending every element of ρ on 1, hence, ρ becomes a map from V^m to $\{0, 1\}$. A relational structure is a pair $\mathcal{R} := (V, (\rho_i)_{i \in I})$ made of a set V , the base or domain of \mathcal{R} , also denoted by $V(\mathcal{R})$ and a family $(\rho_i)_{i \in I}$ of n_i -ary relations ρ_i on V . The family $\mu := (n_i)_{i \in I}$ is the signature of \mathcal{R} . If I is finite, we say that the signature μ is finite. When all the relations ρ_i , $i \in I$, are binary, we have a binary relational structure, (binary structure for short). If one specified relation is a linear order, we have an ordered relational structure. A structure which has the form $\mathcal{R} := (V, \leq, \rho_1, \dots, \rho_k)$, where k is a non negative integer, \leq is a linear order on the set V and each ρ_i is a binary relation on V , is an ordered binary structure of type k . Basic examples of ordered binary structures are chains ($k = 0$), bichains ($k = 1$ and ρ_1 is a linear order on V) and ordered directed graphs ($k = 1$); if in this last case ρ_1 is an irreflexive and symmetric binary relation on V , we just say that the structure is an ordered graph. Let \mathcal{R} be a relational structure. The substructure induced by \mathcal{R} on a subset A of V , simply called the restriction of \mathcal{R} to A , is the relational structure $\mathcal{R} \upharpoonright_A := (A, (\rho_i \upharpoonright_A)_{i \in I})$. The notion of isomorphism between relational structures is defined in the natural way. A local isomorphism of \mathcal{R} is any isomorphism between two restrictions of \mathcal{R} . A relational structure \mathcal{R} is embeddable into a relational structure \mathcal{R}' , in notation $\mathcal{R} \leq \mathcal{R}'$, if \mathcal{R} is isomorphic to some restriction of \mathcal{R}' . Embeddability is a quasi-order on the class of structures having a given signature. We denote by Ω_μ the class of finite relational structures of signature μ ; it is quasi-ordered by embeddability. Most of the time, we consider its members up to isomorphism. The age of a relational structure \mathcal{R} is the set $\text{Age}(\mathcal{R})$ of restrictions of \mathcal{R} to finite subsets of its domain, these restrictions being considered up to isomorphism. A class \mathcal{C} of structures is hereditary if it contains every relational structure which can be embedded in some member of \mathcal{C} (i.e., if $\mathcal{R} \in \mathcal{C}$ and $\mathcal{S} \leq \mathcal{R}$ then $\mathcal{S} \in \mathcal{C}$). In order theoretic terms, a class of finite structures is hereditary iff this is an initial segment of Ω_μ . If \mathfrak{B} is any subset of Ω_μ the set $\text{Forb}(\mathfrak{B}) := \{\mathcal{R} \in \Omega_\mu : \mathcal{B} \not\leq \mathcal{R} \text{ for all } \mathcal{B} \in \mathfrak{B}\}$ is a hereditary class. A bound of a hereditary class \mathcal{C} of finite structures is any element of $\Omega_\mu \setminus \mathcal{C}$ which is minimal w.r.t. embeddability. Each hereditary class \mathcal{C} of Ω_μ is determined by its bounds (in fact $\mathcal{C} = \text{Forb}(\mathfrak{B})$ where \mathfrak{B} is the set of bounds of \mathcal{C}). Any age is an ideal of Ω_μ , that is a non empty initial segment of Ω_μ which is up-directed. One of the most basic result of the theory of relations asserts that the converse (almost) holds.

Lemma 2.1. (Fraïssé, 1954) *If μ is finite, every ideal of Ω_μ is the age of a countable structure.*

Note 2.1. *In the sequel, all relational structures we consider will have a finite signature.*

2.2. Invariant structures

We borrow the following notion and results to [8]; see [5] for an illustration.

Let $C := (L, \leq)$ be a chain. For each integer n , let $[C]^n$ be the set of n -tuples $\vec{a} := (a_1, \dots, a_n) \in L^n$ such that $a_1 < \dots < a_n$. This set will be identified with the set $[L]^n$ of the n -element subsets of L .

For every local automorphism h of C with domain D , set $h(\vec{a}) := (h(a_1), \dots, h(a_n))$ for every $\vec{a} \in [D]^n$. Let $\mathfrak{L} := \langle C, \mathcal{R}, \Phi \rangle$ be a triple made of a chain C on L , a relational structure $\mathcal{R} := (V, (\rho_i)_{i \in I})$ and a set Φ of maps, each one being a map ψ from $[C]^{a(\psi)}$ into V , where $a(\psi)$ is an integer, the *arity* of ψ .

We say that \mathfrak{L} is *invariant* if:

$$\rho_i(\psi_1(\vec{\alpha}_1), \dots, \psi_{m_i}(\vec{\alpha}_{m_i})) = \rho_i(\psi_1(h(\vec{\alpha}_1)), \dots, \psi_{m_i}(h(\vec{\alpha}_{m_i}))) \quad (2.1)$$

for every $i \in I$ and every local automorphism h of C whose domain contains $\vec{\alpha}_1, \dots, \vec{\alpha}_{m_i}$, where m_i is the arity of $\rho_i, \psi_1, \dots, \psi_{m_i} \in \Phi$, $\vec{\alpha}_j \in [C]^{a(\psi_j)}$ for $j = 1, \dots, m_i$.

Condition (2.1) expresses the fact that each ρ_i is invariant under the transformation of the m_i -tuples of V , that are induced on V , by the local automorphisms of C . For example, if ρ is a binary relation and $\Phi = \{\psi\}$ then

$$\rho(\psi(\vec{\alpha}), \psi(\vec{\beta})) = \rho(\psi(h(\vec{\alpha})), \psi(h(\vec{\beta})))$$

means that $\rho(\psi(\vec{\alpha}), \psi(\vec{\beta}))$ depends only upon the relative positions of $\vec{\alpha}$ and $\vec{\beta}$ on the chain C .

If $\mathfrak{L} := \langle C, \mathcal{R}, \Phi \rangle$ and L' is a subset of L , set $\Phi_{\uparrow L'} := \{\psi_{\uparrow [L']^{a(\psi)}} : \psi \in \Phi\}$ and $\mathfrak{L}_{\uparrow L'} := \langle C_{\uparrow L'}, \mathcal{R}, \Phi_{\uparrow L'} \rangle$ the restriction of \mathfrak{L} to L' .

The following result (see [8]) is a consequence of Ramsey's theorem:

Theorem 2.2. *Let $\mathfrak{L} := \langle C, \mathcal{R}, \Phi \rangle$ be a triple such that the domain L of C is infinite, \mathcal{R} consists of finitely many relations and Φ is finite. Then there is an infinite subset L' of L such that $\mathfrak{L}_{\uparrow L'}$ is invariant.*

2.3. Almost multichainable relational structure

A relational structure \mathcal{R} of signature μ is *almost multichainable* if its domain V decomposes into $F \cup (L \times K)$ where F and K are two finite sets and there is a linear order \leq on L such that:

$$\begin{aligned} &\text{For every local isomorphism } f \text{ of } (L, \leq), \text{ the map } (f, Id) \text{ on } L \times K \\ &\text{extended by the identity on } F \text{ induces a local isomorphism of } \mathcal{R}. \end{aligned} \quad (2.2)$$

If F is empty and $|K| = 1$, Condition (2.2) reduces to say that there is a linear order \leq on V such that every local isomorphism of (V, \leq) is a local isomorphism of \mathcal{R} . Relational structures with this property are said *chainable*, a notion invented by Fraïssé [10] (the relationship with monomorphy is given in Section 3.1).

Almost multichainable structures were introduced in [28], see [30]. They fall under the frame of invariant structures. Indeed, let \mathcal{R} be a relational structure; suppose that its domain V decomposes into $V = F \cup (L \times K)$ where F and K are two finite sets and that \leq is a linear order on L . Denote by ψ_k the map from L into V defined by $\psi_k(a) := (a, k)$ if $k \in K$ and $\psi_k(a) := k$ if $k \in F$. Set $\Phi := \{\psi_k : k \in K \cup F\}$ and $C := (L, \leq)$. Then the structure $\langle C, \mathcal{R}, \Phi \rangle$ is said *invariant* iff Condition 2.2 holds. Hence, Theorem 2.2 allows to extract from any \mathcal{R} an almost multichainable structure.

2.4. Well-quasi-ordering

We recall that a poset P is *well-quasi-ordered* (w.q.o.) if P contains no infinite antichain and no infinite descending chain. We recall the following result [29, 3.9 p. 329]. This is a special instance of a property of posets which is similar to Nash-William's lemma on minimal bad sequences [21]).

Lemma 2.3. *If a hereditary class \mathfrak{C} is not w.q.o., it contains an age \mathfrak{D} which is w.q.o. and has infinitely many bounds.*

We recall the following notion and result (see Chapter 13 p. 354, of [10]). A class \mathfrak{C} of finite structures is *very beautiful* if for every integer k , the collection $\mathfrak{C}(k)$ of structures (S, U_1, \dots, U_k) , where $S \in \mathfrak{C}$ and U_1, \dots, U_k are unary relations with the same domain as S , has no infinite antichain w.r.t. embeddability. A crucial property is the following:

Theorem 2.4. [26]. *A very beautiful age has only finitely many bounds.*

In the case of binary structures, Theorem 2.4 has a simple proof.

A straightforward consequence of Higman's theorem on words (see [13]) is that *the age of an almost multichainable structure is very beautiful*. As a consequence:

Lemma 2.5 ([30, Theorem 4.20]). *The age of an almost multichainable structure has only finitely many bounds.*

The finiteness of the bounds is a deep result. Frasnay [11] proved by means of a clever finite combinatorial analysis that chainable relational structures have finitely many bounds. The argument using their very beautiful character is shorter (but less precise: it gives no estimate on the size of bounds).

3. Monomorphic decomposition of a relational structures

We present in this section the notion of monomorphic decomposition of a relational structure. This notion was introduced in [32]. The introduction of an equivalence relation makes the presentation simpler and the proofs easier. This equivalence relation appeared in [24] and for hypergraphs in [31]. Its study is developed in full in [22] (cf. the third part of the thesis).

3.1. Monomorphic decomposition: basic properties

Let $\mathcal{R} := (V, (\rho_i)_{i \in I})$ be a relational structure. A subset B of V is a *monomorphic part* of \mathcal{R} if for every non-negative integer k and every pair A, A' of k -element subsets of $V(\mathcal{R})$, the induced structures on A and A' are isomorphic whenever $A \setminus B = A' \setminus B$. A *monomorphic decomposition* of \mathcal{R} is a partition \mathcal{P} of $V(\mathcal{R})$ into monomorphic parts. Equivalently, it is a partition \mathcal{P} of $V(\mathcal{R})$ into blocks such that for every integer n , the induced structures on two n -element subsets A and A' of $V(\mathcal{R})$ are isomorphic whenever $|A \cap B| = |A' \cap B|$ for every $B \in \mathcal{P}$. Each monomorphic part is included into a maximal one. This monomorphic part is unique and called a *monomorphic component*. The monomorphic components of \mathcal{R} form a monomorphic decomposition of \mathcal{R} of which every monomorphic decomposition of \mathcal{R} is a refinement (Proposition 2.12, page 14 of [32]).

Let x and y be two elements of V . Let F be a finite subset of $V \setminus \{x, y\}$. We say that x and y are F -equivalent and we set $x \simeq_{F, \mathcal{R}} y$ if the restrictions of \mathcal{R} to $\{x\} \cup F$ and $\{y\} \cup F$ are isomorphic. Let k be a non-negative integer, we set $x \simeq_{k, \mathcal{R}} y$ if $x \simeq_{F, \mathcal{R}} y$ for every k -element subset F of $V \setminus \{x, y\}$. We set $x \simeq_{\leq k, \mathcal{R}} y$ if $x \simeq_{k', \mathcal{R}} y$ for every $k' \leq k$. Finally, we say that they are equivalent and we set $x \simeq_{\mathcal{R}} y$ if $x \simeq_{k, \mathcal{R}} y$ for every integer k .

The fundamental property, whose proof is easy (see [31], [24], [22]) is this:

Lemma 3.1. *The relations $\simeq_{k, \mathcal{R}}$, $\simeq_{\leq k, \mathcal{R}}$ and $\simeq_{\mathcal{R}}$ are equivalence relations. Furthermore, the equivalence classes of $\simeq_{\mathcal{R}}$ are the monomorphic components of \mathcal{R} .*

Using this equivalence relation, the proof of the following result, which improves Proposition 2.3 of [32], is straightforward.

Lemma 3.2. *A relational structure \mathcal{R} admits a finite monomorphic decomposition if and only if there is some integer ℓ such that every member of its age $\text{Age}(\mathcal{R})$ admits a monomorphic decomposition into at most ℓ classes.*

Let \mathcal{R} be a relational structure; if for some non-negative integer n all the restrictions to the n -element subsets of its domain are isomorphic, we say that \mathcal{R} is n -monomorphic. We say that \mathcal{R} is $(\leq n)$ -monomorphic if it is m -monomorphic for every $m \leq n$ and that \mathcal{R} is monomorphic if it is n -monomorphic for every integer n . Since two finite chains with the same cardinality are isomorphic, chains are monomorphic. More generally, if there is a linear order \leq on the domain V of a relational structure \mathcal{R} such that the local isomorphisms of $C := (V, \leq)$ are local isomorphisms of \mathcal{R} , then \mathcal{R} is monomorphic. Structures with this property are called *chainable*. Every infinite monomorphic relational structure is chainable (Fraïssé, 1954, [10]). Every monomorphic relational structure of finite cardinality large enough (depending on the signature) is chainable (Frasnay 1965 [11]). It is an easy exercise to show that every binary relational structure on at least 4 elements which is (≤ 3) -monomorphic is chainable. The extension to larger arities is a deep result of Frasnay 1965 ([11]). He shown the existence of an integer p such that every p -monomorphic relational structure R whose maximum of the signature is at most m and domain infinite or sufficiently large is chainable (note that any p -monomorphic relational structure on $2p - 1$ -element is $(\leq p)$ -monomorphic [27]). The least integer p in the sentence above is the *monomorphy threshold*, $p(m)$. Its value was given by Frasnay [12] in 1990: $p(1) = 1$, $p(2) = 3$, $p(m) = 2m - 2$ for $m \geq 3$. For a detailed exposition of this result, see [10] Chapter 13, notably p. 378.

Relational structures with a finite monomorphic decomposition have a form close to the chainable ones, as shown by the following result proved in [32], cf. Theorem 1.8 p. 10.

Theorem 3.3. *A relational structure $\mathcal{R} := (V, (\rho_i)_{i \in I})$ admits a finite monomorphic decomposition if and only if there exists a linear order \leq on V and a finite partition $(V_x)_{x \in X}$ of V into intervals of (V, \leq) such that every local isomorphism of (V, \leq) which preserves each interval is a local isomorphism of \mathcal{R} .*

We may notice that if \mathcal{R} is ordered then the given order \leq has the property stated in Theorem 3.3.

If a structure \mathcal{R} admits a finite monomorphic decomposition, then it has some restriction with the same age which is almost multichainable. As a consequence of Lemma 2.5 we have:

Corollary 3.4. *The age $\text{Age}(\mathcal{R})$ of a relational structure \mathcal{R} with a finite monomorphic decomposition is very beautiful and has finitely many bounds.*

We may give an almost self-contained proof. The w.q.o. character of $\text{Age}(\mathcal{R})$ is Dickson's Lemma. Indeed, if V_1, \dots, V_ℓ is a partition of V into monomorphic blocks, associate to every finite subset A of V the sequence $\vartheta(A) := (|A \cap V_i|)_{i=1, \dots, \ell}$. If A and A' are two finite subsets and $\vartheta(A) \leq \vartheta(A')$ in the direct product \mathbb{N}^ℓ then $\mathcal{R}_{\upharpoonright A}$ is embeddable into $\mathcal{R}_{\upharpoonright A'}$. Since \mathbb{N}^ℓ is w.q.o. by Dickson's Lemma, $\text{Age}(\mathcal{R})$ is w.q.o. Adding k unary predicates to \mathcal{R} amounts to replace each $a_i := |A \cap V_i|$ by a word of length a_i over an alphabet on 2^k letters. The w.q.o. character follows from Higman's Theorem on words. The fact that \mathcal{R} has finitely many bounds follows from Theorem 2.4.

3.2. Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2

Suppose that (1) does not hold. We consider two cases.

a) \mathfrak{C} is w.q.o. by embeddability.

In this case, \mathfrak{C} is a finite union of ideals \mathfrak{C}_i (this is a special case of a general result about posets due to Erdős-Tarski [9]). According to Lemma 2.1, each ideal \mathfrak{C}_i is the age of a structure \mathcal{R}_i . Since \mathfrak{C} does not satisfy (1), there is some \mathcal{R}_i that cannot have a finite monomorphic decomposition. Since \mathfrak{C} is w.q.o., the set of hereditary subclasses is well founded (Higman [13]). Since \mathfrak{C} contains the age of a structure with no finite monomorphic decomposition, it contains a minimal age with this property.

b) \mathfrak{C} is not well-quasi ordered by embeddability.

In this case, it contains an infinite antichain. According to Lemma 2.3, it contains an age \mathfrak{D} which is w.q.o. and has infinitely many bounds. According to Lemma 3.4, no relation having this age can have a finite monomorphic decomposition. Since \mathfrak{D} is w.q.o., it contains an age minimal with this property. \square

Proof of Corollary 1.3

Suppose that \mathfrak{C} is a finite union of ages $\text{Age}(\mathcal{R}_i)$ of relational structures \mathcal{R}_i and each \mathcal{R}_i has a monomorphic decomposition into ℓ_i components. Then, for $\ell := \text{Max}\{\ell_i : i\}$, each member of \mathfrak{C} has a monomorphic decomposition into at most ℓ components.

Now, suppose that there is an integer ℓ such that every member of \mathfrak{C} has a monomorphic decomposition into at most ℓ blocks. In this case \mathfrak{C} cannot satisfy (2) of Theorem 1.2. Otherwise, \mathfrak{C} contains an age \mathfrak{D} such that no relational structure \mathcal{R} having this age has a finite monomorphic decomposition. According to Lemma 3.2, there is no bound on the size of the monomorphic decompositions of the member of \mathfrak{D} , and so the members of \mathfrak{C} . This fact contradicts our hypothesis. Then, \mathfrak{C} satisfy (1) of Theorem 1.2. \square

3.3. Proof of Proposition 1.5

We prove a little more, namely:

The equivalence relation $\simeq_{\mathcal{R}}$ associated to the ordered structure $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$ is the intersection of the equivalence relations $\simeq_{\mathcal{R}_i}$ associated to $\mathcal{R}_i := (V, \leq, \rho_i)$ for $i \in I$.

Clearly $\simeq_{\mathcal{R}}$ is included into $\simeq_{\mathcal{R}_i}$ for each $i \in I$. Conversely, let $x, y \in V$ such that $x \simeq_{\mathcal{R}_i} y$ for every $i \in I$. Let F be a finite subset of $V \setminus \{x, y\}$. Since $x \simeq_{\mathcal{R}_i} y$ there is a local isomorphism f_i of \mathcal{R}_i which carries $F \cup \{x\}$ onto $F \cup \{y\}$. Since f_i is a local isomorphism of $C := (V, \leq)$ which carries $F \cup \{x\}$ onto $F \cup \{y\}$, f_i is independent of i . Hence, this is a local isomorphism of \mathcal{R} proving that $x \simeq_{\mathcal{R}} y$. \square

4. The case of ordered binary structures

In this section, and the next, we consider ordered binary structures. A crucial property of ordered structures is that if two finite substructures are isomorphic, there is just one isomorphism from one to the other. A repeated use of this property allows us to describe the form of equivalence classes when these structures are binary. We start by describing the case of one class.

Lemma 4.1. *An ordered binary structure $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$ is monomorphic iff it is (≤ 2) -monomorphic iff each relation ρ_i which is reflexive is either a chain which coincides with \leq or its dual, a reflexive clique or an antichain and every relation ρ_i which is irreflexive is either an acyclic tournament which coincides with the strict order $<$ or its dual, a clique or an independent set.*

Proof. The first equivalence is a special case of a more general result. It says that if $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$ is an ordered structure and m is the maximum of the arities of the relations $(\rho_i)_{i \in I}$, then \mathcal{R} is monomorphic iff it is $(\leq m)$ -monomorphic. We give the proof of this general form. If \mathcal{R} is monomorphic then it is $(\leq k)$ -monomorphic for every non negative integer k . Now, suppose that \mathcal{R} is $(\leq m)$ -monomorphic. As every chainable structure is monomorphic (Fraïssé [10]), it suffices to prove that \mathcal{R} is chainable. Let then f be a local automorphism of the chain $C := (V, \leq)$. To prove that f is a local automorphism of \mathcal{R} it suffices to show that any restriction of f to a subset X , with at most m elements, of its domain is a local automorphism of \mathcal{R} . Let X be such a subset. As \mathcal{R} is $(\leq m)$ -monomorphic, the restrictions $\mathcal{R}_{\upharpoonright X}$ and $\mathcal{R}_{\upharpoonright f(X)}$ are isomorphic. such an isomorphism, say φ , is an isomorphism from $C_{\upharpoonright X}$ onto $C_{\upharpoonright f(X)}$. As C is a chain, this isomorphism is unique, then $f_{\upharpoonright X} = \varphi$. Hence f is an isomorphism from $\mathcal{R}_{\upharpoonright X}$ onto $\mathcal{R}_{\upharpoonright f(X)}$.

For the second equivalence, it is easy to verify that if each relation ρ_i has one of the given forms then \mathcal{R} is (≤ 2) -monomorphic. For the converse, suppose that \mathcal{R} is (≤ 2) -monomorphic and consider a relation ρ_i , for some $i \in I$. Since \mathcal{R} is 1-monomorphic then ρ_i is either reflexive or irreflexive. Suppose, to fix our ideas, that it is reflexive. Since \mathcal{R} is 2-monomorphic, then either the restrictions of ρ_i to 2-elements subsets of V are symmetric in which case ρ_i is a reflexive clique or an antichain. Or the restrictions of ρ_i to 2-elements subsets of V are asymmetric in which case ρ_i is a linear order which coincides with \leq or its dual. The case where ρ_i is irreflexive yields to four other situations in which ρ_i is either a clique, an independent, set if it is symmetric, or an acyclic tournament which coincides with the strict order $<$ or its dual if it is asymmetric. \square

In order to describe the equivalence classes in general, we introduce some notation. Let $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$ be an ordered binary structure. Identifying a relation to its characteristic function, we set $d_i(x, y) := (\rho_i(x, y), \rho_i(y, x))$ for all $x, y \in V$ and $i \in I$ and $d(x, y) := (d_i(x, y))_{i \in I}$. We may note that the value of $d(x, y)$ determines the value of $d(y, x)$. In fact, set $\bar{u} := (\beta, \alpha)$ for every $u := (\alpha, \beta) \in \{0, 1\} \times \{0, 1\}$ and $(u_i)_{i \in I} := (\bar{u}_i)_{i \in I}$ for every sequence of members of $\{0, 1\} \times \{0, 1\}$. Doing so, we have $d(y, x) = \bar{d}(x, y)$.

Set $C := (V, \leq)$. Let $x, y \in V$. If $x < y$ in V , we set $[x, y] := \{z \in V : x \leq z \leq y\}$ and $]x, y[:= \{z \in V : x < z < y\}$. If the order between x and y is not given, we denote by $I_{\leq}(x, y)$ the least interval of C containing x and y . We recall that a subset A of V is an *interval* of \mathcal{R} if it is an interval of C and

$$d(x, y) = d(x', y) \quad (4.1)$$

for every $x, x' \in A$ and $y \notin A$.

The following fact is obvious:

Fact 1. *Two elements x, y of V such that $x < y$ are 1-equivalent if and only if*

$$d(x, z) = d(z, y) \text{ for every } z \in]x, y[\quad (4.2)$$

and

$$d(x, z) = d(y, z) \text{ for every } z \notin [x, y]. \quad (4.3)$$

Lemma 4.2. *Let $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$ be an ordered binary structure and x, y be two elements of V such that $x < y$. Then, the following properties are equivalent:*

(i) *x and y are (≤ 2) -equivalent;*

(ii) *$[x, y]$ is a 1-monomorphic interval of \mathcal{R} and the restrictions of \mathcal{R} to any two 2-element subsets of $[x, y]$ distinct of $\{x, y\}$ are isomorphic.*

(iii) *x and y are F -equivalent for every finite subset F of $V \setminus \{x, y\}$.*

Proof. We prove $(i) \Rightarrow (ii) \Rightarrow (iii)$. The implication $(iii) \Rightarrow (i)$ is obvious.

$(i) \Rightarrow (ii)$. Suppose that x and y are (≤ 2) -equivalent. We claim that $[x, y]$ is a 1-monomorphic interval of \mathcal{R} . If $[x, y] = \{x, y\}$ then since x and y are 0-equivalent, $\mathcal{R}_{\uparrow\{x\}}$ and $\mathcal{R}_{\uparrow\{y\}}$ are isomorphic and thus $[x, y]$ is 1-monomorphic. Since x and y are 1-equivalent, the fact that $[x, y]$ is an interval of \mathcal{R} follows from (4.3) of Fact 1. If $[x, y] \neq \{x, y\}$, let $z \in]x, y[$ and $a \notin [x, y]$ (if any). Since x and y are 1-equivalent, $\mathcal{R}_{\uparrow\{x, z\}}$ and $\mathcal{R}_{\uparrow\{z, y\}}$ are isomorphic (and the only isomorphism sends x to z and z to y), in particular $\mathcal{R}_{\uparrow\{x\}}$ and $\mathcal{R}_{\uparrow\{z\}}$ are isomorphic, hence $[x, y]$ is 1-monomorphic. Since x and y are 2-equivalent, the restrictions of \mathcal{R} to $\{x, a, z\}$ and $\{y, a, z\}$ are isomorphic; the only isomorphism sends a onto a , x to z and z to y , hence $d(a, z) = d(a, x) = d(a, y)$, proving that $[x, y]$ is an interval of \mathcal{R} . We look at the 2-element subsets of $[x, y]$ which are distinct from $\{x, y\}$. If $[x, y] = \{x, y\}$ there are none and there is nothing to prove. Otherwise, $[x, y] \neq \{x, y\}$. Let $z \in]x, y[$. Since x and y are 1-equivalent, $\mathcal{R}_{\uparrow\{x, z\}}$ and $\mathcal{R}_{\uparrow\{z, y\}}$ are isomorphic. If there is no other element than z this yields the conclusion of the lemma. If there is an other element z' we may suppose $z < z'$. Since x and y are 2-equivalent, there is an isomorphism of $\mathcal{R}_{\uparrow\{x, z, z'\}}$ onto $\mathcal{R}_{\uparrow\{z, z', y\}}$ and in fact a unique one. It sends x to z , z to z' and z' to y . It follows that the five restrictions $\mathcal{R}_{\uparrow\{z, z'\}}$, $\mathcal{R}_{\uparrow\{x, z\}}$, $\mathcal{R}_{\uparrow\{x, z'\}}$, $\mathcal{R}_{\uparrow\{y, z\}}$, $\mathcal{R}_{\uparrow\{y, z'\}}$ are isomorphic. This property yields immediately the conclusion of the lemma if there is no other element. If there are others, then this property also says that all the restrictions of \mathcal{R} to all pairs $\{x, z\}$ for which $z \neq y$ are isomorphic; since the restrictions of \mathcal{R} to pairs $\{z, z'\}$ are isomorphic to the restrictions of \mathcal{R} to such pairs $\{x, z\}$, the restrictions of \mathcal{R} to all pairs distinct of $\{x, y\}$ are isomorphic, as claimed.

(ii) \Rightarrow (iii). Suppose that $[x, y]$ is a 1-monomorphic interval of \mathcal{R} and the restrictions of \mathcal{R} to any two 2-element subsets of $[x, y]$ distinct of $\{x, y\}$ are isomorphic. Let F be a finite subset of $V \setminus \{x, y\}$ and φ be the unique order-isomorphism from $F \cup \{x\}$ onto $F \cup \{y\}$. Let $z, z' \in F$. If $z \in F \setminus [x, y]$ then $\varphi(z) = z$. If $z' \in [x, y]$ then $\varphi(z') \in [x, y]$ and since $[x, y]$ is a 1-monomorphic interval of \mathcal{R} , the map φ induces an isomorphism of $\mathcal{R}_{\upharpoonright\{z, z'\}}$ onto $\mathcal{R}_{\upharpoonright\{\varphi(z), \varphi(z')\}}$. If $z, z' \in [x, y]$ then $\varphi(z), \varphi(z') \in [x, y]$. From the 1-monomorphy of $[x, y]$ and the fact that the 2-element subsets of $[x, y]$ distinct of $\{x, y\}$ are isomorphic, φ induces an isomorphism of $\mathcal{R}_{\upharpoonright\{z, z'\}}$ onto $\mathcal{R}_{\upharpoonright\{\varphi(z), \varphi(z')\}}$. It follows that φ is an isomorphism of $\mathcal{R}_{\upharpoonright F \cup \{x\}}$ onto $\mathcal{R}_{\upharpoonright F \cup \{y\}}$ hence $x \simeq_{F, \mathcal{R}} y$. \square

As a corollary we have

Theorem 4.3. *On an ordered binary structure $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$, the equivalence relations $\simeq_{\leq 2, \mathcal{R}}$ and $\simeq_{\mathcal{R}}$ coincide.*

An other consequence of Lemma 4.2 is this

Corollary 4.4. *Let $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$ be an ordered binary structure and A be a subset of V . If A is an interval of (V, \leq) included into some (≤ 1) -equivalence class then it is included into a (≤ 2) -equivalence class.*

Proof. We apply Lemma 4.2. Let $x, y \in A$ with $x < y$. Since A is an interval of (V, \leq) and all the elements of A are (≤ 1) -equivalent, $[x, y]$ is a 1-monomorphic interval of \mathcal{R} and the restrictions of \mathcal{R} to any two 2-element subsets of $[x, y]$ distinct of $\{x, y\}$ are isomorphic, hence by Lemma 4.2, x and y are (≤ 2) -equivalent. \square

Remark 4.5. *The fact that two elements x and y are (≤ 2) -equivalent does not imply that the interval $[x, y]$ is 2-monomorphic. Indeed, let $\mathcal{R} := (V, \leq, \rho)$ where ρ coincides with \leq except on one ordered pair (x, y) with $x < y$. If the interval $]x, y[$ is non-empty, the set V decomposes into four equivalence classes, namely $\{x, y\}$ and the three open intervals of (V, \leq) : $] \leftarrow x[$, $]x, y[$ and $]x \rightarrow [$ determined by x and y . In this case, $[x, y]$ is not 2-monomorphic.*

The form of the equivalence classes for an arbitrary ordered binary structure is given in Theorem 4.6 below.

Theorem 4.6. *An equivalence class A of an ordered binary structure $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$ is either an interval of (V, \leq) or consists of two distinct elements x, y with $x < y$ such that the interval $]x, y[$ forms an other equivalence class.*

Proof. Let A be an equivalence class. Item (ii) of Lemma 4.2 ensures that for every $x < y$ in A the open interval $]x, y[$ is included into a (≤ 2) -equivalence class. From this fact, it follows that if $|A| \geq 3$ then A is an interval of (V, \leq) . Indeed, suppose that this is not the case, that is there is some $z \notin A$ which is between two elements of A . Since A has at least three elements, there are at least two elements, say u, v , of A on one side of z and an other element, say w , on the other side. The open interval determined by the extreme elements, e.g. u and w contains z and v . These two elements are equivalent. Contradiction. Thus, if A is not an interval of (V, \leq) then A is made of two elements x, y with $x < y$ and there is some $z \notin A$ such that $x < z < y$. By Lemma 4.2, the open interval $]x, y[$ is included into a (≤ 2) -equivalence class distinct from A . To conclude, we need to prove that $]x, y[$ is equal to this equivalence class. We prove the following.

Claim 1. *Let $x < x' < y < y'$ with $x \simeq_{\leq 2, \mathcal{R}} y$ and $x' \simeq_{\leq 2, \mathcal{R}} y'$, then $x \simeq_{\leq 2, \mathcal{R}} y'$.*

Proof. a) We prove that the conclusion holds if there is a fifth element z in the interval $[x, y']$. For example, suppose $z \in]x, y[$. In this case, from Lemma 4.2, we have $x' \simeq_{\leq 2, \mathcal{R}} z$. Now, if $x' < z < y$, we have $z \simeq_{\leq 2, \mathcal{R}} y$ from Lemma 4.2; if not, then $x < z < x'$; since $z \simeq_{\leq 2, \mathcal{R}} y'$, we have $x' \simeq_{\leq 2, \mathcal{R}} y$ from Lemma 4.2. In both cases we have $x' \simeq_{\leq 2, \mathcal{R}} y$. If $y < z < y'$ the conclusion is similar.

b) From a) we may suppose that $]x, y'[= \{x', y\}$. In this case, it suffices to prove that $[x, y']$ is a (≤ 2) -monomorphic \mathcal{R} -interval. That is the restrictions of \mathcal{R} on the six ordered pairs $\{x, x'\}$, $\{x, y\}$, $\{x, y'\}$, $\{x', y\}$, $\{x', y'\}$, $\{y, y'\}$ are isomorphic. From the fact that $x \simeq_{\leq 2, \mathcal{R}} y$ the restrictions of \mathcal{R} to $\{x, x', y'\}$ and to $\{x', y, y'\}$ are isomorphic. Similarly, from the fact that $x' \simeq_{\leq 2, \mathcal{R}} y'$ the restrictions of \mathcal{R} to $\{x, x', y\}$ and to $\{x, y, y'\}$ are isomorphic. The first isomorphism yields that the restrictions of \mathcal{R} on $\{x, x'\}$ and $\{x', y\}$ are isomorphic, the same for $\{x, y'\}$ and $\{x', y'\}$ and also for $\{x', y'\}$ and $\{y, y'\}$. Hence, at least the restrictions of \mathcal{R} to three of these pairs, namely $\{x, y'\}$, $\{x', y'\}$, $\{y, y'\}$ are isomorphic. The second isomorphism yields that the restrictions of \mathcal{R} on $\{x, x'\}$ and $\{x, y\}$ are isomorphic, the same for $\{x, y\}$ and $\{x, y'\}$ and also for $\{x', y\}$ and $\{y, y'\}$. Also, three of these pairs, namely $\{x, x'\}$, $\{x, y\}$, $\{x, y'\}$ are isomorphic. The pair $\{x, y'\}$ belonging to these two sets, the restrictions of \mathcal{R} to these five pairs are all isomorphic; since the restrictions of \mathcal{R} to the remaining pairs $\{y, y'\}$ and to $\{x', y\}$ are isomorphic, the restrictions of \mathcal{R} to all these pairs are isomorphic as claimed. By exchanging x, x' and y, y' , we get the same conclusion if $x' < x < y' < y$. With these two properties follows that $[x, y[$ is an equivalence class. □

With this, the proof of the theorem is complete. □

Remark 4.7. *Here is an example for which none of the 2-element equivalence classes is an interval. Let $V := \mathbb{N} \times \{0, 1, 2\}$. Order V by $(n, i) \leq (n', i')$ if either $n < n'$ or $n = n'$ and $i \leq i'$; the order type of (V, \leq) is the lexicographical product $3 \cdot \omega$. Let ρ be the lexicographical product $C_3 \cdot \omega$, that is $(n, i) \rho (n', i')$ if either $n < n'$ or $n = n'$ and $i' = i + 1$ (modulo 3). Let $\mathcal{R} := (V, \leq, \rho)$. Then, the equivalence classes are the pairs $\{(n, 0), (n, 2)\}$ and the singletons $\{(n, 1)\}$ for $n \in \mathbb{N}$.*

From Lemma 4.1 and Theorem 4.6, Theorem 3.3 restricted to binary ordered structures follows. Indeed, let $\mathcal{R} := (V, \leq, (\rho_i)_{i \in I})$ be such a structure. Then, each equivalence class of $\simeq_{\mathcal{R}}$ which is not an interval of (V, \leq) has just two elements; replacing each of these classes by two blocks made of these two elements will give a partition of V into monomorphic parts. On each part, says A , every local isomorphism of $(A, \leq \upharpoonright A)$, extended by the identity outside, is a local isomorphism of \mathcal{R} , hence every local isomorphism of (V, \leq) which preserves each interval of this new partition is a local isomorphism of \mathcal{R} . If $\simeq_{\mathcal{R}}$ has only finitely many classes, the new partition of V has only finitely many blocks and the conclusion of Theorem 3.3 holds.

If the number of relations is finite, we have the following separation lemma;

Lemma 4.8. *If an ordered binary structure $\mathcal{R} := (V, \leq, \rho_1, \dots, \rho_k)$ of type k has infinitely many equivalence classes then one of these two cases occurs:*

1. *There is an infinite subset $A \subseteq V$ such that any two distinct elements of A are 0-equivalent but not 1-equivalent.*

2. There are two disjoint infinite subsets A_1, A_2 of V such that any two distinct elements of $A_i, i \in \{1, 2\}$ are 1-equivalent but not 2-equivalent and for every $x, y \in A_i$, with $i \in \{1, 2\}$ and $x < y$, we have $[x, y] \cap A_j \neq \emptyset$, for $i \neq j$.

Proof.

Case 1. \mathcal{R} has infinitely many classes of 1-equivalence.

Since \mathcal{R} is made of finitely many relations, V consists of only finitely many classes of 0-equivalence. Since V is made of infinitely many classes of 1-equivalence, one class of 0-equivalence, say X_0 , contains infinitely many classes of 1-equivalence. Pick one element from every class of 1-equivalence belonging to X_0 to form a subset A of V . The set A satisfies the Assertion (1) of Lemma 4.8.

Case 2. \mathcal{R} has finitely many classes of 1-equivalence.

According to Theorem 4.3, every (≤ 2) -equivalence class is an equivalence class, hence \mathcal{R} has infinitely many (≤ 2) -equivalence classes. Since V is made of finitely many classes of 1-equivalence, some 1-equivalence class, say X_1 , contains infinitely many (≤ 2) -equivalence classes. Pick an element from each class of (≤ 2) -equivalence class included into X_1 . Let A be the resulting set. Let $a, b \in A$ with $a < b$. Then the interval $[a, b]$ cannot be contained in X_1 , otherwise by Corollary 4.4, a and b would be (≤ 2) -equivalent. Hence, there exists $c \in [a, b]$ which belongs to an other class of 1-equivalence $X' \neq X_1$. We can extract from A a sequence $(a_i)_{i \geq 0}$ which is monotonic w.r.t. \leq . With no loss of generality, we may suppose this sequence increasing. According to the above remark, for every $i \geq 0$, there exists $c_i \in [a_i, a_{i+1}]$ with c_i belonging to a class of 1-equivalence which is different from X_1 . Since the number of 1-equivalence classes is finite, we can then find an infinite subsequence $(c'_i)_{i \geq 0}$ of $(c_i)_{i \geq 0}$ whose elements are in the same class of 1-equivalence. Let then $(a'_i)_{i \geq 0}$ be a subsequence of $(a_i)_{i \geq 0}$ such that $c'_i \in [a'_i, a'_{i+1}]$. Set $A_1 = \{a'_i, i \in \mathbb{N}\}$ and $A_2 = \{c'_i, i \in \mathbb{N}\}$. The sets A_1 and A_2 satisfy Assertion (2) of Lemma 4.8. □

5. Proof of the first part of Theorem 1.6

We give a proof of the first part of Theorem 1.6. We prove that the collection \mathfrak{S}_k of ordered binary structures of type k which do not have a finite monomorphic decomposition, has a finite basis \mathfrak{A}_k . The proof of the second part is given in section 6.

The proof of Theorem 1.6 goes as follows. Let $\mathcal{R} := (V, \leq, \rho_1, \dots, \rho_k)$ be an ordered binary structure which has infinitely many equivalence classes. According to Lemma 4.8, we have two cases.

5.1. Case 1

\mathcal{R} satisfies Assertion (1) of Lemma 4.8.

In this case, let $f : \mathbb{N} \rightarrow V$ be a 1-to-1 map such that $f(\mathbb{N}) = A$ where A is the set given by Assertion (1) of Lemma 4.8. Since $f(n)$ and $f(m)$ are not 1-equivalent for every $n < m$, we may find some element $g(n, m)$ witnessing this fact, meaning that the restrictions of \mathcal{R} to $\{f(n), g(n, m)\}$ and $\{f(m), g(n, m)\}$ are not isomorphic.

Let $\Phi := \{f, g\}$ and $\mathfrak{L} := \langle \omega, \mathcal{R}, \Phi \rangle$, where ω is the chain made of \mathbb{N} and the natural order.

Ramsey's theorem in the version of Theorem 2.2, allows us to find an infinite subset $X \subseteq \mathbb{N}$ such that $\mathfrak{L} \upharpoonright_X$ is invariant. By relabeling X with non-negative integers, we may suppose $X = \mathbb{N}$ and hence that \mathfrak{L} is invariant.

Claim 2. *The maps f and g satisfy the following properties:*

1. $f(n) \leq f(m) \Leftrightarrow f(n') \leq f(m'), \forall n < m, n' < m'$.
2. $d(f(n), f(m)) = d(f(n'), f(m')), \forall n < m, n' < m'$.
3. $g(n, m) \leq g(k, l) \Leftrightarrow g(n', m') \leq g(k', l'), \forall n < m \leq k < l, n' < m' \leq k' < l'$.
4. $d(g(n, m), g(k, l)) = d(g(n', m'), g(k', l')), \forall n < m \leq k < l, n' < m' \leq k' < l'$.
5. $g(n, m) \in [f(n), f(m)] \Leftrightarrow g(n', m') \in [f(n'), f(m')], \forall n < m, n' < m'$.
6. $g(n, m) \leq f(k)$ for some integers $n < m < k \Leftrightarrow g(n, m) \leq f(l)$ for every $l > m$.
7. $d(f(n), g(n, m)) = d(f(k), g(k, l)), \forall n < m, k < l$.
8. $d(g(n, m), f(k)) = d(g(p, q), f(l)), \forall n < m < k, p < q < l$.
9. *If the restrictions $\mathcal{R} \upharpoonright_{\{f(n), g(n, m)\}}$ and $\mathcal{R} \upharpoonright_{\{f(k), g(n, m)\}}$ are isomorphic for some integers $n < m < k$ then $\mathcal{R} \upharpoonright_{\{f(n'), g(n', m')\}}$ and $\mathcal{R} \upharpoonright_{\{f(k'), g(n', m')\}}$ are isomorphic for every $n' < m' < k'$.*
10. $g(n, m)$ and $f(k)$ are different for every distinct integers $n < m$ and k .
11. $g(n, m) \neq g(n', m')$ for every $n < m < n' < m'$.

Proof. The nine first items follow from invariance. To prove Item (10), suppose that there are integers n, m, k with $n < m$ such that $g(n, m) = f(k)$. By construction of the functions f and g , the sets $\{f(n), g(n, m)\}$ and $\{f(m), g(n, m)\}$ have two elements each, hence k cannot be equal to n or to m . According to Item (1) and (2), the restrictions of \mathcal{R} to $\{f(n), f(k)\}$ and $\{f(m), f(k)\}$ are isomorphic; thus, if $g(n, m) = f(k)$ we get that the restrictions of \mathcal{R} to $\{f(n), g(n, m)\}$ and to $\{f(m), g(n, m)\}$ are isomorphic, contradicting the choice of $g(n, m)$. For the proof of Item (11), suppose that there are integers $n < m < n' < m'$ such that $g(n, m) = g(n', m')$. The transformation fixing n, m and sending n' onto m' is a local isomorphism of the chain, hence the restrictions of \mathcal{R} to $\{f(n'), g(n, m)\}$ and $\{f(m'), g(n, m)\}$ are isomorphic. Replacing $g(n, m)$ by $g(n', m')$, we get that the restrictions to $\{f(n'), g(n', m')\}$ and $\{f(m'), g(n', m')\}$ are isomorphic which is a contradiction with the choice of $g(n', m')$. \diamond

We define a map $F : \mathbb{N} \times \{0, 1\} \longrightarrow V(\mathcal{R})$ and an ordered binary structure of type k , $\mathcal{R}^{(1)} := (V^{(1)}, \leq^{(1)}, \rho_1^{(1)}, \dots, \rho_k^{(1)})$ with vertex set $V^{(1)} := \mathbb{N} \times \{0, 1\}$ such that F is an embedding from $\mathcal{R}^{(1)}$ into \mathcal{R} .

We define first F . We set $F(n, 1) := g(2n, 2n + 1)$ for $n \in \mathbb{N}$. From Item (9) of Claim 2, we have two cases:

Case (a). $\mathcal{R} \upharpoonright_{\{f(n), g(n, m)\}}$ and $\mathcal{R} \upharpoonright_{\{f(k), g(n, m)\}}$ are isomorphic for some integers $n < m < k$; from invariance, this property holds for all $n < m < k$.

Case (b). Case (a) does not hold.

In Case (a), we set $F(n, 0) := f(2n + 1)$ for every $n \in \mathbb{N}$. In Case b, we set $F(n, 0) := f(2n)$ for every $n \in \mathbb{N}$.

The map F is 1-to-1 (by construction f is 1-to-1, hence the restriction of F to $\mathbb{N} \times \{0\}$ is 1-to-1; the restriction of F to $\mathbb{N} \times \{1\}$ is also 1-to-1 by (11) of Claim 2; the images of $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{1\}$ are disjoint by (10) of Claim 2).

Since F is 1-to-1, we take for $\mathcal{R}^{(1)}$ the inverse image of \mathcal{R} .

This amounts to

$$x \leq^{(1)} y \Leftrightarrow F(x) \leq F(y) \text{ and } d^{(1)}(x, y) = d(F(x), F(y))$$

for every $x, y \in \mathbb{N} \times \{0, 1\}$, where $d^{(1)}(x, y) := (d_i^{(1)}(x, y))_{i=1, \dots, k}$ and $d_i^{(1)}(x, y) := (\rho_i^{(1)}(x, y), \rho_i^{(1)}(y, x))$ for every $1 \leq i \leq k$.

Claim 3. For every $n < m \in \mathbb{N}$, $(n, 0)$ and $(m, 0)$ are not 1-equivalent, hence $\mathcal{R}^{(1)}$ has infinitely many equivalence classes.

Proof.

It suffices to prove that:

$$\mathcal{R}^{(1)} \upharpoonright_{\{(n,0),(n,1)\}} \text{ and } \mathcal{R}^{(1)} \upharpoonright_{\{(m,0),(n,1)\}} \text{ are not isomorphic} \quad (5.1)$$

By definition, $\mathcal{R} \upharpoonright_{\{f(2n),g(2n,2n+1)\}}$ and $\mathcal{R} \upharpoonright_{\{f(2n+1),g(2n,2n+1)\}}$ are not isomorphic. In Case (a), $\mathcal{R} \upharpoonright_{\{f(2n),g(2n,2n+1)\}}$ and $\mathcal{R} \upharpoonright_{\{f(2m+1),g(2n,2n+1)\}}$ are isomorphic, hence $\mathcal{R} \upharpoonright_{\{f(2n+1),g(2n,2n+1)\}}$ and $\mathcal{R} \upharpoonright_{\{f(2m+1),g(2n,2n+1)\}}$ are not isomorphic; since $F(n, 0) := f(2n + 1)$, $F(m, 0) = f(2m + 1)$ and $F(n, 1) = g(2n, 2n + 1)$ that means that $\mathcal{R}^{(1)} \upharpoonright_{\{(n,0),(n,1)\}}$ and $\mathcal{R}^{(1)} \upharpoonright_{\{(m,0),(n,1)\}}$ are not isomorphic, proving that (5.1) holds. In Case (b), $\mathcal{R} \upharpoonright_{\{f(2n),g(2n,2n+1)\}}$ and $\mathcal{R} \upharpoonright_{\{f(2m+1),g(2n,2n+1)\}}$ are not isomorphic. By invariance, $\mathcal{R} \upharpoonright_{\{f(2m),g(2n,2n+1)\}}$ and $\mathcal{R} \upharpoonright_{\{f(2m+1),g(2n,2n+1)\}}$ are isomorphic, hence $\mathcal{R} \upharpoonright_{\{f(2n),g(2n,2n+1)\}}$ and $\mathcal{R} \upharpoonright_{\{f(2m),g(2n,2n+1)\}}$ are not isomorphic; since $F(n, 0) := f(2n)$, $F(m, 0) = f(2m)$ and $F(n, 1) = g(2n, 2n + 1)$ that means that $\mathcal{R}^{(1)} \upharpoonright_{\{(n,0),(n,1)\}}$ and $\mathcal{R}^{(1)} \upharpoonright_{\{(m,0),(n,1)\}}$ are not isomorphic, proving that (5.1) holds. \diamond

Claim 4. The set \mathfrak{A}_k^1 of ordered binary structures $\mathcal{R}^{(1)}$ obtained by this process is finite.

Proof. According to Claim 2, $\mathcal{R}^{(1)}$ is entirely defined by its values on the pairs $((i, j), (i', j'))$ of vertices taken among the four vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$; the values on the other pairs will be deduced by taking local isomorphisms of $C := (\mathbb{N} \times \{0, 1\}, \leq^{(1)})$ and using Claim 2. \diamond

Let us give a hint about the form of the structures which arise (the full description in the case of a single binary relation is given in Section 6).

Due to its invariance, the map $d^{(1)}$ is determined by its values on the five ordered pairs $((0, 0), (1, 0))$, $((0, 1), (1, 1))$, $((0, 0), (0, 1))$, $((0, 0), (1, 1))$, $((0, 1), (1, 0))$. The only requirement for those pairs, due to Condition (5.1), is that

$$d^{(1)}((0, 0), (0, 1)) \neq d^{(1)}((0, 1), (1, 0)). \quad (5.2)$$

On each ordered pair, $d^{(1)}$ can take 4^k values. On five pairs, this gives 4^{5k} possibilities, but 4^{4k} are forbidden (those for which $d^{(1)}$ takes the same values on the pairs $((0, 0), (0, 1))$ and $((0, 1), (1, 0))$). This gives $4^{4k} \cdot (4^k - 1)$ possibilities. In fact, some of the resulting structures embed into some others. Thus the number of non equimorphic structures is a bit less.

5.2. Case 2

\mathcal{R} satisfies the Assertion (2) of Lemma 4.8.

Since A_1 is infinite, there is a countable sequence $(x_n)_{n \in \mathbb{N}}$ of elements of A_1 which is either increasing or decreasing. Let $(y_n)_{n \in \mathbb{N}}$ such that $y_n \in I_{\leq}(x_n, x_{n+1}) \cap A_2$. Set $A'_1 := \{x_n : n \in \mathbb{N}\}$, $A'_2 := \{y_n : n \in \mathbb{N}\}$.

Then $(A'_1 \cup A'_2, \leq_{A'_1 \cup A'_2})$ is ordered as ω or ω^* . Each of the sets A'_1 and A'_2 is contained into a class of 1-equivalence. Conditions (4.2) and (4.3) of Fact 1 are satisfied for every $x, y \in A_i$ and $z \in A_j$, $j \neq i$. We have then two situations:

First situation. There exists $x_0 \in V \setminus (A'_1 \cup A'_2)$ which witnesses the fact that A'_1 and A'_2 are contained in two different classes of 1-equivalence. As $A'_1 \cup A'_2$ is ordered as ω or ω^* , we may suppose that we have either $x_0 \leq a$ or $x_0 \geq a$, for every $a \in A'_1 \cup A'_2$, because otherwise, we can find some cofinite subset of $A'_1 \cup A'_2$ for which we have this condition.

We may then find maps $f', g', g'' : \mathbb{N} \rightarrow V$ such that, $f'(\mathbb{N}) = A'_1$, $g'(\mathbb{N}) = A'_2$, $g''(\mathbb{N}) := \{x_0\}$, $g'(n) \in I_{\leq}(f'(n), f'(n+1))$, $\forall n \in \mathbb{N}$ and the restrictions of \mathcal{R} to $\{f'(n), g'(n), g''(0)\}$ and $\{f'(m), g'(n), g''(0)\}$ are not isomorphic for any $n, m \in \mathbb{N}$ with $n \neq m$ (in fact the restrictions of \mathcal{R} to $\{f'(n), g''(0)\}$ and $\{g'(m), g''(0)\}$ are not isomorphic for every $n, m \in \mathbb{N}$).

As in Case 2 (subsection 5.1), we define a map $F' : \{a\} \cup (\mathbb{N} \times \{0, 1\}) \rightarrow V(\mathcal{R})$, with $a \notin \mathbb{N} \times \{0, 1\}$, such that $F'(a) := g''(0)$, $F'(n, 0) := f'(n)$ and $F'(n, 1) := g'(n)$.

The map F' being 1-to-1, we may define an ordered binary structure of type k , $\mathcal{R}^{(2)} := (V^{(2)}, \leq^{(2)}, \rho_1^{(2)}, \dots, \rho_k^{(2)})$ with vertex set $V^{(2)} := \{a\} \cup (\mathbb{N} \times \{0, 1\})$ such that F' is an embedding from $\mathcal{R}^{(2)}$ into \mathcal{R} . This amounts to

$$x \leq^{(2)} y \Leftrightarrow F'(x) \leq F'(y) \text{ and } d^{(2)}(x, y) = d(F'(x), F'(y))$$

for every $x, y \in \{a\} \cup \mathbb{N} \times \{0, 1\}$, where $d^{(2)}(x, y) := (d_i^{(2)}(x, y))_{i=1, \dots, k}$ and $d_i^{(2)}(x, y) := (\rho_i^{(2)}(x, y), \rho_i^{(2)}(y, x))$ for every $1 \leq i \leq k$.

By construction, $\mathcal{R}^{(2)}$ satisfies Condition (5.3) below, hence has infinitely many equivalence classes.

$$\forall n < m \in \mathbb{N}, \mathcal{R}^{(2)} \upharpoonright_{\{a, (n, 0), (n, 1)\}} \text{ and } \mathcal{R}^{(2)} \upharpoonright_{\{a, (n, 1), (m, 0)\}} \text{ are not isomorphic.} \quad (5.3)$$

The ordered binary structure $\mathcal{R}^{(2)}$ satisfies Observation 1 stated below which follows directly from the fact that the elements of A'_i for $i \in \{1, 2\}$ are 1-equivalent.

Observation 1. 1. $a \leq^{(2)} (n, i) \Leftrightarrow a \leq^{(2)} (m, i)$, $\forall n < m$, $i \in \{0, 1\}$.

2. $a \leq^{(2)} (n, 0) \Leftrightarrow a \leq^{(2)} (n, 1)$, $\forall n \in \mathbb{N}$.

3. $d^{(2)}(a, (n, i)) = d^{(2)}(a, (m, i))$, $\forall n < m$, $i \in \{0, 1\}$.

4. $d^{(2)}((n, i), (m, i)) = d^{(2)}((n', i), (m', i)), \forall n < m, n' < m', i \in \{0, 1\}$.
5. $d^{(2)}((n, 0), (n, 1)) = d^{(2)}((n, 1), (m, 0)), \forall n < m$.
6. $d^{(2)}((n, 0), (n, 1)) = d^{(2)}((m, 0), (m, 1)), \forall n < m$.
7. $d^{(2)}((n, 0), (m, 1)) = d^{(2)}((m, 1), (m + 1, 0)), \forall n < m$.

With this we obtain a finite subset \mathfrak{B}_k^1 of ordered binary structures with the same vertex set $\mathbb{N} \times \{0, 1\} \cup \{a\}$.

Second situation. There is no vertex x_0 as above. Then, since, by Lemma 4.8, the elements of $A'_i, i \in \{1, 2\}$, are 1-equivalent but not 2-equivalent, we can deduce that if x, y are two vertices of $A'_i, i \in \{1, 2\}$, then there are two elements z, z' which witness the fact that x, y are not 2-equivalent (i.e. the restrictions of \mathcal{R} to $\{x, z, z'\}$ and to $\{y, z, z'\}$ are not isomorphic). From Fact 1, it follows that the two vertices z, z' can not be both in $I_{\leq}(x, y)$ or both outside it. Hence, $z \in I_{\leq}(x, y) \cap A'_j$ with $j \neq i$ and $z' \notin I_{\leq}(x, y)$ (in fact if one has $z \in I_{\leq}(x, y) \cap A'_i$, the restrictions of \mathcal{R} to $\{x, z, z'\}$ and to $\{y, z, z'\}$ will be isomorphic). In this case and by Lemma 4.8, the relation between two elements of A'_i , for at least one $i = 1, 2$, is different from the relation between two elements s, t , with $s \in A'_1$ and $t \in A'_2$ (i.e. if $a, b \in A'_1$ and $c, d \in A'_2$ then at least two among the restrictions $\mathcal{R}_{\uparrow\{a,b\}}, \mathcal{R}_{\uparrow\{a,c\}}$ and $\mathcal{R}_{\uparrow\{c,d\}}$ are not isomorphic). Indeed, since $(A'_1 \cup A'_2, \leq_{A'_1 \cup A'_2})$ is ordered as ω or ω^* , we can suppose $x < z < y < z'$, then if $z' \in A'_j$, so z and z' are 1-equivalent, in this case and according to Fact 1, we must have $d(x, z) \neq d(z, z')$ (otherwise the restrictions of \mathcal{R} to $\{x, z, z'\}$ and to $\{y, z, z'\}$ are isomorphic). And if $z' \in A'_i$, so x, y and z' are 1-equivalent, in this case $d(x, z) = d(z, y) = d(z, z')$ and $d(x, z') = d(x, y) = d(y, z')$, then, necessarily $d(x, z) \neq d(x, z')$ because otherwise the restrictions of \mathcal{R} to $\{x, z, z'\}$ and to $\{y, z, z'\}$ will be isomorphic. We can then define two maps $f_1, g_1 : \mathbb{N} \rightarrow V(\mathcal{R})$ such that, $f_1(\mathbb{N}) = A'_1, g_1(\mathbb{N}) = A'_2, g_1(n) \in I_{\leq}(f_1(n), f_1(n + 1)), \forall n \in \mathbb{N}$.

Set $F'' : \mathbb{N} \times \{0, 1\} \rightarrow V(\mathcal{R})$ such that $F''(n, 0) := f_1(n)$ and $F''(n, 1) := g_1(n)$.

We can define an ordered binary structure $\mathcal{R}^{(3)} := (V^{(3)}, \leq^{(3)}, \rho_1^{(3)}, \dots, \rho_k^{(3)})$ with vertex set $V^{(3)} = \mathbb{N} \times \{0, 1\}$ such that

$$x \leq^{(3)} y \Leftrightarrow F''(x) \leq F''(y) \text{ and } d^{(3)}(x, y) = d(F''(x), F''(y))$$

for every $x, y \in \mathbb{N} \times \{0, 1\}$, where $d^{(3)}(x, y) := (d_i^{(3)}(x, y))_{i=1, \dots, k}$ and $d_i^{(3)}(x, y) := (\rho_i^{(3)}(x, y), \rho_i^{(3)}(y, x))$ for every $1 \leq i \leq k$. As we said before, by construction of $\mathcal{R}^{(3)}$, the order $\leq^{(3)}$ is isomorphic to ω or ω^* with $(n, 1) \in I_{\leq^{(3)}}((n, 0), (n + 1, 0))$ and for every $n < m$, we have either

$$d^{(3)}((n, 0), (m, 0)) \neq d^{(3)}((n, 0), (m, 1)),$$

or

$$d^{(3)}((n, 0), (m, 0)) \neq d^{(3)}((n, 1), (m, 1)),$$

or

$$d^{(3)}((n, 0), (m, 1)) \neq d^{(3)}((n, 1), (m, 1)).$$

Then, with the fact that A'_1 and A'_2 are, each one, included into a class of 1-equivalence, $\mathcal{R}^{(3)}$ satisfies the following observation.

Observation 2. 1. $d^{(3)}((n, 0), (n, 1)) = d^{(3)}((m, 0), (m, 1)), \forall n < m.$

2. $d^{(3)}((n, 0), (n, 1)) = d^{(3)}((n, 1), (m, 0)), \forall n < m.$

3. $d^{(3)}((n, 0), (m, 1)) = d^{(3)}((m, 1), (m + 1, 0)), \forall n < m.$

4. $d^{(3)}((n, i), (m, i)) = d^{(3)}((n', i), (m', i)), \forall n < m, n' < m', i \in \{0, 1\}.$

It is then clear that $\mathcal{R}^{(3)}$ has infinitely many equivalence classes. By construction, the set \mathfrak{B}_k^2 of ordered binary structure obtained in this case is finite.

First, we conclude that $\mathfrak{A}_k^1 \cup \mathfrak{B}_k^1 \cup \mathfrak{B}_k^2$ is finite. Hence, the set \mathfrak{A}_k of minimal ordered binary structures (w.r.t embeddability) of $\mathfrak{A}_k^1 \cup \mathfrak{B}_k^1 \cup \mathfrak{B}_k^2$ is finite. Next, by construction, \mathfrak{A}_k is a basis. With that, the proof of the first part of Theorem 1.6 is complete.

6. Description of the ordered directed graphs

We give in this section the proof of the second part of Theorem 1.6.

Let \mathfrak{A}_1 be the set of ordered binary structures $\mathcal{R} := (V, \leq, \rho)$ defined in the previous section and let $\check{\mathfrak{A}}_1$ be the subset made of the ordered reflexive directed graphs $\mathcal{G} := (V, \leq, \rho)$ of the set \mathfrak{A}_1 . The members of $\check{\mathfrak{A}}_1$ are almost multichains on $F \cup (L \times K)$ such that $L := \mathbb{N}, |K| = 2$ and $|F| \leq 1$. We prove that the set $\check{\mathfrak{A}}_1$ contains one thousand two hundred and forty two members, such that $|\check{\mathfrak{A}}_1 \cap \mathfrak{A}_1^1| = 1122, |\check{\mathfrak{A}}_1 \cap \mathfrak{B}_1^1| = 48$ and $|\check{\mathfrak{A}}_1 \cap \mathfrak{B}_1^2| = 72$. We also prove the following result.

Lemma 6.1. *No directed ordered graph of the set $\check{\mathfrak{A}}_1$ embeds into an other one.*

Before giving the description of the graphs $\mathcal{G} := (V, \leq, \rho)$ of $\check{\mathfrak{A}}_1$, let us give a hint about their forms, in particular the order type of (V, \leq) (we recall that if $\mathcal{G} \in \mathfrak{B}_1^1 \cup \mathfrak{B}_1^2$ then V is ordered as ω or ω^*).

Due to invariance property of the relations, to construct \mathcal{G} we just decide on relations between the four vertices $(0, 0), (0, 1), (1, 0), (1, 1)$. For example, suppose that $(0, 1)$ is between $(0, 0)$ and $(1, 0)$ w.r.t \leq , then we have two cases.

Case 1. If $(0, 0) < (1, 0)$ then we necessarily have $(0, 0) < (0, 1) < (1, 0)$. The invariance of the structure yields $(1, 0) < (1, 1)$. By transitivity of \leq we get $(0, 1) < (1, 1)$ and $(0, 0) < (1, 1)$. Let $n < m$ two non negative integers, by invariance the above inequalities hold if we replace 0 and 1 in the first component of each of the pairs above by n and m respectively. Then $(n, 0) < (n, 1) < (m, 0) < (m, 1)$ for every $n < m$. Thus V is ordered as ω .

Case 2. If $(1, 0) < (0, 0)$, the same considerations as in Case 1 give V ordered as ω^* .

And if we suppose that $(0, 1)$ is not between $(0, 0)$ and $(1, 0)$ w.r.t \leq , then with the same arguments \leq is isomorphic to one among the following orders: the lexicographical products $\underline{2}^*.\omega$ and $\underline{2}.\omega^*$, where $\underline{2}$ is the chain made of two element $\{0, 1\}$ ordered so that $0 < 1$ and $\underline{2}^*$ is its dual, $\omega + \omega, \omega^* + \omega, \omega + \omega^*, \omega^* + \omega^*$. The same work is done to obtain the relation ρ .

Set $A := \mathbb{N} \times \{0\}$ and $B := \mathbb{N} \times \{1\}$. To describe our reflexive ordered directed graphs we introduce three parameters p, ℓ and k . These are non-negative integers where p indicates the forms of the restrictions $\mathcal{G}_{\downarrow A}$

and $\mathcal{G}_{\downarrow B}$ each of which may be isomorphic to an antichain, a chain isomorphic to ω or ω^* or a reflexive clique, ℓ gives the order-type of the linear order \leq and k is a counter which gives the number of graphs with the same p and ℓ .

Denote by $\mathcal{G}_{\ell,k}^{(p)} := (V_{\ell,k}^{(p)}, \leq_{\ell,k}^{(p)}, \rho_{\ell,k}^{(p)})$ the ordered directed graphs of $\check{\mathfrak{A}}_1$ such that $1 \leq p \leq 10$, $1 \leq \ell \leq 6$. The set of vertices $V_{\ell,k}^{(p)}$ is either $\mathbb{N} \times \{0, 1\} = A \cup B$ (if $\mathcal{G}_{\ell,k}^{(p)}$ is in $\mathfrak{A}_1^1 \cup \mathfrak{B}_1^2$) or $\mathbb{N} \times \{0, 1\} \cup \{a\} = A \cup B \cup \{a\}$ (if $\mathcal{G}_{\ell,k}^{(p)}$ is in \mathfrak{B}_1^1).

The ordered directed graphs with the same value of p are said of *class* p , their restrictions to A are identical and their restrictions to B also. If they have the same value of ℓ , then the linear orders (V, \leq) have the same order-type, ℓ takes values from 1 to 6 if the linear order is isomorphic to respectively ω , ω^* , $\omega + \omega$, $\omega^* + \omega$, $\omega + \omega^*$, $\omega^* + \omega^*$. We do not consider the cases where the linear order \leq is isomorphic to $\underline{2}^*.\omega$, or to $\underline{2}.\omega^*$ because all the ordered directed graphs which are obtained in this case are isomorphic to some ones for which the order \leq is isomorphic to ω or ω^* . The integer k enumerates the graphs for all values of p and ℓ . Different classes have not necessarily the same cardinalities.

For $p = 1$ if $\ell = 1, 2$ we have $1 \leq k \leq 18$ and if $3 \leq \ell \leq 6$ we have $1 \leq k \leq 15$. For $p = 2, 3, 4$, if $\ell = 1, 2$ we have $1 \leq k \leq 21$ and if $3 \leq \ell \leq 6$ we have $1 \leq k \leq 15$.

For $5 \leq p \leq 10$, if $\ell = 1, 2$ we have $1 \leq k \leq 22$ and if $3 \leq \ell \leq 6$ we have $1 \leq k \leq 24$. The total is one thousand two hundred and forty two as claimed.

In each class p , when $\ell = 1$, the linear order $\leq_{\ell,k}^{(p)}$ is isomorphic to ω . In this case we have, $(0, 0) <_{\ell,k}^{(p)} (0, 1) <_{\ell,k}^{(p)} (1, 0)$ when the vertex set is $\mathbb{N} \times \{0, 1\}$, and $a <_{\ell,k}^{(p)} (0, 0) <_{\ell,k}^{(p)} (0, 1) <_{\ell,k}^{(p)} (1, 0)$ when this set is $\mathbb{N} \times \{0, 1\} \cup \{a\}$. The order is reversed when $\ell = 2$. If $\ell \geq 3$ or $p \geq 5$, the vertex set is $\mathbb{N} \times \{0, 1\}$ and the elements of A are less than those of B according to $\leq_{\ell,k}^{(p)}$. All the ordered directed graphs given for $\ell \geq 3$ belong to \mathfrak{A}_1^1 and for $p \geq 5$ they all belong to $\mathfrak{A}_1^1 \cup \mathfrak{B}_1^2$.

We will give a graphical representations for some classes. All these representations are done on the following six vertices $(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1)$ for the graphs of $\mathfrak{A}_1^1 \cup \mathfrak{B}_1^2$ and on the following seven vertices $a, (0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1)$ for those of \mathfrak{B}_1^1 (the loops are not shown). These representations are given for $\ell = 1$ (the linear order $\leq_{\ell,k}^{(p)}$ is isomorphic to ω).

Recall that a graph G which is isomorphic to its dual is said *self-dual*, and if G is a directed graph, the *symmetrized* of G is the directed graph G' obtained from G by adding every edge $u := (x, y)$ such that $u^{-1} := (y, x)$ is an edge of G . Thus G' may be considered as an undirected graph.

Let $\mathcal{G} := (V, \leq, \rho)$ be an ordered reflexive directed graph. The subset E of V^2 such that $(x, y) \in E$ if and only if $\rho(x, y) = 1$ is the edge set of \mathcal{G} and $G := (V, E)$ is the *directed graph associated* to the ordered directed graph \mathcal{G} . For $x, y \in V$, set $d(x, y) := (\rho(x, y), \rho(y, x))$.

We are now ready to describe our ordered reflexive directed graphs of $\check{\mathfrak{A}}_1$ given in Theorem 1.6. To make this description, we first start by setting the value of p from 1 to 10, then we set the value of ℓ , ($1 \leq \ell \leq 6$) and for each of these values we consider all the possible values of k .

According to our notations, we have just to describe the associated directed graphs $G_{\ell,k}^{(p)} = (V_{\ell,k}^{(p)}, E_{\ell,k}^{(p)})$.

For $n \in \mathbb{N}$, set $\alpha_n := ((n, 0), (n, 1))$.

Class $p = 1$: The restrictions of $G_{\ell,k}^{(1)}$ to sets $A := \mathbb{N} \times \{0\}$ and $B := \mathbb{N} \times \{1\}$ are both antichains.

I) If $\ell = 1, 2$ then $1 \leq k \leq 18$.

The ordered directed graphs $\mathcal{G}_{\ell,k}^{(1)}$ for $1 \leq k \leq 9$ are in \mathfrak{A}_1^1 , they are in \mathfrak{B}_1^2 for $10 \leq k \leq 12$ and in \mathfrak{B}_1^1 for $13 \leq k \leq 18$.

• For $1 \leq k \leq 12$, the vertex set of the graph is $A \cup B$. A pair (x, x') of vertices, where $x = (n, i)$, $x' = (n', i')$, is

- an edge of $G_{\ell,1}^{(1)}$ if $n = n'$ and $i < i'$;
- an edge of $G_{\ell,2}^{(1)}$ if (x', x) is an edge of $G_{\ell,1}^{(1)}$. Thus $G_{\ell,2}^{(1)}$ is the dual of $G_{\ell,1}^{(1)}$;
- an edge of $G_{\ell,3}^{(1)}$ if $n = n'$ and $i \neq i'$. The graph $G_{\ell,3}^{(1)}$ is self-dual;
- an edge of $G_{\ell,4}^{(1)}$ if $n \leq n'$ and $i < i'$;
- an edge of $G_{\ell,5}^{(1)}$ if (x', x) is an edge of $G_{\ell,4}^{(1)}$. Thus $G_{\ell,5}^{(1)}$ is the dual of $G_{\ell,4}^{(1)}$;
- an edge of $G_{\ell,6}^{(1)}$ if it is either an edge of $G_{\ell,4}^{(1)}$ or an edge of $G_{\ell,5}^{(1)}$. Thus $G_{\ell,6}^{(1)}$ is the symmetrized of $G_{\ell,4}^{(1)}$ (and of $G_{\ell,5}^{(1)}$), it is self-dual;
- an edge of $G_{\ell,7}^{(1)}$ if $i < i'$. The graph $G_{\ell,7}^{(1)}$ is equimorphic to its dual.
- an edge of $G_{\ell,8}^{(1)}$ if either $(n \leq n'$ and $i < i')$ or $(n > n'$ and $i < i')$ or $(n < n'$ and $i > i')$.
- an edge of $G_{\ell,9}^{(1)}$ if (x', x) is an edge of $G_{\ell,8}^{(1)}$. Thus $G_{\ell,9}^{(1)}$ is the dual of $G_{\ell,8}^{(1)}$;
- an edge of $G_{\ell,10}^{(1)}$ if either $(n \leq n'$ and $i < i')$ or $(n < n'$ and $i > i')$;
- an edge of $G_{\ell,11}^{(1)}$ if (x', x) is an edge of $G_{\ell,10}^{(1)}$. Thus $G_{\ell,11}^{(1)}$ is the dual of $G_{\ell,10}^{(1)}$;
- an edge of $G_{\ell,12}^{(1)}$ if $i \neq i'$. The graph $G_{\ell,12}^{(1)}$ is the symmetrized of $G_{\ell,10}^{(1)}$ (and of $G_{\ell,11}^{(1)}$).

We recall that $\underline{2}$ is the poset made of $2 := \{0, 1\}$ ordered so that $0 < 1$. The poset $\underline{2}^*$ is its dual. Denote by K_2 the reflexive clique on two vertices and by $\Delta_{\mathbb{N}}$ the antichain with \mathbb{N} as vertex set. With this notation, $G_{\ell,1}^{(1)}$ is isomorphic to $\underline{2} \cdot \Delta_{\mathbb{N}}$, the lexicographic product of $\underline{2}$ by $\Delta_{\mathbb{N}}$ (that is the antichain $\Delta_{\mathbb{N}}$ where every vertex is replaced by the chain $\underline{2}$). The graph $G_{\ell,2}^{(1)}$ is isomorphic to $\underline{2}^* \cdot \Delta_{\mathbb{N}}$, the graph $G_{\ell,3}^{(1)}$ is isomorphic to $K_2 \cdot \Delta_{\mathbb{N}}$, the graph $G_{\ell,6}^{(1)}$ is the half complete bipartite graph of Schmerl- Trotter [33] and the graph $G_{\ell,7}^{(1)}$ is isomorphic to the ordinal sum $\Delta_{\mathbb{N}} + \Delta_{\mathbb{N}}$.

• For $13 \leq k \leq 18$, the vertex set of the graph is $\{a\} \cup A \cup B$.

A pair (x, x') of vertices is

- an edge of $G_{\ell,13}^{(1)}$ if $x = a$, $x' = (n, 1)$;
- an edge of $G_{\ell,14}^{(1)}$ if (x', x) is an edge of $G_{\ell,13}^{(1)}$. Thus $G_{\ell,14}^{(1)}$ is the dual of $G_{\ell,13}^{(1)}$;

- an edge of $G_{\ell,15}^{(1)}$ if it is either an edge of $G_{\ell,13}^{(1)}$ or an edge of $G_{\ell,14}^{(1)}$. Thus $G_{\ell,15}^{(1)}$ is the symmetrized of $G_{\ell,13}^{(1)}$ (and also of $G_{\ell,14}^{(1)}$);
- an edge of $G_{\ell,16}^{(1)}$ if either $x = a$ and $x' = (n, 0)$ or $x = (n, 1)$ and $x' = a$; this graph is self-dual;
- an edge of $G_{\ell,17}^{(1)}$ if it is either an edge of $G_{\ell,16}^{(1)}$ or an edge of $G_{\ell,13}^{(1)}$;
- an edge of $G_{\ell,18}^{(1)}$ if (x', x) is an edge of $G_{\ell,17}^{(1)}$. Thus $G_{\ell,18}^{(1)}$ is the dual of $G_{\ell,17}^{(1)}$;

The graphical representations of $G_{1,k}^{(1)}$, $1 \leq k \leq 18$ are given in Figure 1.

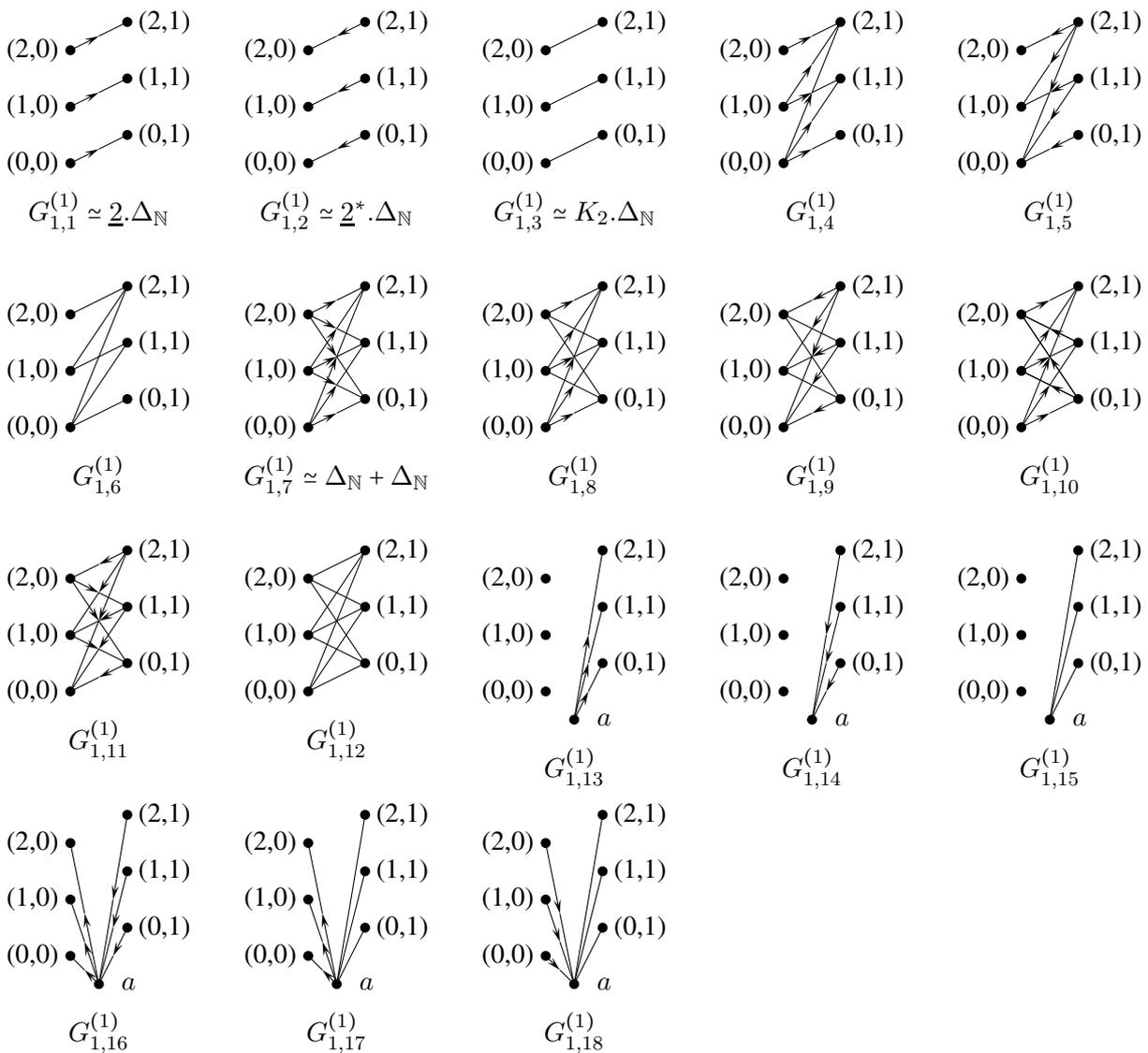


Figure 1. Minimal graphs of class $p = 1$ for $\ell = 1$. The ordered directed graphs $\mathcal{G}_{1,k}^{(1)}$ are in \mathfrak{A}_1^1 for $1 \leq k \leq 9$, in \mathfrak{B}_1^2 for $10 \leq k \leq 12$ and in \mathfrak{B}_1^1 for $13 \leq k \leq 18$.

II) If $3 \leq \ell \leq 6$, we have the same examples for each value of ℓ and their number is 15, according to the linear order $\leq_{\ell,k}^{(1)}$, the elements of A are placed before those of B . In these cases, $\mathcal{G}_{\ell,k}^{(1)} \in \mathfrak{A}_1^1$ for every $1 \leq k \leq 15$.

- $G_{\ell,k}^{(1)} = G_{1,k}^{(1)}$ for every $1 \leq k \leq 6$.
- $G_{\ell,k}^{(1)} = G_{1,k+1}^{(1)}$ for every $7 \leq k \leq 10$.
- $G_{\ell,11}^{(1)}$ is obtained from $G_{1,7}^{(1)}$ by adding all edges $((n, 1), (m, 0))$ for $n \geq m$.
- $G_{\ell,12}^{(1)}$ is obtained from $G_{1,10}^{(1)}$ by adding all edges $((n, 1), (m, 0))$ for $n \geq m$, the graph $G_{\ell,12}^{(1)}$ is the dual of $G_{\ell,11}^{(1)}$.
- $G_{\ell,13}^{(1)}$ is undirected. Its edge set is $\{(n, 0), (n', 1)\}; n \neq n'\}$.
- $G_{\ell,14}^{(1)}$ is obtained from $G_{\ell,13}^{(1)}$ by adding all edges α_n for $n \in \mathbb{N}$.
- $G_{\ell,15}^{(1)}$ is obtained from $G_{\ell,13}^{(1)}$ by adding all edges α_n^{-1} for $n \in \mathbb{N}$.

Class $p = 2$: In this case, the restrictions of $G_{\ell,k}^{(2)}$ to A and B are both chains isomorphic to ω .

I) If $\ell = 1, 2$ we have $1 \leq k \leq 21$, the ordered directed graphs $\mathcal{G}_{\ell,k}^{(2)}$ for $1 \leq k \leq 12$ are in \mathfrak{A}_1^1 , they are in \mathfrak{B}_1^1 for $13 \leq k \leq 18$ and in \mathfrak{B}_1^2 for $19 \leq k \leq 21$.

- For $1 \leq k \leq 9$ the graph $G_{\ell,k}^{(2)}$ coincides with $G_{\ell,k}^{(1)}$ on pairs of $A \times B$. Thus $G_{\ell,7}^{(2)}$ is a chain isomorphic to $\omega + \omega$ and the ordered directed graph $\mathcal{G}_{\ell,7}^{(2)}$ is one of the bichains given in [20].

- For $10 \leq k \leq 12$, the graph $G_{\ell,k}^{(2)}$ coincides with $G_{\ell,10}^{(1)}$ on pairs of $A \times B$ with

1. suppressing edges α_n , $n \in \mathbb{N}$ if $k = 10$; then $G_{\ell,10}^{(2)}$ is isomorphic to $\Delta_2.\omega$, the lexicographical product of the antichain on two vertices Δ_2 with ω .
2. replacing α_n by α_n^{-1} , $n \in \mathbb{N}$ if $k = 11$; then $G_{\ell,11}^{(2)}$ is isomorphic to $\underline{2}^*.\omega$, the ordered directed graph $\mathcal{G}_{\ell,11}^{(2)}$ is, in this case, one of the bichains given in [20].
3. adding the edges α_n^{-1} , $n \in \mathbb{N}$ if $k = 12$; then $G_{\ell,12}^{(2)}$ is isomorphic to $K_2.\omega$.

- For $13 \leq k \leq 18$, the edge set on $(\{a\} \cup A) \times (\{a\} \cup B)$ of the graph $G_{\ell,k}^{(2)}$ is the union of edge sets of $G_{\ell,10}^{(1)}$ and $G_{\ell,k}^{(1)}$.

- The edge sets of graphs $G_{\ell,19}^{(2)}$ and $G_{\ell,20}^{(2)}$ on $A \times B$ coincide with those of $G_{\ell,11}^{(1)}$ and $G_{\ell,12}^{(1)}$ respectively.

- The edge set of graph $G_{\ell,21}^{(2)}$ on $A \times B$ is empty. Then $G_{\ell,21}^{(2)}$ is isomorphic to $\omega \oplus \omega$, the direct sum of two chains isomorphic to ω .

The graphical representations of $G_{1,k}^{(2)}$, $1 \leq k \leq 21$ are given in Figure 2.

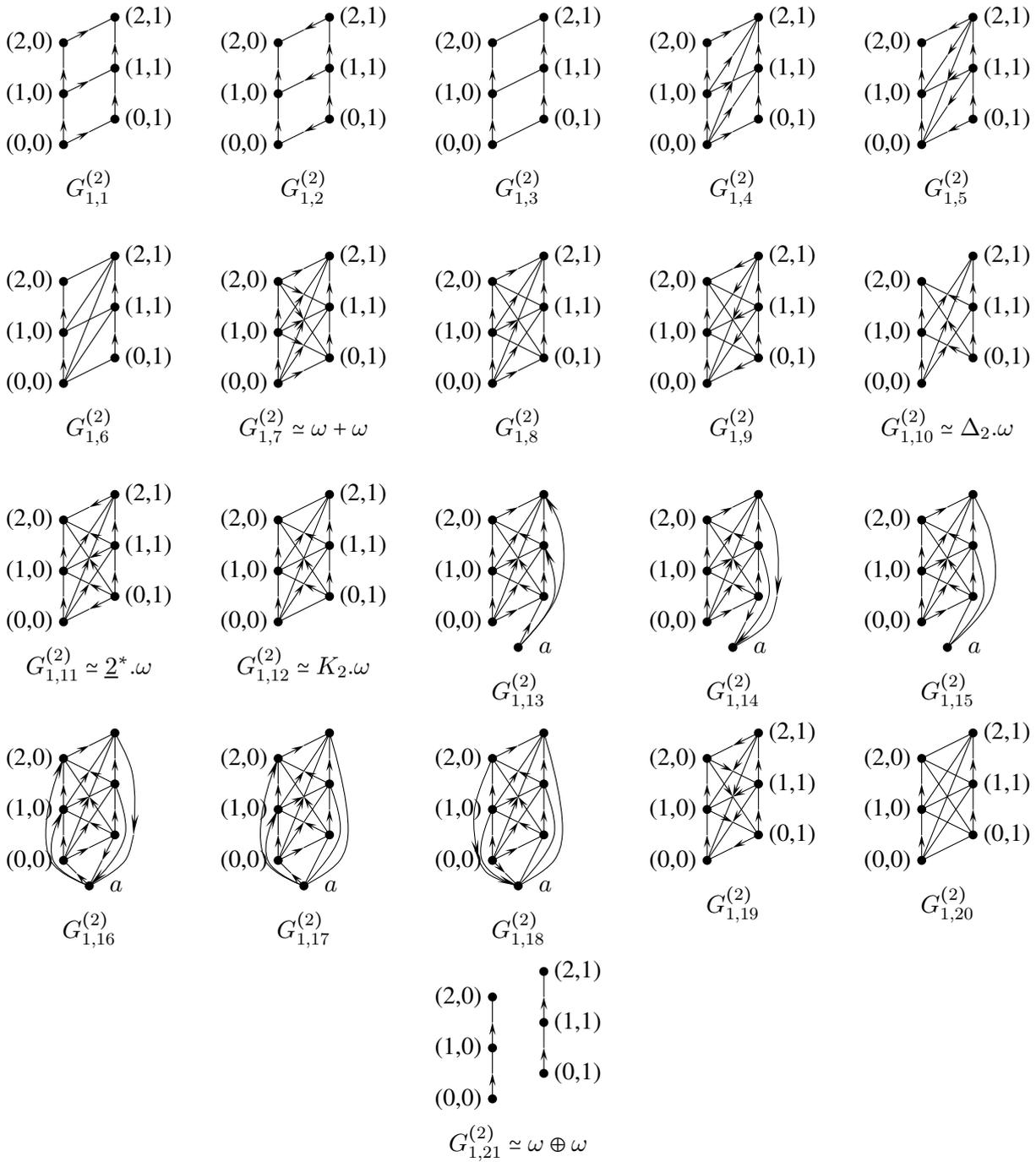


Figure 2. Minimal graphs of class $p = 2$ for $\ell = 1$. The ordered directed graphs $\mathcal{G}_{1,k}^{(2)}$ are in \mathfrak{A}_1^1 for $1 \leq k \leq 12$, in \mathfrak{B}_1^1 for $13 \leq k \leq 18$ and in \mathfrak{B}_1^2 for $19 \leq k \leq 21$.

II) If $3 \leq \ell \leq 6$, we have the same examples for each value of ℓ and their number is 15. In these cases, $\mathcal{G}_{\ell,k}^{(2)} \in \mathfrak{A}_1^1$ for every $1 \leq k \leq 15$.

- $G_{\ell,k}^{(2)} = G_{1,k}^{(2)}$ for every $1 \leq k \leq 6$ and for every $8 \leq k \leq 9$.
- $G_{\ell,7}^{(2)}$ is a linear order isomorphic to ω . The ordered directed graph $\mathcal{G}_{\ell,7}^{(2)}$ in this case is one of the bichains given in [20].
- $G_{\ell,10}^{(2)} = G_{1,19}^{(2)}$.
- For $11 \leq k \leq 15$, the graph $G_{\ell,k}^{(2)}$ coincides on $A \times B$ with $G_{\ell,k}^{(1)}$.

Class $p = 3$: In this case, the restrictions of $G_{\ell,k}^{(3)}$ to A and B are both chains isomorphic to ω^* .

I) If $\ell = 1, 2$ we have $1 \leq k \leq 21$, the ordered directed graphs $\mathcal{G}_{\ell,k}^{(3)}$ for $1 \leq k \leq 12$ are in \mathfrak{A}_1^1 , they are in \mathfrak{B}_1^1 for $13 \leq k \leq 18$ and in \mathfrak{B}_1^2 for $19 \leq k \leq 21$.

- For $1 \leq k \leq 9$, the graph $G_{\ell,k}^{(3)}$ coincides on $A \times B$ with $G_{\ell,k}^{(1)}$. Then $G_{1,5}^{(3)}$ is the dual of $G_{1,4}^{(2)}$, the directed graph $G_{\ell,7}^{(3)}$ is a chain isomorphic to $\omega^* + \omega^*$ and the ordered directed graph $\mathcal{G}_{\ell,7}^{(3)}$ is one of the bichains given in [20].

- For $10 \leq k \leq 21$, the graph $G_{\ell,k}^{(3)}$ is the dual of $G_{\ell,k}^{(2)}$. Thus the graph $G_{\ell,10}^{(3)}$ is isomorphic to $\Delta_2 \cdot \omega^*$, the graph $G_{\ell,11}^{(3)}$ is isomorphic to $\underline{2} \cdot \omega^*$ and hence the ordered directed graph $\mathcal{G}_{\ell,11}^{(3)}$ is one of the bichains given in [20]. The graph $G_{\ell,12}^{(3)}$ is isomorphic to $K_2 \cdot \omega^*$ and the graph $G_{\ell,21}^{(3)}$ is isomorphic to $\omega^* \oplus \omega^*$.

II) If $3 \leq \ell \leq 6$, we have the same examples for each value of ℓ and their number is 15. In these cases, $\mathcal{G}_{\ell,k}^{(3)} \in \mathfrak{A}_1^1$ for every $1 \leq k \leq 15$.

- $G_{\ell,k}^{(3)} = G_{1,k}^{(3)}$ for every $1 \leq k \leq 6$ and for every $8 \leq k \leq 9$.
- $G_{\ell,7}^{(3)}$ is a linear order isomorphic to ω^* . The corresponding ordered directed graph $\mathcal{G}_{\ell,7}^{(3)}$ is one of the bichains given in [20].
- $G_{\ell,10}^{(3)} = G_{1,19}^{(3)}$.
- For $11 \leq k \leq 15$, the graph $G_{\ell,k}^{(3)}$ coincides on $A \times B$ with $G_{\ell,k}^{(1)}$.

Class $p = 4$: In this case, A and B are both reflexive cliques.

I) If $\ell = 1, 2$ we have $1 \leq k \leq 21$, the ordered directed graphs $\mathcal{G}_{\ell,k}^{(4)}$ for $1 \leq k \leq 12$ are in \mathfrak{A}_1^1 , they are in \mathfrak{B}_1^1 for $13 \leq k \leq 18$ and in \mathfrak{B}_1^2 for $19 \leq k \leq 21$.

- For $1 \leq k \leq 9$, the graph $G_{\ell,k}^{(4)}$ coincides with $G_{\ell,k}^{(1)}$ on pairs of $A \times B$. Then $G_{\ell,7}^{(4)}$ is isomorphic to $K_{\mathbb{N}} + K_{\mathbb{N}}$, the ordinal sum of two reflexive cliques with the same vertex set \mathbb{N} .

- The graph $G_{\ell,10}^{(4)}$ is the symmetrized of $G_{\ell,10}^{(2)}$.
- The graph $G_{\ell,11}^{(4)}$ (respectively $G_{\ell,12}^{(4)}$) is obtained from $G_{\ell,10}^{(4)}$ by adding edges $\alpha_n, n \in \mathbb{N}$ (respectively $\alpha_n^{-1}, n \in \mathbb{N}$). The graph $G_{\ell,12}^{(4)}$ is the dual of $G_{\ell,11}^{(4)}$.

- For $13 \leq k \leq 18$, the graph $G_{\ell,k}^{(4)}$ is obtained from $G_{\ell,k}^{(2)}$ by taking its symmetrized on $A \cup B$, the remaining edges (ie, those for which one extremity is a) being the same as in $G_{\ell,k}^{(2)}$.

- The graph $G_{\ell,19}^{(4)}$ coincides with $G_{\ell,11}^{(1)}$ on pairs of $A \times B$.
- The graph $G_{\ell,20}^{(4)}$ is the dual of $G_{\ell,19}^{(4)}$.
- The graph $G_{\ell,21}^{(4)}$ is the symmetrized of $G_{\ell,21}^{(2)}$, it is isomorphic to $K_{\mathbb{N}} \oplus K_{\mathbb{N}}$ the direct sum of two reflexive cliques with the same vertex set \mathbb{N} .

II) If $3 \leq \ell \leq 6$, we have the same examples for each value of ℓ and their number is 15. In these cases, $\mathcal{G}_{\ell,k}^{(4)} \in \mathfrak{A}_1^1$ for every $1 \leq k \leq 15$.

- $G_{\ell,k}^{(4)} = G_{1,k}^{(4)}$ for every $1 \leq k \leq 6$.
- $G_{\ell,k}^{(4)} = G_{1,k+1}^{(4)}$ for every $7 \leq k \leq 8$.
- $G_{\ell,9}^{(4)} = G_{1,19}^{(4)}$.
- $G_{\ell,10}^{(4)} = G_{1,20}^{(4)}$.
- For $11 \leq k \leq 15$, the graph $G_{\ell,k}^{(4)}$ coincides on $A \times B$ with $G_{\ell,k}^{(1)}$.

Class $p = 5$: In this case all the ordered directed graphs $\mathcal{G}_{\ell,k}^{(5)}$ have the same vertex set which is $A \cup B$ such that one of the restrictions to A or B is a chain isomorphic to ω , the other being an antichain.

I) If $\ell = 1, 2$ we have $1 \leq k \leq 22$, in these cases $\mathcal{G}_{\ell,k}^{(5)} \in \mathfrak{A}_1^1$ for $1 \leq k \leq 9$ and $13 \leq k \leq 21$, the ordered directed graph $\mathcal{G}_{\ell,k}^{(5)} \in \mathfrak{B}_1^2$ for $10 \leq k \leq 12$ and $k = 22$.

- For $1 \leq k \leq 12$, the graph $G_{\ell,k}^{(5)}$ is such that its restriction to A is a chain isomorphic to ω , its restriction to B is an antichain and the remaining edges (those on $A \times B$) being the same as in $G_{\ell,k}^{(1)}$.

- For $13 \leq k \leq 21$, the graph $G_{\ell,k}^{(5)}$ is such that its restriction to B is a chain isomorphic to ω , its restriction to A is an antichain, the remaining edges (those on $A \times B$) being the same as in $G_{\ell,k-12}^{(1)}$.

- The graph $G_{\ell,22}^{(5)}$ is such that its restriction to A is a chain isomorphic to ω and its restriction to B is an antichain, there are no other edges. Thus $G_{\ell,22}^{(5)}$ is isomorphic to $\omega \oplus \Delta_{\mathbb{N}}$ the direct sum of a chain and an antichain with the same vertex set \mathbb{N} .

The graphical representations of $G_{1,k}^{(5)}$, $1 \leq k \leq 22$ are given in Figure 3.

II) If $3 \leq \ell \leq 6$, we have the same examples for each value of ℓ and their number is 24. In these cases, $\mathcal{G}_{\ell,k}^{(5)} \in \mathfrak{A}_1^1$ for every $1 \leq k \leq 24$.

- $G_{\ell,k}^{(5)} = G_{1,k}^{(5)}$ for every $1 \leq k \leq 6$.
- $G_{\ell,k}^{(5)} = G_{1,k+1}^{(5)}$ for every $7 \leq k \leq 10$.
- $G_{\ell,k}^{(5)} = G_{1,k+2}^{(5)}$ for every $11 \leq k \leq 16$.
- $G_{\ell,k}^{(5)} = G_{1,k+3}^{(5)}$ for every $17 \leq k \leq 18$.
- The graphs $G_{\ell,19}^{(5)}$, $G_{\ell,20}^{(5)}$ and $G_{\ell,21}^{(5)}$ coincide on $A \times B$ respectively with $G_{\ell,13}^{(1)}$, $G_{\ell,14}^{(1)}$ and $G_{\ell,15}^{(1)}$ such that the set A is ordered as ω and the set B is an antichain.
- The graphs $G_{\ell,22}^{(5)}$, $G_{\ell,23}^{(5)}$ and $G_{\ell,24}^{(5)}$ coincide on $A \times B$ respectively with $G_{\ell,13}^{(1)}$, $G_{\ell,14}^{(1)}$ and $G_{\ell,15}^{(1)}$ such that the set B is ordered as ω and the set A is an antichain.

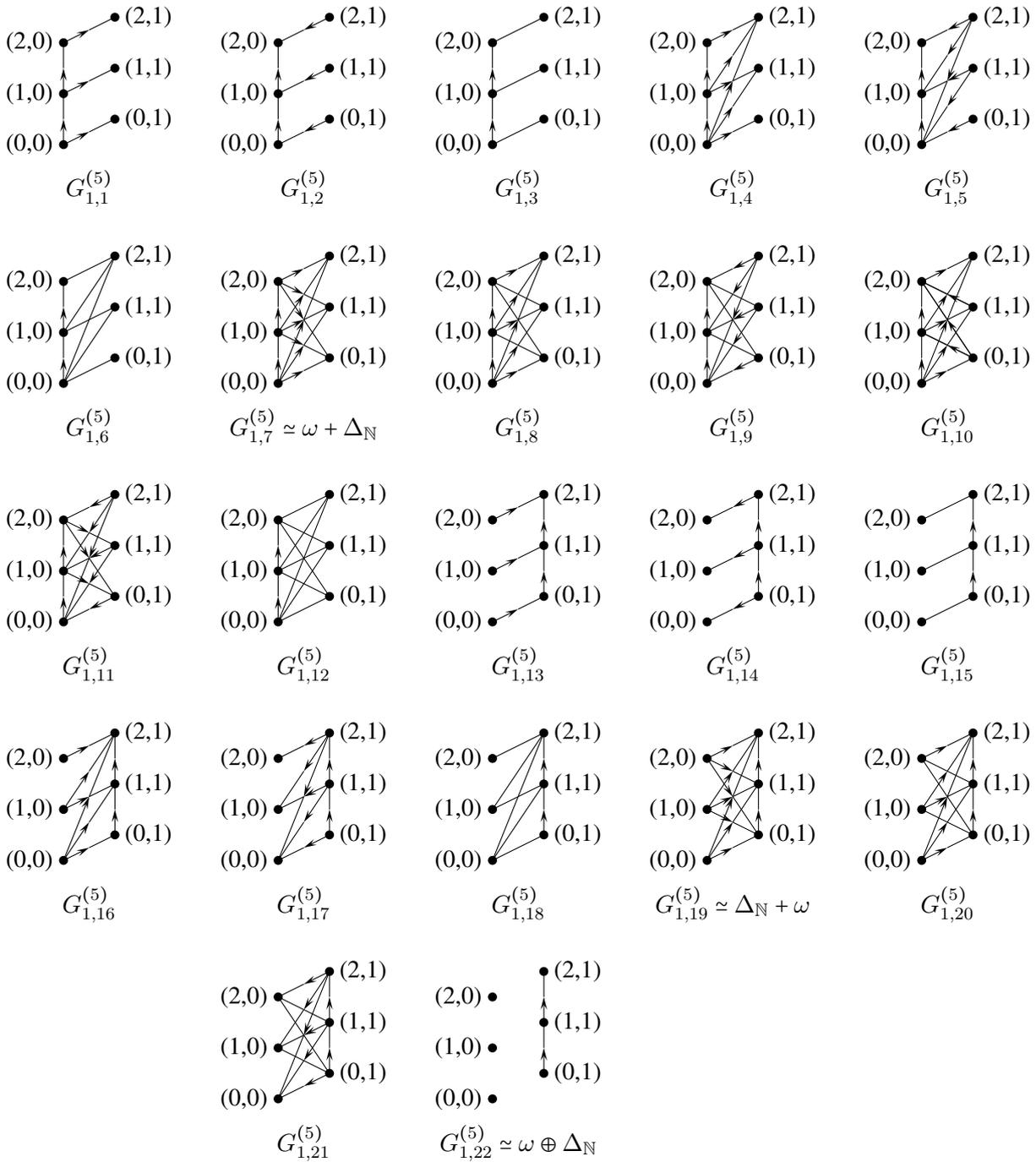


Figure 3. The minimal graphs of class $p = 5$ for $\ell = 1$. The ordered directed graph $G_{1,k}^{(5)}$ are in \mathfrak{A}_1^1 for $1 \leq k \leq 9$ and $13 \leq k \leq 21$ and in \mathfrak{B}_1^2 for $10 \leq k \leq 12$ and $k = 22$.

Classes $6 \leq p \leq 10$: In these cases all the ordered directed graphs have the same vertex set which is $A \cup B$. On pairs of $A \times B$, the graphs $G_{\ell,k}^{(p)}$ are obtained in the same way as in case $p = 5$, i.e., the graph $G_{\ell,k}^{(p)}$ coincides with $G_{\ell,k}^{(5)}$, with the following differences. For $p = 6$, the chain on A or B is replaced by a chain isomorphic to ω^* . Hence, every graph in this class is the dual of one graph of the class $p = 5$. For $p = 7$, the chain on A or B is replaced by a reflexive clique. If $p = 8$, the antichain on A or B is replaced by a chain isomorphic to ω^* . If $p = 9$, the antichain on A or B is replaced by a reflexive clique. And in case $p = 10$, the chain on A or B is replaced by a chain isomorphic to ω^* and the antichain is replaced by a reflexive clique.

In these cases, we also obtain bichains among those given in [20], they are the ordered directed graphs $\mathcal{G}_{\ell,k}^{(8)}$ with $\ell \in \{1, 2\}$ and $k \in \{7, 19\}$.

We obtain, among all these ordered directed graphs, the twenty bichains $\mathcal{B} := (V, \leq, \leq')$ of Monteil and Pouzet [20] and [6].

Proof of Lemma 6.1. Suppose that there is an embedding f of $\mathcal{G}_{\ell,k}^{(p)}$ into $\mathcal{G}_{\ell',k'}^{(p')}$ for some values of p, p', ℓ, ℓ', k and k' . Then f is an embedding of $(V_{\ell,k}^{(p)}, \leq_{\ell,k}^{(p)})$ into $(V_{\ell',k'}^{(p')}, \leq_{\ell',k'}^{(p')})$. It follows that, except for $\ell = 1$ and $\ell = 2$ (the chain ω is embeddable in $\omega, \omega + \omega, \omega + \omega^*, \omega^* + \omega$ and ω^* is embeddable in $\omega^*, \omega^* + \omega^*, \omega + \omega^*, \omega^* + \omega$), if $\ell \geq 3$ then necessarily $\ell = \ell'$ (because none of the orders $\omega + \omega, \omega + \omega^*, \omega^* + \omega, \omega^* + \omega^*$ is embeddable into the other). As, by construction, in these cases every vertex of A is less than every vertex of B w.r.t the linear orders $\leq_{\ell,k}^{(p)}$ and $\leq_{\ell',k'}^{(p')}$, then f sends A into A and B into B which leads to $p = p'$. Then, by invariance, we may consider f as an isomorphism from the restriction of $\mathcal{G}_{\ell,k}^{(p)}$ to $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$ onto the restriction of $\mathcal{G}_{\ell',k'}^{(p')}$ to the same set and this gives $k = k'$.

Now, suppose $\ell = 1$. According to the fact that the restrictions of each directed graph associated to an ordered directed graph of $\tilde{\mathfrak{A}}_1$ to A and to B are monomorphic and to the fact that, by construction, the restrictions of every member \mathcal{G} of $\tilde{\mathfrak{A}}_1$ to $\{(0, 0), (0, 1)\}$ and to $\{(0, 1), (1, 0)\}$ if $\mathcal{G} \in \mathfrak{A}_1^1 \cup \mathfrak{B}_1^2$ and its restrictions to $\{a, (0, 0), (0, 1)\}$ and to $\{a, (0, 1), (1, 0)\}$ if $\mathcal{G} \in \mathfrak{B}_1^1$ are not isomorphic (then a subset which contains both elements of A and B cannot be monomorphic), then either f sends A into A and B into B or f sends A into B and B into A . Then $p = p'$. It follows that $\ell' = 1 = \ell$, otherwise f is not an embedding of $(V_{\ell,k}^{(p)}, \leq_{\ell,k}^{(p)})$ into $(V_{\ell',k'}^{(p')}, \leq_{\ell',k'}^{(p')})$ (because in case $\ell = 1$ we have $(0, 0) \leq_{\ell,k}^{(p)} (0, 1) \leq_{\ell,k}^{(p)} (1, 0) \leq_{\ell,k}^{(p)} (1, 1)$ and if $\ell' \geq 3$ then $(0, 0) \leq_{\ell',k'}^{(p')} (1, 0) \leq_{\ell',k'}^{(p')} (0, 1) \leq_{\ell',k'}^{(p')} (1, 1)$, i.e. every vertex of A is less than every vertex of B).

Since these ordered directed graphs are invariant, it suffices to f to be an embedding from the fourth vertices set $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$, if $\mathcal{G}_{\ell,k}^{(p)} \in \mathfrak{A}_1^1 \cup \mathfrak{B}_1^2$ or from the set $\{a, (0, 0), (1, 0), (0, 1), (1, 1)\}$, if $\mathcal{G}_{\ell,k}^{(p)} \in \mathfrak{B}_1^1$ such that $f(0, 0)$ and $f(1, 0)$ are in A and $f(0, 1)$ and $f(1, 1)$ are in B , if f sends A into A and B into B , or such that $f(0, 0)$ and $f(1, 0)$ are in B and $f(0, 1)$ and $f(1, 1)$ are in A , if f sends A into B and B into A , and fixing a if any. We have no such embedding if $k \neq k'$. Then $k = k'$. If $\ell = 2$, by an analogous argument we have the same conclusion. \square

With this, the proof of Theorem 1.6 is complete.

7. Profiles of members of $\tilde{\mathfrak{A}}_1$ and a proof of Proposition 1.7

For all values of the integers p, ℓ and k , denote by $\varphi_{\ell,k}^{(p)}$ the profile of the ordered directed graph $\mathcal{G}_{\ell,k}^{(p)}$. We recall that the graph $G_{\ell,k}^{(p)}$ is the directed graph associated to the ordered directed graph $\mathcal{G}_{\ell,k}^{(p)}$. The proof of Proposition 1.7 follows from Lemmas 7.1 to 7.7.

Lemma 7.1. *The profile of the ordered directed graph $\mathcal{G}_{\ell,k}^{(p)}$ for $1 \leq \ell \leq 2$ and $(p = 1$ and $1 \leq k \leq 3)$ or $(2 \leq p \leq 4$ and $10 \leq k \leq 12)$ grows as the Fibonacci sequence.*

Proof. There are twenty four ordered directed graphs to consider. The proof given here take into account all these graphs. The reader can get some help by looking at the graphs $G_{1,1}^{(1)}, G_{1,2}^{(1)}, G_{1,3}^{(1)}$ represented

Figure 1 and the graphs $G_{1,10}^{(2)}$, $G_{1,11}^{(2)}$, $G_{1,12}^{(2)}$ represented Figure 2. Let $\mathcal{G}_{\ell,k}^{(p)} = (V_{\ell,k}^{(p)}, \leq_{\ell,k}^{(p)}, \rho_{\ell,k}^{(p)})$ be an ordered directed graph of our list, then $V_{\ell,k}^{(p)} = A \cup B$. As $\ell \in \{1, 2\}$, then $\leq_{\ell,k}^{(p)}$ is ordered as ω or ω^* . By invariance, the restrictions of $\mathcal{G}_{\ell,k}^{(p)}$ to the pairs $\{(n, 0), (n, 1)\}$, $n \in \mathbb{N}$, are all isomorphic. According to the description of the graphs given previously, all other pairs of $A \times B$ are isomorphic together (see $G_{1,1}^{(1)}$ and $G_{1,1}^{(2)}$ for an example). To calculate $\varphi_{\ell,k}^{(p)}(r)$ for $r \in \mathbb{N}$, consider r distinct vertices ordered w.r.t $\leq_{\ell,k}^{(p)}$. Then, either this chain ends by a pair of the form $\{(n, 0), (n, 1)\}$ with $n \in \mathbb{N}$ and, in this case, the number of non isomorphic subgraphs with r vertices is $\varphi_{\ell,k}^{(p)}(r - 2)$, or not. And in this latter case, the number of such subgraphs of order r is $\varphi_{\ell,k}^{(p)}(r - 1)$. We then get:

$$\begin{cases} \varphi_{\ell,k}^{(p)}(0) = \varphi_{\ell,k}^{(p)}(1) = 1. \\ \varphi_{\ell,k}^{(p)}(r) = \varphi_{\ell,k}^{(p)}(r - 2) + \varphi_{\ell,k}^{(p)}(r - 1) \text{ for } r \geq 2. \end{cases}$$

Lemma 7.2. *The profile of the ordered directed graph $\mathcal{G}_{\ell,k}^{(p)}$ for $1 \leq \ell \leq 2$ with $(p = 1 \text{ and } 7 \leq k \leq 12)$ or $(p = 2 \text{ and } k \in \{5, 6, 9\})$ or $(p = 3 \text{ and } k \in \{4, 6, 8\})$ or $(p = 4 \text{ and } k \in \{4, 5, 7\})$ is given by $\varphi_{\ell,k}^{(p)}(r) = 2^r - 1$, $r \geq 1$.* □

Proof. There are thirty ordered directed graphs. Nine associated directed graphs $G_{\ell,k}^{(p)}$ are represented in Figure 1 and Figure 2, namely $G_{1,k}^{(1)}$, for $7 \leq k \leq 12$, and $G_{1,k}^{(2)}$ for $k \in \{5, 6, 9\}$. In these cases, $V_{\ell,k}^{(p)} = A \cup B$ and every vertex of A is linked in the same way to all the vertices of B which are less than it and it is linked in the same way to all the vertices of B which are greater than it w.r.t $\leq_{\ell,k}^{(p)}$. Then, we can encode every subgraph with r vertices by a word of length r made of the two letters $\{0, 1\}$. Consider r distinct vertices ordered by $\leq_{\ell,k}^{(p)}$. To each vertex we associate 0 if it belongs to A and 1 if it belongs to B , the letters being from left to right. Except the two words $00 \dots 0$ and $11 \dots 1$ of the same length which yield isomorphic subgraphs (as $p \leq 4$ the restrictions of $G_{1,k}^{(1)}$ to A and B are isomorphic), two different words of length r yield two non isomorphic subgraphs. Hence the number of non isomorphic subgraphs with r vertices is equal to the number of different words of length r minus 1. □

Lemma 7.3. *The profile of the ordered directed graph $\mathcal{G}_{\ell,k}^{(p)}$ for $1 \leq \ell \leq 2$ with $(p = 1, k \in \{4, 5, 6\})$ or $(p = 2, k \in \{4, 7, 8\})$ or $(p = 3, k \in \{5, 7, 9\})$ or $(p = 4, k \in \{6, 8, 9\})$ is given by: $\varphi_{\ell,k}^{(p)}(r) = 2^r - r$, $r \geq 1$.*

Proof. There are twenty four ordered directed graphs to consider. Six associated directed graphs $G_{\ell,k}^{(p)}$ are represented in Figure 1 and Figure 2, namely $G_{1,k}^{(1)}$, for $k = 4, 5, 6$ and $G_{1,k}^{(2)}$, for $k = 4, 7, 8$. In these cases we have also $V_{\ell,k}^{(p)} = A \cup B$. In each of these cases, we can also encode, in the same order, every subgraph with r vertices by a binary word of length r as in the proof of lemma 7.2. For $p = 1$, we associate 0 to each vertex of A and 1 to each vertex of B . If $p = 2, 3, 4$ we do the converse (we associate 1 to each vertex of A and 0 to each vertex of B). Then all the words of length r of the form $\underbrace{1 \dots 1}_q \underbrace{0 \dots 0}_{r-q}$ with q ($0 \leq q \leq r$) are associated to isomorphic subgraphs (for example, for $p = 1$ these subgraphs are all isomorphic to an independent set with r vertices). Hence, the number of non isomorphic subgraphs with r vertices is at most the number of different words of length r minus r . In fact it is equal. □

Lemma 7.4. *The profile of the ordered directed graph $\mathcal{G}_{\ell,k}^{(p)}$ for $1 \leq \ell \leq 2$ with $(p = 1 \text{ and } k \in \{13, 14, 15\})$ or $(2 \leq p \leq 3 \text{ and } k \in \{13, 16, 17, 19, 20, 21\})$ or $(p = 4 \text{ and } k \in \{15, 17, 18, 19, 20, 21\})$ is given by $\varphi_{\ell,k}^{(p)}(r) = 2^{r-1}$, $r \geq 1$.*

Proof. There are forty two ordered directed graphs. Nine associated directed graphs $G_{\ell,k}^{(p)}$ are represented in Figure 1 and Figure 2, namely $G_{1,k}^{(1)}$, for $k = 13, 14, 15$, and $G_{1,k}^{(2)}$ for $k = 13, 16, 17, 19, 20, 21$.

If $k \notin \{19, 20, 21\}$, the vertex set for all other graphs cited in the lemma is $\mathbb{N} \times \{0, 1\} \cup \{a\}$. These graphs have the particularity to be monomorphic on $\mathbb{N} \times \{0, 1\}$ (that is the restrictions to two subsets with the same cardinality are isomorphic). We can encode every subgraph of length r by a word over the alphabet $\{a, 0, 1\}$. We associate 0 to each vertex of $\mathbb{N} \times \{0\}$ and 1 to each one of $\mathbb{N} \times \{1\}$ and we add a in the beginning of the word if the subgraph contains the vertex a . All words of length r made only with the two letters 0 and 1 yield isomorphic subgraphs. Depending on the graph, these subgraphs are isomorphic to those associated to words of length r which begin by a and whose remaining letters are identical (identical to 0 when $p = 1$ or $2 \leq p \leq 3$ and $k = 16, 17$ and identical to 1 for $p = 4$ or $2 \leq p \leq 3$ and $k = 13$). Then, the number of non isomorphic subgraphs of r vertices is equal to the number of different words of length r beginning by a . This number is 2^{r-1} .

If $k \in \{19, 20, 21\}$, the vertex set of the graphs is $A \cup B$. These graphs are such that every subgraph with q vertices from A and $r - q$ vertices from B is isomorphic to one of subgraphs with $r - q$ vertices from A and q vertices from B . In term of words, the graphs encoded by $a_1 a_2 \dots a_r$ and by its complement $\bar{a}_1 \bar{a}_2 \dots \bar{a}_r$ where $\bar{0} = 1$ and $\bar{1} = 0$, are isomorphic. The result follows. \square

Lemma 7.5. *The profile of the ordered directed graph $\mathcal{G}_{\ell,k}^{(p)}$ for $1 \leq \ell \leq 2$ and ($p = 1, k \in \{16, 17, 18\}$) or ($2 \leq p \leq 3, k \in \{14, 15, 18\}$) or ($p = 4, k \in \{13, 14, 16\}$) is given by $\varphi_{\ell,k}^{(p)}(r) = 2^{r-1} + 1, r \geq 2$.*

Proof. There are twenty four ordered directed graphs to consider. Six associated directed graphs $G_{\ell,k}^{(p)}$ are represented in Figure 1 and Figure 2, namely $G_{1,k}^{(1)}$, for $k = 16, 17, 18$, and $G_{1,k}^{(2)}$ for $k = 14, 15, 18$. For all the graphs cited in the lemma the vertex set is $\mathbb{N} \times \{0, 1\} \cup \{a\}$. They also have the particularity to be monomorphic on $\mathbb{N} \times \{0, 1\}$. As previously done in the proof of Lemma 7.4, we can encode each subgraph on r vertices by a word over the alphabet $\{a, 0, 1\}$, with the difference that in this case, the subgraphs of order r whose associated words begin by a and are made with only one letter (0 or 1) are not isomorphic to those whose associated word do not begin by a . Then, the number of non isomorphic subgraphs of order $r \geq 2$ is equal to the number of words of length r beginning by a plus one. This gives $2^{r-1} + 1$. \square

Lemma 7.6. $\varphi_{\ell,k}^{(p)}(r) \geq 2^{r-1}, r \geq 2$ for $1 \leq \ell \leq 2$ and ($2 \leq p \leq 4, k \in \{1, 2, 3\}$).

Proof. There are eighteen ordered directed graphs. Three associated directed graphs are represented in Figure 2, namely $G_{1,k}^{(2)}$, for $k = 1, 2, 3$. Note that the first values of the profile of these ordered graphs are: $\varphi_{\ell,k}^{(p)}(0) = \varphi_{\ell,k}^{(p)}(1) = 1, \varphi_{\ell,k}^{(p)}(2) \in \{2, 3\}$ and $\varphi_{\ell,k}^{(p)}(3) \in \{6, 8\}$ (eg. $\varphi_{1,1}^{(2)}(2) = 2, \varphi_{1,1}^{(2)}(3) = 6$ and $\varphi_{1,2}^{(2)}(2) = 3, \varphi_{1,2}^{(2)}(3) = 8$). Now, to each subgraph with r vertices, ordered according to $\leq_{\ell,k}^{(p)}$, we can associate a word of length r on $\{0, 1\}$ (we associate 0 if the vertex is in A and 1 otherwise). If $p = 2, k \in \{2, 3\}$ or $p = 3, k \in \{1, 3\}$ or $p = 4, k \in \{1, 2\}$, two different words are associated to non isomorphic subgraphs, except for the two words of the forme $\underbrace{0 \dots 0}_q \underbrace{1 \dots 1}_{r-q}$ and $\underbrace{1 \dots 1}_q \underbrace{0 \dots 0}_{r-q}$ with ($0 \leq q \leq r$) which, for each value of q are associated to two isomorphic subgraphs, but each of the words containing the factor 01 is associated to two non isomorphic subgraphs. Indeed, the factor 01 corresponds to two vertices $(n, 0)$ and $(m, 1)$ which comes successively according to $\leq_{\ell,k}^{(p)}$, the case $m = n$ leads to a subgraph which is different from those obtained in the case $m > n$. The result follows.

If $p = 2, k = 1$ or $p = 3, k = 2$ or $p = 4, k = 3$, all words of the form $\underbrace{0 \dots 0}_q \underbrace{1 \dots 1}_{r-q}$ with $(0 \leq q \leq r)$ gives the same subgraph if the factor 01 corresponds to vertices $(n, 0)$ and $(m, 1)$ with $m = n$. But for each factor 01 contained in a given word we have two different subgraphs. As there are more than r words with factors 01 the result follows. \square

Lemma 7.7. *For $\ell \geq 3$ or $p \geq 5$, the profile of the ordered directed graph $\mathcal{G}_{\ell,k}^{(p)}$ is greater or equal to one of the five functions: $\varphi_1(n) := 2^n - 1$, $\varphi_2(n) := 2^n - n$, $\varphi_3(n) := 2^{n-1}$, $\varphi_4(n) := 2^{n-1} + 1$ and the Fibonacci function.*

Proof. There are one thousand and eighty ordered directed graphs. Twenty two associated directed graphs, corresponding to $\ell = 1, p = 5$ are represented Figure 3. All graphs for $p \geq 5$ are deduced from graphs for $p \leq 4$ with the restrictions to $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{1\}$ which are not isomorphic. Hence, the number of subgraphs on r vertices is greater than those obtained in case $p \leq 4$ where the profiles are given by one of the above five functions. For $\ell \geq 3$, the graphs are obtained from those for which $\ell \leq 2$ by changing the linear order $\leq_{\ell,k}^{(p)}$ (for $3 \leq \ell \leq 6$ the linear order is one of the orders $\omega + \omega, \omega^* + \omega, \omega + \omega^*, \omega^* + \omega^*$), the arguments used in the proofs of previous lemmas remain valid. \square

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